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Cancellation for 4-manifolds with virtually abelian fundamental group

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ABSTRACT

Suppose X and Y are compact connected topological 4-manifolds with fundamental group π. For any $r \geqslant 0$, X **is** r**-stably homeomorphic to** Y if $X\#r(S^2 \times S^2)$ is homeomorphic to $Y\#r(S^2 \times S^2)$. How close is stable homeomorphism to homeomorphism?

When the common fundamental group π is virtually abelian, we show that large r can be diminished to $n+2$, where π has a finite-index subgroup that is free-abelian of rank n. In particular, if π is finite then $n = 0$, hence X and Y are 2-stably homeomorphic, which is one $S^2 \times S^2$ summand in excess of the cancellation theorem of Hambleton–Kreck [12].

The last section is a case study of the homeomorphism classification of closed manifolds in the tangential homotopy type of $X = X_-\#X_+$, where $X_\pm$ are closed nonorientable topological 4-manifolds with order-two fundamental groups [13].

1. Introduction

Suppose X is a compact connected smooth 4-manifold, with fundamental group π and orientation character $\omega : \pi \to \{\pm 1\}$. Our motivation herein is the Cappell–Shaneson stable surgery sequence [7, 3.1], whose construction involves certain stable diffeomorphisms. These explicit self-diffeomorphisms lead to a modified version of Wall realization rel ∂X:

$$L_5^s(\mathbf{Z}[\pi^\omega]) \times \mathcal{S}^s_{\mathrm{DIFF}}(X) \longrightarrow \overline{\mathcal{S}}^s_{\mathrm{DIFF}}(X), \tag{1}$$

where $\mathcal{S}$ is the simple smooth structure set and $\overline{\mathcal{S}}$ is the stable structure set. Recall that the equivalence relation on these structure sets is smooth s-bordism of smooth manifold homotopy structures. The actual statement of [7, Theorem 3.1] is sharper in that the amount of stabilization, that is, the number of connected summands of $S^2 \times S^2$, depends only on the rank of a representative of a given element of the odd L-group.

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In the case X is *sufficiently large*, in that it contains a two-sided incompressible smooth 3-submanifold Σ, a periodicity argument using Cappell's decomposition [6, 7] shows that the restriction of the above action on $\mathcal{S}^s_{\mathrm{DIFF}}(X)$ to the summand UNil^s_5 of $L^s_5(\mathbf{Z}[\pi^\omega])$ is free. Therefore for each nonzero element of this exotic UNil-group, there exists a distinct, stable, smooth homotopy structure on X, restricting to a diffeomorphism on ∂X, which is not $\mathbf{Z}[\pi_1(\Sigma)]$-homology splittable along Σ. If Σ is the 3-sphere, the TOP case is [17]. Furthermore, when X is a connected sum of two copies of $\mathbf{RP}^4$, see [15] and [5].

For any $r \geqslant 0$, denote the r**-stabilization of** X by

$$X_r := X\#r(S^2 \times S^2).$$

2. On the topological classification of 4-manifolds

The main result (2.4) of this section is an upper bound on the number of $S^2 \times S^2$ connected summands sufficient for a *stable homeomorphism,* where the fundamental group of X lies in a certain class of good groups. By using Freedman–Quinn surgery [10, §11], if X is also sufficiently large (2.3 for example), each nonzero element ϑ of the UNil-group and simple DIFF homotopy structure $(Y, h : Y \to X)$ pair to form a distinct TOP homotopy structure $(Y_\vartheta, h_\vartheta)$ that represents the DIFF homotopy structure $\vartheta \cdot (Y, h)$ obtained from (1).

2.1. Statement of results

For finite groups π, the theorem's conclusion and the proof's topology are similar to Hambleton–Kreck [12]. However, the algebra is quite different.

Theorem 2.1. *Suppose π is a good group (in the sense of [10]) with orientation character $\omega : \pi \to \{\pm 1\}$. Consider $A := \mathbf{Z}[\pi^\omega]$, a group ring with involution: $\overline{g} = \omega(g)g^{-1}$. Select an involution-invariant subring R of the commutative* Center(A). *Its norm subring is*

$$R_0 \;:=\; \left\{ \sum_i x_i \overline{x}_i \;\middle|\; x_i \in R \right\}.$$

Suppose A is a finitely generated R_0-module, R_0 is noetherian, and the dimension d is finite:

$$d := \dim(\operatorname{maxspec} R_0) < \infty.$$

Now suppose that X is a compact connected TOP *4-manifold with*

$$(\pi_1(X), w_1) = (\pi, \omega)$$

and that it has the form

$$(X, \partial X) = (X_{-1}, \partial X)\#(S^2 \times S^2).$$

If X_r is homeomorphic to Y_r for some $r \geqslant 0$, then X_d is homeomorphic to Y_d.

