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Rigidity of pseudo-free group actions on contractible manifolds

By

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In this article, we announce joint work with Frank Connolly and Jim Davis [6]. This follows an earlier case study of pseudo-free involutions on the n-torus carried out in [5]. The author is grateful to the organizers of the RIMS conferences where these results were disseminated in Asia: *Transformation Groups and Surgery Theory* (Masayuki Yamasaki, August 2010), *Transformation Groups and Combinatorics* (Mikiya Masuda, June 2011).

§ 1. History and the Main Theorem

Definition. Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of a group Γ. We say that Γ satisfies *Property* $C_{\mathcal{F}\subset\mathcal{G}}$ if every element $H \in \mathcal{G} - \mathcal{F}$ has its centralizer $C_\Gamma(H)$ in $\mathcal{G}$. One says that Γ satisfies *Property* $M_{\mathcal{F}\subset\mathcal{G}}$ if every element $H \in \mathcal{G} - \mathcal{F}$ is contained in a unique maximal element H_{max} of $\mathcal{G}$. Furthermore, one says that Γ satisfies *Property* $NM_{\mathcal{F}\subset\mathcal{G}}$ if Γ satisfies $M_{\mathcal{F}\subset\mathcal{G}}$ and each H_{max} is self-normalizing in Γ.

Below we consider the increasing chain $\{1\} \subset \mathtt{fin} \subset \mathtt{fbc} \subset \mathtt{vc}$ of families, where $\{1\}$ consists of the trivial subgroup, $\mathtt{fin}$ consists of the finite subgroups, $\mathtt{fbc}$ consists of the finite-by-cyclic subgroups, and $\mathtt{vc}$ consists of the virtually cyclic subgroups.

Definition. Let Γ be a group. We define $\mathscr{S}(\Gamma)$ as the set of Γ-homeomorphism classes of contractible manifolds equipped with an effective cocompact proper Γ-action.

For any Γ-space X, consider the free part of the action:

$$X_{free} := \{x \in X \mid gx = x \text{ implies } g = 1 \in \Gamma\}.$$

Our Main Theorem parameterizes $\mathscr{S}(\Gamma)$, and determines when it is one element.

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Theorem 1.1 (Main Theorem). *Let Γ be a group. Assume:*

1. *Γ satisfies Property $C_{\{1\}\subset \mathtt{fin}}$,*
2. *Γ satisfies Property $M_{\mathtt{fbc}\subset \mathtt{vc}}$,*
3. *Γ is virtually torsion-free with $n := \mathrm{vcd}(\Gamma) > 4$,*
4. *there exists $[X, \Gamma] \in \mathscr{S}(\Gamma)$ where X_{free}/Γ has the homotopy type of a finite complex,*
5. *Γ satisfies the Farrell–Jones Conjecture in lower K-theory and in L-theory.*

Write $\varepsilon := (-1)^n$. There is a bijection of sets, with $0 \mapsto [X, \Gamma]$, given by Wall realization:

$$\bigoplus_{(\mathtt{mid})(\Gamma)} \mathrm{UNil}_{n+\varepsilon}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \xrightarrow{\approx} \mathscr{S}(\Gamma). \tag{1.1}$$

Here $(\mathtt{mid})(\Gamma)$ is the set of conjugacy classes of maximal infinite dihedral subgroups of Γ. Furthermore, each element of $\mathscr{S}(\Gamma)$ has a locally conelike representative with the same Γ-homeomorphism type of links of singularities.

In particular, if $n \equiv 0, 1 \pmod 4$, or if Γ has no element of order two, then $\mathscr{S}(\Gamma)$ has only one element. In this case, for any cocompact Γ-manifold M, every Γ-homotopy equivalence $f : M \to X$ is Γ-homotopic to a Γ-homeomorphism.

For the proof, see the full article [6]. Notably, for the topological actions $\Gamma \curvearrowright M$, Smith theory was used to get isolated fixed points from Hypothesis (1), and Siebenmann theory was used to conclude the action must be locally conelike from Hypothesis (4).

The vanishing result of the last paragraph of Theorem 1.1 is immediate from the following calculation [7, 4] of the Cappell groups that occur as the summands in (1.1).