Here are the class of examples of good fundamental groups promised in the paper's title.

Proposition 2.2. *Suppose π is a finitely generated, virtually abelian group, with any homomorphism ω : $\pi \to \{\pm 1\}$. For some R, the pair (π, ω) satisfies the above hypotheses: π is good, A is a finitely generated*

R_0-module, R_0 is noetherian, and d is finite. Furthermore, $d = n + 1$, where π contains a finite-index subgroup that is free-abelian of finite rank $n \geqslant 0$.

The author's original motivations are infinite virtually cyclic groups of the second kind.

Corollary 2.3. *Let X be a compact connected* TOP *4-manifold whose fundamental group is an amalgamated product $G_- *_F G_+$ with F a finite common subgroup of $G_\pm$ of index two. If Y is stably homeomorphic to X, then $Y\#3(S^2 \times S^2)$ is homeomorphic to $X\#3(S^2 \times S^2)$.*

Proof. Division of $G_\pm$ by the normal subgroup F yields a short exact sequence of groups:

$$1 \longrightarrow F \longrightarrow G_- *_F G_+ \longrightarrow \mathbf{C}_2 * \mathbf{C}_2 \cong \mathbf{C}_\infty \rtimes_{-1} \mathbf{C}_2 \longrightarrow 1.$$

So $\pi_1(X)$ contains an infinite cyclic group of finite index (namely, twice the order of F). Now apply Proposition 2.2 with $n = 1$ ($d = 2$). Then apply Theorem 2.1 to $X\#(S^2 \times S^2)$. □

Given the full strength of the proposition, we generalize the above specialized corollary.

Corollary 2.4. *Let X be a compact connected* TOP *4-manifold whose fundamental group is virtually abelian: say $\pi_1(X)$ contains a finite-index subgroup that is free-abelian of rank $n < \infty$. If Y is stably homeomorphic to X, then Y is $(n+2)$-stably homeomorphic to X.* □

More generally, can we reach the same conclusion if π has a finite-index subgroup Γ that is *polycyclic* of Hirsch length n? The example $\pi = \mathbf{Z}^2 \rtimes_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)} \mathbf{Z}$ is not virtually abelian.

2.2. Definitions and lemmas

The following concepts with applications are more fully expounded in Bak's book [1], though below we refer to Bass's book [3] as they first appeared there. We assume the reader knows the more standard notions.

Definition 2.5 *([3, I:4.1]).* A **unitary ring** (A, λ, Λ) consists of a ring with involution A, an element

$$\lambda \in \mathrm{Center}(A) \quad \text{satisfying} \quad \lambda\overline{\lambda} = 1,$$

and a **form parameter** Λ. This is an abelian subgroup of A satisfying

$$\{\, a + \lambda\overline{a} \mid a \in A \,\} \subseteq \Lambda \subseteq \{\, a \in A \mid a - \lambda\overline{a} = 0 \,\}$$

and

$$r a \overline{r} \in \Lambda \quad \text{for all} \quad r \in A \text{ and } a \in \Lambda.$$

Here is a left-handed classical definition discussed in the equivalence after its reference.

Definition 2.6 *([3, I:4.4]).* We regard a **quadratic module** over a unitary ring (A, λ, Λ) as a triple $(M, \langle \cdot, \cdot \rangle, \mu)$ consisting of a left A-module M, a bi-additive function

$$\langle \cdot, \cdot \rangle : M \times M \longrightarrow A \quad \text{such that} \quad \langle ax, by \rangle = a \,\langle x, y \rangle\, \overline{b} \quad \text{and} \quad \langle y, x \rangle = \lambda\, \overline{\langle x, y \rangle}$$

(called a **λ-hermitian form**), and a function (called a **Λ-quadratic refinement**)

$$\mu : M \longrightarrow A/\Lambda \quad \text{such that} \quad \mu(ax) = a\,\mu(x)\,\overline{a} \quad \text{and} \quad [\langle x, y\rangle] = \mu(x+y) - \mu(x) - \mu(y).$$

The following unitary automorphisms can be realized by diffeomorphisms [7, 1.5].