Theorem 1.2 (Connolly–Davis–Ranicki). *Let n be an integer. Set $\varepsilon := (-1)^n$. Then there is an isomorphism of abelian groups:*

$$\mathrm{UNil}_{n+\varepsilon}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \equiv 1 \pmod 4 \\ (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty & \text{if } n \equiv 2 \pmod 4 \\ (\mathbb{Z}/2)^\infty & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

The parameterization of (1.1) is achieved away from the singularities by a smooth handle construction, where gluing instructions are given by generalized Arf invariants.

In Section 2, we show that the above five properties are satisfied by certain actions on CAT(0) manifolds. In Section 3, we provide a family of exotic CAT(0) examples which cannot come from a Riemannian manifold of nonpositive sectional curvatures.

§ 2. Geometric consequences

Definition. A proper action $\Gamma \curvearrowright X$ is *pseudo-free* if the singular set is discrete:

$$X_{sing} \;:=\; \{x \in X \mid gx = x \text{ for some } g \neq 1 \in \Gamma\}.$$

Theorem 1.1 was originally established in [5] for the special case of the family of crystallographic groups $\Gamma = \mathbb{Z}^n \rtimes_{-1} C_2$ for all $n > 3$. More generally, we conclude:

Corollary 2.1. *Let Γ be a pseudo-free, cocompact, discrete group of isometries of Euclidean space $\mathbb{E}^n$ or hyperbolic space $\mathbb{H}^n$ with $n > 4$. The bijection* (1.1) *holds.*

Proof. This will be immediate from Corollary 2.2 below and Selberg's lemma. □

Euclidean and hyperbolic spaces fit into a broader class, CAT(0) spaces (see [3]):

Corollary 2.2. *Let X be a* CAT(0) *topological manifold of dimension $n > 4$. Suppose Γ is a virtually torsion-free, locally conelike, pseudo-free, cocompact discrete proper group of isometries of X. Then the bijection* (1.1) *holds.*

This can be viewed as a generalization of [2, Theorem A]. Both rigidity results rely on the truth of the Farrell–Jones Conjecture for these groups, [2, Theorem B].

Proof. By assumption, Hypothesis (3) holds. Since any two points in X are joined by a unique geodesic segment, X is contractible. Also, since $\Gamma \curvearrowright X$ is locally conelike, the quotient X_{free}/Γ is the interior of a compact topological ∂-manifold. Hence Hypothesis (4) holds. By a recent theorem of Bartels–Lück [2], Hypothesis (5) holds.

Let H be a nontrivial finite subgroup of Γ. Since the action $\Gamma \curvearrowright X$ is pseudo-free, the fixed set X^H is a single point. Note the proper action $\Gamma \curvearrowright X$ restricts to a proper action $C_\Gamma(H) \curvearrowright X^H$. So $C_\Gamma(H)$ is finite. Therefore Hypothesis (1) holds.

Let $D \in \mathsf{vc}(\Gamma) - \mathsf{fbc}(\Gamma)$. There is a unique D-invariant geodesic line $\ell_D \subset X$, as follows. It follows from Hypothesis (1), see [6], that D is isomorphic to the infinite dihedral group, $D_\infty = C_2 * C_2$. Write $D = \langle a, b \mid a^2 = b^2 = 1\rangle$. Let x and y be the unique fixed points in X of a and b. Since ab has infinite order, x and y are distinct, joined by a unique geodesic segment $\sigma \subset X$. Note that $D\sigma$ is homeomorphic to $\mathbb{R}$ and is a closed subset of X. Suppose $\ell \subset X$ is a D-invariant geodesic line. Then $D\sigma \approx \mathbb{R}$ is a closed subset of $\ell \approx \mathbb{R}$. Hence $D\sigma = \ell$. Therefore, any such ℓ is unique.

It remains to prove such an ℓ exists, that is, the D-invariant embedded line $D\sigma \subset X$ is geodesic. It suffices to show that the segment $\sigma \cup b\sigma \subset X$ joining x and bx is geodesic. Let $\tau \subset X$ be the unique geodesic segment joining x and bx. Since $b^2 = 1$ and the action of b is isometric, τ is b-invariant and its midpoint m, with respect to the arclength parameterization, is fixed by b. Hence $y = m$ and so $\sigma \cup b\sigma = \tau$ is geodesic. Therefore $\ell_D := D\sigma$ is the unique D-invariant geodesic line in X.