Definition 2.7 *([3, I:5.1])*. Let $(M, \langle\cdot,\cdot\rangle, \mu)$ be a quadratic module over a unitary ring (A, λ, Λ). A **transvection** $\sigma_{u,a,v}$ is an isometry of this structure defined by the formula

$$\sigma_{u,a,v} : M \longrightarrow M; \qquad x \longmapsto x + \langle v, x\rangle\, u - \overline{\lambda}\,\langle u, x\rangle\, v - \overline{\lambda}\,\langle u, x\rangle\, au$$

where $u, v \in M$ and $a \in A$ are elements satisfying

$$\langle u, v\rangle = 0 \in A \quad \text{and} \quad \mu(u) = 0 \in A/\Lambda \quad \text{and} \quad \mu(v) = [a] \in A/\Lambda.$$

The following lemmas involve, for any finitely generated projective A-module $P = P^{**}$, a nonsingular $(+1)$-quadratic form over A called the **hyperbolic construction**

$$\mathscr{H}(P) := (P \oplus P^*, \langle\cdot,\cdot\rangle, \mu) \text{ where } \langle x+f, y+g\rangle := f(y) + \overline{g(x)} \text{ and } \mu(x+f) := [f(x)].$$

Topologically, $\mathscr{H}(A)$ is the equivariant intersection form of $S^2 \times S^2$ with coefficients in A.

Lemma 2.8. *Consider a compact connected* TOP *4-manifold X with good fundamental group π and orientation character $\omega : \pi \to \{\pm 1\}$. Define a ring with involution $A := \mathbf{Z}[\pi^\omega]$. Suppose that there is an orthogonal decomposition*

$$K := \operatorname{Ker} w_2(X) = V_0 \perp V_1$$

as quadratic submodules of the intersection form of X over A, with a nonsingular restriction to V_0. Define a homology class and a free A-module

$$p_+ := [S^2_+ \times \mathrm{pt}]$$
$$P_+ := Ap_+.$$

Consider the summand

$$\mathscr{H}(P_+) = H_2(S^2_+ \times S^2; A)$$

of

$$H_2(X\#(S^2_+ \times S^2)\#(S^2_- \times S^2); A).$$

Then for any transvection $\sigma_{p,a,v}$ on the quadratic module $K \perp \mathscr{H}(P_+)$ with $p \in V_0 \oplus P_+$ and $v \in K$, the stabilized isometry $\sigma_{p,a,v} \oplus \mathbf{1}_{H_2(2(S^2\times S^2);A)}$ can be realized by a self-homeomorphism of $X\#3(S^2 \times S^2)$ which restricts to the identity on ∂X.

Remark 2.9. In the case that ∂X is empty and $\pi_1(X)$ is finite, then Lemma 2.8 is exactly [12, Corollary 2.3]. Although it turns out that their proof works in our generality, we include a full exposition, providing details absent from Hambleton–Kreck [12].

Lemma 2.10. *Suppose X and p satisfy the hypotheses of Lemma 2.8. If p is unimodular in $V_0 \oplus P_+$, then the summand $X_1 = X\#(S^2_+ \times S^2)$ of X_2 can be topologically re-split so that $S^2 \times \mathrm{pt}$ represents p.*

Proof. Since $V_0 \perp \mathscr{H}(P_+)$ is nonsingular, there exists an element $q \in V_0 \perp \mathscr{H}(P_+)$ such that (p, q) is a hyperbolic pair. Since $p, q \in \operatorname{Ker} w_2(X_1)$ and w_2 is the sole obstruction to framing the normal bundle in the universal cover, each homology class is represented by a canonical regular homotopy class of framed immersion

$$\alpha, \beta : S^2 \times \mathbf{R}^2 \longrightarrow X_1$$

with transverse double-points. Since the self-intersection number of α vanishes, all its double-points pair to yield framed immersed Whitney discs; consider each disc separately:

$$W : D^2 \times \mathbf{R}^2 \longrightarrow X_1.$$

Upon performing finger-moves to regularly homotope W, assume that one component of

$$\alpha(S^2 \times 0) \setminus W(\partial D^2 \times \mathbf{R}^2)$$

is a framed embedded disc

$$V : D^2 \times \mathbf{R}^2 \longrightarrow X_1$$

and, by an arbitrarily small regular homotopy of β, that $\beta | S^2 \times 0$ is transverse to $W | \operatorname{int} D^2 \times 0$ with algebraic intersection number 1 in $\mathbf{Z}[\pi_1(X_1)]$. Hence W is a framed properly immersed disc in

$$\overline{X}_1 := X_1 \setminus \operatorname{Im} V.$$

So, since $\pi_1(\overline{X}_1) \cong \pi_1(X)$ is a good group, by Freedman's disc theorem [10, 5.1A], there exists a framed properly TOP embedded disc

$$W' : D^2 \times \mathbf{R}^2 \longrightarrow \overline{X}_1$$

such that

$$W' = W \text{ on } \partial D^2 \times \mathbf{R}^2 \quad \text{and} \quad \operatorname{Im} W' \subset \operatorname{Im} W.$$

Therefore, by performing a Whitney move along W', we obtain that α is regularly homotopic to a framed immersion with one fewer pair of self-intersection points. Thus α is regularly homotopic to a framed TOP embedding α'. A similar argument, allowing an arbitrarily small regular homotopy of α', shows that β is regularly homotopic to a framed TOP embedding β' transverse to α', with a single intersection point

$$\alpha'(x_0 \times 0) = \beta'(y_0 \times 0)$$

such that the open disc

$$\Delta := \beta'(y_0 \times \mathbf{R}^2) \subset \alpha'(S^2 \times 0).$$

Define a closed disc

$$\Delta' := S^2 \setminus (\alpha')^{-1}(\Delta).$$

Surgery on X_1 along β' yields a compact connected TOP 4-manifold X'. Hence X_1 is recovered by surgery on X' along the framed embedded circle

$$\gamma : S^1 \times \mathbf{R}^3 \approx \mathrm{nbhd}_{S^2}(\partial\Delta') \times \mathbf{R}^2 \xrightarrow{\alpha'} X_1 \setminus \mathrm{Im}\,\beta' \subset X'.$$

But the circle γ is trivial in X', since it extends via α' to a framed embedding of the disc Δ' in X'. Therefore we obtain a TOP re-splitting of the connected sum

$$X_1 \approx X' \# (S^2 \times S^2)$$

so that $S^2 \times \mathrm{pt}$ of the right-hand side represents the image of p. □

The next algebraic lemma decomposes certain transvections so that the pieces fit into the previous topological lemma.

Lemma 2.11. *Suppose (A, λ, Λ) is a unitary ring such that: the additive monoid of A is generated by a subset S of the unit group $(A^\times, \cdot)$. Let $K = V_0 \perp V_1$ be a quadratic module over (A, λ, Λ) with a nonsingular restriction to V_0, and let $P_\pm$ be free left A-modules of rank one. Then any stabilized transvection*

$$\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)} \quad \text{on} \quad K \perp \mathscr{H}(P_+) \perp \mathscr{H}(P_-)$$

with $p \in V_0 \oplus P_+$ and $v \in K$ is a composite of transvections $\sigma_{p_i,0,v_j}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_j \in K \oplus \mathscr{H}(P_-)$.

Proof. Using a symplectic basis $\{ p_\pm, q_\pm \}$ of each hyperbolic plane $\mathscr{H}(P_\pm)$, define elements of $K \oplus \mathscr{H}(P_+ \oplus P_-)$:

$$\begin{aligned} v_0 &:= v + p_- - aq_- \\ v_1 &:= -p_- \\ v_2 &:= aq_-. \end{aligned}$$

Then

$$v = \sum_{i=0}^{2} v_i.$$

Observe that each $v_i \in K \oplus \mathscr{H}(P_-)$ is isotropic with $\langle v_i, p \rangle = 0$. So transvections $\sigma_{p,0,v_j}$ are defined. Note, by Definition 2.7, for all $x \in K \oplus \mathscr{H}(P_+ \oplus P_-)$, that

$$\begin{aligned} (\sigma_{p,0,v_2} \circ \sigma_{p,0,v_1} \circ \sigma_{p,0,v_0})(x) &= x + \sum_i \langle v_i, x \rangle\, p - \sum_i \overline{\lambda}\, \langle p, x \rangle\, v_i - \overline{\lambda}\, \langle p, x \rangle \sum_{i<j} \langle v_j, v_i \rangle\, p \\ &= x + \langle v, x \rangle\, p - \overline{\lambda}\, \langle p, x \rangle\, v - \overline{\lambda}\, \langle p, x \rangle\, ap \\ &= \sigma_{p,a,v}(x) \oplus \mathbf{1}_{\mathscr{H}(P_-)}. \end{aligned}$$

Therefore it suffices to consider the case that $v \in K \oplus \mathscr{H}(P_-)$ is isotropic. Write

$$p = p' \oplus p'' \in V_0 \oplus P_+.$$

Define a unimodular element

$$p_0 := p' \oplus 1p_+.$$

Note, since P_+ has rank one and by hypothesis, there exist $n \in \mathbf{Z}_{\geqslant 0}$ and unimodular elements $p_1, \dots, p_n \in Sp_+ \subseteq P_+$ such that

$$p - p_0 = p'' - 1p_+ = \sum_{i=1}^{n} p_i.$$

For each $1 \leqslant i \leqslant n$, write

$$p_i := s_i p_+ \quad \text{for some} \quad s_i \in S.$$

Observe for all $1 \leqslant i, j \leqslant n$ that

$$\begin{aligned} \langle v, p_i \rangle &= 0 \\ \mu(p_i) &= s_i \mu(p_+) \overline{s_i} = 0 \\ \langle p_i, p_j \rangle &= s_i \langle p_+, p_+ \rangle \overline{s_j} = 0. \end{aligned}$$