If $D' \in \mathsf{vc}(\Gamma) - \mathsf{fbc}(\Gamma)$ satisfies $D \subseteq D'$, then $\ell_{D'}$ is D-invariant, hence $\ell_{D'} = \ell_D$, so $D' \subseteq \mathrm{Stab}_\Gamma(\ell_D)$. Therefore, since $\mathrm{Stab}_\Gamma(\ell_D)$ has a proper isometric action on $\ell_D \approx \mathbb{R}$, it is the unique maximal virtually cyclic subgroup of Γ containing D. Thus Hypothesis (2) holds. Now apply Theorem 1.1 in order to obtain the bijection (1.1). $\square$

§ 3. Geometric examples

Indeed, such CAT(0) examples of (X, Γ) exist which cannot be Riemannian. A natural source for such infinite Γ with 2-torsion are reflection groups of convex polytopes. Thanks go to Mike Davis for feedback on this exposition and a guide to define Γ below.

Let K be an abstract simplicial complex with finite vertex set S. In [8, Section 1.2], Davis constructs a cubical cell complex P_K and right-angled Coxeter system (W_K, S):

$$P_K \;:=\; \bigcup_{\sigma \in K} [-1,1]^\sigma \times \{-1,1\}^{S-\sigma} \;\subset\; [-1,1]^S \tag{3.1}$$

$$W_K \;=\; \left\langle S \;\middle|\; \{s^2 = 1\}_{s \in S} \;,\; \{[s,t] = 1\}_{\{s,t\} \in K} \right\rangle . \tag{3.2}$$

Herein, we use the set-theoretic notation $B^A := \{\text{functions } f : A \longrightarrow B\}$.

The link of each vertex of P_K, hence of each vertex of the universal cover $\widetilde{P_K}$, is isomorphic to the geometric realization $|K| \subset [0,1]^S$. There is a cocompact, proper, isometric action $W_K \curvearrowright \widetilde{P_K}$ covering the natural reflection action $W_K \curvearrowright [-1,1]^S$. From these actions, Davis obtains an identification and an exact sequence of groups:

$$1 \longrightarrow \pi_1(P_K) = [W_K, W_K] \longrightarrow W_K \xrightarrow{\ \varphi\ } \{-1,1\}^S \longrightarrow 1. \tag{3.3}$$

The *barycentric subdivision* bK is the abstract simplicial complex whose n-simplices are all linearly ordered subsets of K of cardinality $n+1$. A simplicial complex is *flag* if, whenever a finite subset of vertices are pairwise joined by edges, they span a simplex. Since bK is flag, by [8, Proposition 1.2.3], the induced metric on $X := \widetilde{P_{bK}}$ is CAT(0). Then, since P_{bK} is aspherical, (3.3) implies that $W := W_{bK}$ is virtually torsion-free.

Lemma 3.1. *Let K be an abstract simplicial complex with finite vertex set. Recall the right-angled Coxeter group W and the cubical complex X defined above. There is a virtually torsion-free subgroup $\Gamma \trianglelefteq W$ with torsion such that $\Gamma \curvearrowright X$ is pseudo-free.*

Proof. Note bK has vertex set K. Write $n := \dim K$. Consider the epimorphism

$$\theta : \{-1,1\}^K \longrightarrow \{-1,1\}^{n+1} \;; \qquad f \longmapsto \left(\prod_{\dim \sigma = i} f(\sigma) \right)_{i=0}^{n} .$$

Define a normal subgroup

$$\Gamma \;:=\; (\theta \circ \varphi)^{-1}\langle(-1,\dots,-1)\rangle \;\trianglelefteq\; W.$$

Note Γ is virtually torsion-free: in fact, (3.3) restricts to an exact sequence

$$1 \longrightarrow [W,W] \longrightarrow \Gamma \xrightarrow{\;\varphi\;} \theta^{-1}\langle(-1,\dots,-1)\rangle \longrightarrow 1.$$