Hence, we also have

$$\begin{aligned} \langle v, p_0 \rangle &= 0 \\ \mu(p_0) &= 0. \end{aligned}$$

Then transvections $\sigma_{p_i,0,v}$ are defined and commute, so note

$$\sigma_{p,0,v} = \prod_{i=0}^{n} \sigma_{p_i,0,v}. \qquad \square$$

Proof of Lemma 2.8. Define a homology class and a free A-module

$$\begin{aligned} p_- &:= [S^2_- \times \mathrm{pt}] \\ P_- &:= A p_-. \end{aligned}$$

Consider the A-module decomposition

$$H_2(X_2; A) = H_2(X; A) \oplus \mathscr{H}(P_+) \oplus \mathscr{H}(P_-).$$

Observe that the unitary ring

$$(A, \lambda, \Lambda) = (\mathbf{Z}[\pi^\omega], +1, \{\, a - \overline{a} \mid a \in A \,\})$$

satisfies the hypothesis of Lemma 2.11 with the multiplicative subset

$$S = \pi \cup -\pi.$$

Therefore the stabilized transvection

$$\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)}$$

is a composite of transvections $\sigma_{p_i,0,v_i}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_i \in K \oplus \mathscr{H}(P_-)$. Then by Lemma 2.10, for each i, a TOP re-splitting

$$f_i : X_1 \approx X^i \#(S^2 \times S^2)$$

of the connected sum, for some 4-manifold X^i, can be chosen so that $S^2 \times \mathrm{pt}$ represents p_i. So by the Cappell–Shaneson realization theorem [7, 1.5],[1] for each i, the pullback under $(f_i)_*$ of the stabilized transvection

$$\sigma_{p_i,0,v_i} \oplus \mathbf{1}_{H_2(S^2 \times S^2;A)} = \sigma_{p_i \oplus 0,0,v_i \oplus 0}$$

is an isometry induced by a self-diffeomorphism of

$$(X^i \#(S^2 \times S^2))\#(S^2 \times S^2).$$

Hence, by conjugation with the homeomorphism f_i, this isometry is induced by a self-homeomorphism of

$$X_2 = X_1 \#(S^2 \times S^2).$$

Thus the stabilized transvection

$$\left(\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)}\right) \oplus \mathbf{1}_{H_2(S^2 \times S^2;A)}$$

is induced by the stabilized composite self-homeomorphism of

$$X_3 = X_2 \#(S^2 \times S^2). \qquad \square$$

2.3. Proof of the main theorem

Now we modify the induction of [12, Proof B]; our result will be one $S^2 \times S^2$ connected summand less efficient than Hambleton–Kreck [12] in the case that π is finite. The main algebraic technique is a theorem of Bass [3, IV:3.4] on the transitivity of a certain subgroup of isometries on the set of hyperbolic planes. We refer the reader to [3, §IV:3] for the terminology used in our proof. The main topological technique is a certain clutching construction of an s-cobordism.

Proof of Theorem 2.1. We may assume $r \geqslant d + 1$. Let

$$f : X\#r(S^2 \times S^2) \longrightarrow Y\#r(S^2 \times S^2)$$

be a homeomorphism. We show that

$$\overline{X} := X\#(r-1)(S^2 \times S^2)$$

is homeomorphic to

$$\overline{Y} := Y\#(r-1)(S^2 \times S^2),$$

thus the result follows by backwards induction on r.

Consider Definition 2.15 and [3, Hypotheses IV:3.1]. By our hypothesis and Lemma 2.16, the minimal form parameter

$$\Lambda := \{\, a - \overline{a} \mid a \in A \,\}$$

[1] Their theorem realizes any transvection of the form $\sigma_{p_+,a,v}$ by a diffeomorphism of the 1-stabilization.

makes (A, Λ) a quasi-finite unitary $(R, +1)$-algebra. Note, since

$$X = X_{-1}\#((S')^2 \times S^2)$$

by hypothesis, that the rank $r + 1$ free A-module summand

$$P := H_2\left((S')^2 \times \mathrm{pt} \sqcup r(S^2 \times \mathrm{pt}); A\right)$$

of

$$\operatorname{Ker} w_2(\overline{X}\#(S^2 \times S^2))$$

satisfies [3, Case IV:3.2(a)]. Then, by [3, Theorem IV:3.4], the subgroup G of the group $U(\mathscr{H}(P))$ of unitary automorphisms defined by

$$G := \langle\, \mathscr{H}(E(P)),\ EU(\mathscr{H}(P))\,\rangle$$

acts transitively on the set of hyperbolic pairs in $\mathscr{H}(P)$. So, by [3, Corollary IV:3.5] applied to the quadratic module

$$V := \operatorname{Ker} w_2(X_{-1}),$$

the subgroup G_1 of $U(V \perp \mathscr{H}(P))$ defined by

$$G_1 := \left\langle\, \mathbf{1}_V \perp G,\ EU(\mathscr{H}(P), P; V),\ EU(\mathscr{H}(P), \overline{P}; V)\,\right\rangle$$

acts transitively on the set of hyperbolic pairs in $V \perp \mathscr{H}(P)$. Let

$$(p_0, q_0) \quad \text{and} \quad (p_0', q_0')$$

be the standard basis of the summand $H_2(S^2 \times S^2; A)$ of

$$H_2(\overline{X}\#(S^2 \times S^2); A) \quad \text{and} \quad H_2(\overline{Y}\#(S^2 \times S^2); A).$$

Therefore there exists an isometry $\varphi \in G_1$ of

$$V \perp \mathscr{H}(P) = \operatorname{Ker} w_2(\overline{X}\#(S^2 \times S^2))$$

such that

$$\varphi(p_0, q_0) = (f_*)^{-1}(p_0', q_0').$$

Lemma 2.12. *The isometry*

$$\varphi \oplus \mathbf{1}_{H_2(3(S^2 \times S^2); A)}$$

is induced by a self-homeomorphism g of

$$\overline{X}\#4(S^2 \times S^2).$$

Then the homeomorphism

$$h := (f\#\mathbf{1}_{3(S^2\times S^2)}) \circ g : \overline{X}\#4(S^2\times S^2) \longrightarrow \overline{Y}\#4(S^2\times S^2)$$

satisfies the equation

$$h_*(p_i, q_i) = (p_i', q_i') \quad \text{for all} \quad 0 \leqslant i \leqslant 3.$$

Here the hyperbolic pairs

$$\{\,(p_i, q_i)\,\}_{i=1}^{3} \quad \text{and} \quad \{\,(p_i', q_i')\,\}_{i=1}^{3}$$

in the last three $S^2\times S^2$ summands are defined similarly to (p_0, q_0) and (p_0', q_0').

Lemma 2.13. *The manifold triad* $(W; \overline{X}, \overline{Y})$ *is a compact* TOP s*-cobordism* $\mathrm{rel}\,\partial X$*:*

$$W^5 := \overline{X}\times[0,1]\natural\, 4(S^2\times D^3) \bigcup_h \overline{Y}\times[0,1]\natural\, 4(S^2\times D^3).$$

Therefore, since $\pi_1(\overline{X}) \cong \pi_1(X)$ is a good group, by the TOP s-cobordism theorem [10, 7.1A], $\overline{X}$ is homeomorphic to $\overline{Y}$. This proves the theorem by induction on r. □

Remark 2.14. The reason for restriction to the A-submodule

$$K = \mathrm{Ker}\, w_2(\overline{X}\#(S^2\times S^2))$$

is two-fold. Geometrically [7, p. 504], a unique quadratic refinement of the intersection form exists on K, hence K is maximal. Also, the inverse image of (p_0', q_0') under the isometry f_* is guaranteed to be a hyperbolic pair in K, hence K is simultaneously minimal.

2.4. Remaining lemmas and proofs

Definition 2.15 *([3, IV:1.3])*. An R_0-algebra A is **quasi-finite** if, for each maximal ideal $\mathfrak{m} \in \mathrm{maxspec}(R_0)$, the following containment holds:

$$\mathfrak{m}A_{\mathfrak{m}} \subseteq \mathrm{rad}\, A_{\mathfrak{m}}$$

and that the following ring is left artinian:

$$A[\mathfrak{m}] := A_{\mathfrak{m}}/\mathrm{rad}\, A_{\mathfrak{m}}.$$

Here

$$A_{\mathfrak{m}} := (R_0)_{\mathfrak{m}} \otimes_{R_0} A$$

is the localization of A at $\mathfrak{m}$, and $\mathrm{rad}\, A_{\mathfrak{m}}$ is its Jacobson radical. The pair (A, Λ) is a **quasi-finite unitary** (R, λ)**-algebra** if (A, λ, Λ) is a unitary ring, A is an R-algebra with involution, and A is a quasi-finite R_0-algebra. Here R_0 is the subring of R generated by **norms**:

$$R_0 = \left\{ \sum_i r_i\bar{r}_i \;\middle|\; r_i \in R \right\}.$$

Lemma 2.16. *Suppose A is an algebra over a ring R_0 such that A is a finitely generated left R_0-module. Then A is a quasi-finite R_0-algebra.*

Proof. Let $\mathfrak{m} \in \operatorname{maxspec}(R_0)$. By [2, Corollary III:2.5] to Nakayama's lemma,

$$A_{\mathfrak{m}} \cdot \mathfrak{m} = A_{\mathfrak{m}} \cdot \operatorname{rad}(R_0)_{\mathfrak{m}} \subseteq \operatorname{rad} A_{\mathfrak{m}}.$$

Then

$$A[\mathfrak{m}] = (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}})/\left((\operatorname{rad} A_{\mathfrak{m}})/\mathfrak{m}A_{\mathfrak{m}}\right)$$

and is a finitely generated module over the field

$$(R_0)_{\mathfrak{m}}/\mathfrak{m}(R_0)_{\mathfrak{m}},$$

by hypothesis. Therefore $A[\mathfrak{m}]$ is left artinian, hence A is quasi-finite. □

The existence of the realization g is proven algebraically; refer to [3, §II:3].