Observe the reflection action $W_{\Delta^n} = \{-1,1\}^{n+1} \curvearrowright [-1,1]^{n+1} = P_{\Delta^n}$ restricts to a pseudo-free action $\langle(-1,\dots,-1)\rangle \curvearrowright P_{\Delta^n}$. There is a cubical map $P_{bK} \to P_{\Delta^K} \to P_{\Delta^n}$, induced on P-constructions by an inclusion and a projection, that is equivariant with respect to the homomorphism $\theta\circ\varphi : W \longrightarrow W_{\Delta^n}$ and is injective on each cube $[-1,1]^\sigma$. Then $W \curvearrowright P_{bK}$ restricts to a pseudo-free action $\Gamma \curvearrowright P_{bK}$. So, since the map $X \longrightarrow P_{bK}$ is W-equivariant and is injective on each cube, the action $\Gamma \curvearrowright X$ is pseudo-free. $\square$

Example 3.2. Now we proceed to specify the exotic CAT(0) examples $W \curvearrowright X$ of Davis–Januskiewicz, recounted in [8, Example 10.5.3]. The key feature is that X is a topological manifold of any given dimension $n \geq 7$, but it not simply connected at infinity. Hence X is a contractible n-dimensional manifold, not homeomorphic to $\mathbb{R}^n$.

Let $3 \leq m \leq n-4$. Start with a triangulated homology m-sphere M with fundamental group $\pi \neq 1$. (Recall a *homology m-sphere* is a closed manifold with the same integral homology groups as S^m.) For example, M can be the Poincaré homology 3-sphere. Write C for the complement of the open star of a vertex in M. Then C is a compact, triangulated ∂-manifold of dimension m, with the fundamental group π, and $\partial C \approx S^{m-1}$. Thicken C into a compact, triangulated ∂-manifold

$$A := C \times D^{n-m-1} \;\text{ with }\; \partial A \;\approx\; (C \times S^{n-m-2}) \cup_{(S^{m-1}\times S^{n-m-2})} (S^{m-1} \times D^{n-m-1}).$$

Note the induced map $\pi_1(\partial A) \to \pi_1(A) = \pi$ of fundamental groups is an isomorphism. Furthermore, ∂A is a homology $(n-2)$-sphere, since M is a homology m-sphere.

Define a simply connected homology-manifold L of dimension $n-1$ by

$$L \;:=\; A \cup_{\partial A} \mathrm{Cone}(\partial A).$$

Observe that L is not a manifold since the link of the cone point c is not a sphere. Nonetheless, by a theorem of Edwards, the suspension of L is a topological manifold. More generally, this is true for any triangulated homology-manifold with simply connected links. Write K for the abstract simplicial complex of L. Consider the cubical Davis complex X, right-angled Coxeter system (W,S), and subgroup Γ from Lemma 3.1. Each vertex link is $|bK| \approx |K| = L$. Thus X is a topological manifold. Therefore, Corollary 2.2 calculates $\mathscr{S}(\Gamma)$. But, by [8, Theorem 9.2.2], $\pi_1(L-c) \neq 1$ implies $\pi_1^\infty(X) \neq 1$.

Finally, the axiomatic formulation of Theorem 1.1 is worthwhile; it removes the reliance on convex geometry in the proof. Here is a non-convex example to illustrate the axioms; thanks go to David Speyer for pointing it out on `http://mathoverflow.net`.

Example 3.3. For any commutative ring R, recall the R-Heisenberg group

$$\mathrm{Hei}(R) \ := \ \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ \middle| \ x, y, z \in R \right\} \ \subset \ GL(3, R).$$

Consider the Eisenstein integers $\mathbb{Z}[\omega]$, where $\omega := \exp(2\pi i/3) \in \mathbb{C}$ is a primitive third root of unity. Also consider the diagonal matrix $D := \mathrm{diag}(1, \omega, \omega^2) \in GL(3, \mathbb{C})$. Define a semidirect product $\Gamma = \mathrm{Hei}(\mathbb{Z}[\omega]) \rtimes C_3$, where the C_3-action is given by conjugation by D in $GL(3, \mathbb{C})$. Take $X = \mathrm{Hei}(\mathbb{C})$. Then Γ satisfies Hypotheses (1–5), using a theorem of Bartels–Farrell–Lück [1]. Therefore: $\mathscr{S}(\Gamma) = \{[X, \Gamma]\}$, by Theorem 1.1.

Recall the Solvable Subgroup Theorem [3, II.7.8]: if a virtually solvable group Γ admits a cocompact proper action by isometries on a CAT(0) space, then Γ must be virtually abelian. However, our group Γ is virtually solvable but not virtually abelian. Therefore our Γ cannot act cocompactly and properly by isometries on a CAT(0) space.

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