Proof of Lemma 2.12. Consider Lemma 2.8 applied to

$$\overline{X}\#(S^2 \times S^2) \qquad V_0 = \mathscr{H}(P) \qquad V_1 = V.$$

It suffices to show that the group G_1 is generated by a subset of the transvections $\sigma_{p,a,v}$ with $p \in \mathscr{H}(P)$ and $v \in V \oplus \mathscr{H}(P)$.

By [3, Cases II:3.10(1–2)], the group

$$EU(\mathscr{H}(P))$$

is generated by all transvections $\sigma_{u,a,v}$ with $u, v \in \overline{P}$ or $u, v \in P$. By [3, Case II:3.10(3)], the group

$$\mathscr{H}(E(P))$$

is generated by a subset of the transvections $\sigma_{u,a,v}$ with $u \in P, v \in \overline{P}$ or $u \in \overline{P}, v \in P$. By [12, Definition 1.4], the group

$$EU(\mathscr{H}(P), P; V)$$

is generated by all transvections $\sigma_{u,a,v}$ with $u \in P, v \in V$, and the group

$$EU(\mathscr{H}(P), \overline{P}; V)$$

is generated by all $\sigma_{u,a,v}$ with $u \in \overline{P}, v \in V$. In any case, $p \in \mathscr{H}(P)$ and $v \in V \oplus \mathscr{H}(P)$. □

The assertion is essentially that $(W; \overline{X}, \overline{Y})$ is a h-cobordism with zero Whitehead torsion.

Proof of Lemma 2.13. By the Seifert–vanKampen theorem, we have a pushout diagram

$$\begin{array}{ccc}
\pi_1\left(\overline{X} \times 1 \,\#\, 4(S^2 \times S^2)\right) & \xrightarrow[\cong]{h_*} & \pi_1\left(\overline{Y} \times [0,1] \natural 4(S^2 \times D^3)\right) \\
\Big\downarrow \mathbf{1} & & \Big\downarrow \\
\pi_1\left(\overline{X} \times [0,1] \natural 4(S^2 \times D^3)\right) & \longrightarrow & \pi_1(W).
\end{array}$$

So the maps induced by the inclusion $\overline{X} \sqcup \overline{Y} \to W$ are isomorphisms:

$$\begin{aligned} i_* &: \pi_1(\overline{X} \times 0) \longrightarrow \pi_1(W) \\ j_* &: \pi_1(\overline{Y} \times 0) \longrightarrow \pi_1(W). \end{aligned}$$

Denote π as the common fundamental group using these identifications.

Observe that the nontrivial boundary map ∂_3 of the cellular chain complex

$$C_*(j; \mathbf{Z}[\pi]) \,:\, 0 \longrightarrow \bigoplus_{0 \leqslant k < 4} \mathbf{Z}[\pi] \cdot (S^2 \times D^3) \xrightarrow{h_\# \circ \partial} \bigoplus_{0 \leqslant l < 4} \mathbf{Z}[\pi] \cdot (S^2 \times S^2) \longrightarrow 0$$

is obtained as follows. First, attach thickened 2-cells to kill 4 copies of the trivial circle in $\overline{Y}$. Then, onto the resultant manifold

$$\overline{Y} \# 4(S^2 \times S^2),$$

attach thickened 3-cells to kill certain belt 2-spheres, which are the images under h of the normal 2-spheres to the 4 copies of the trivial circle in $\overline{X}$. Hence, as morphisms of based left $\mathbf{Z}[\pi]$-modules, the boundary map

$$\partial_3 = h_\# \circ \partial$$

is canonically identified with the morphism

$$h_* = \mathbf{1} : H_2(4(S^2 \times S^2); \mathbf{Z}[\pi]) \longrightarrow H_2(4(S^2 \times S^2); \mathbf{Z}[\pi])$$

on homology induced by the attaching map h. This last equality holds by the construction of h, since

$$h_*(p_i, q_i) = (p'_i, q'_i) \quad \text{for all} \quad 0 \leqslant i < 4.$$

So the inclusion $j : \overline{Y} \to W$ has torsion

$$\tau(C_*(j; \mathbf{Z}[\pi])) \;=\; [h_\#] \;=\; [h_*] \;=\; [\mathbf{1}] \;=\; 0 \in \mathrm{Wh}(\pi).$$

A similar argument using h^{-1} shows that the inclusion $i : \overline{X} \to W$ has zero torsion in $\mathrm{Wh}(\pi)$. Therefore $(W; \overline{X}, \overline{Y})$ is a compact TOP s-cobordism rel ∂X. □

The final proof of this section employs the theory of commutative rings and subrings (including invariant theory), as well as language from algebraic geometry (spec and maxspec).

Proof of Proposition 2.2. Since π is virtually polycyclic, it is a good group [10, 5.1A]. Since π is virtually abelian, by intersection with finitely many conjugates of a finite-index abelian subgroup, we find an exact sequence of groups with Γ normal abelian and G finite:

$$1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow G \longrightarrow 1.$$

This induces an action $G \curvearrowright \Gamma$. Consider these rings with involution and norm subring R_0:

$$\begin{aligned} A &:= \mathbf{Z}[\pi^\omega] \quad \text{where } \forall g \in \pi : \overline{g} = \omega(g)g^{-1} \\ A_0 &:= \mathbf{Z}[\Gamma^\omega] \quad \text{which is a commutative ring} \\ R &:= (A_0)^G \;=\; \{x \in A_0 \mid \forall g \in G : gx = x\} \\ R_0 &:= \left\{ \textstyle\sum_i x_i\overline{x}_i \;\middle|\; x_i \in R \right\}. \end{aligned}$$

Note $\mathbf{Z}[\Gamma^G] \subseteq R \subseteq \mathrm{Center}(A)$. Since π is finitely generated, by Schreier's lemma, so is Γ. So, by enlarging G as needed, we may assume Γ is a free-abelian group of a finite rank n.

First, we show that A is a finitely generated R_0-module. Since Γ has only finitely many right cosets in π, the group ring A is a finitely generated A_0-module. Since G is finite and A_0 is a finitely generated commutative ring and $\mathbf{Z}$ is a noetherian ring, by Bourbaki [4, §V.1: Theorem 9.2], A_0 is a finitely generated R-module and R is a finitely generated ring. By Bass [3, Intro IV:1.1], the commutative ring R is integral over its norm subring R_0. So, since R is a finitely generated integral R_0-algebra, it follows that R is a finitely generated R_0-module [9, Corollary 4.5]. Therefore, A is a finitely generated R_0-module.

Second, we show that R_0 is a noetherian ring. It follows from Hilbert's basis theorem [9, Corollary 1.3] that the finitely generated commutative $\mathbf{Z}$-alegebra A_0 is noetherian. So R_0 is too, by Eakin's theorem [9, A3.7a], since A_0 is a finitely generated R_0-module.

Third, we show that the irreducible-dimension of the Zariski topology on $\mathfrak{P} := \mathrm{spec}(R_0)$ is $n+1$. Here, by **irreducible-dimension** of a topological space, we mean the supremum of the lengths of proper chains of closed irreducible subsets, where **reducible** means being the union of two nonempty closed proper subsets [14, §I:1]. Krull dimension of a ring equals irreducible-dimension of its spec [14, II:3.2.7]; in particular $\dim(\mathfrak{P}) = \dim(R_0)$. Since A_0 is a finitely generated R_0-module, A_0 is integral over R_0 by [9, Corollary 4.5]. So $\dim(R_0) = \dim(A_0)$, by the Cohen–Seidenberg theorems [9, 4.15, 4.18; Axiom D3]. Note $\dim(A_0) = n+1$ since $\dim(\mathbf{Z}) = 1$, by [9, Exercise 10.1]. Thus $\dim(\mathfrak{P}) = n+1$.

Last, we show the topological space $\mathfrak{P} = \mathrm{spec}(R_0)$ and its subspace $\mathfrak{M} := \mathrm{maxspec}(R_0)$ have equal irreducible-dimensions. Since R is a finitely generated commutative ring and R is a finitely generated R_0-module, by the Artin–Tate lemma [9, Exercise 4.32], also R_0 is a finitely generated ring. Then R_0 is a Jacobson ring, by the generalized Nullstellensatz [9, Theorem 4.19], since $\mathbf{Z}$ is Jacobson. So we obtain an isomorphism of posets:

$$\mathrm{ClosedSets}(\mathfrak{P}) \longrightarrow \mathrm{ClosedSets}(\mathfrak{M}); \quad C \longmapsto C \cap \mathfrak{M} \quad \text{with inverse} \quad D \longmapsto \mathrm{closure}_{\mathfrak{P}}(D).$$

This correspondence is worked out by Grothendieck [11, §IV.10: Proposition 1.2(c'); Définitions 1.3, 3.1, 4.1; Corollaire 4.6]. Hence $\dim(\mathfrak{M}) = \dim(\mathfrak{P})$. Thus $d = n+1$. □

3. Manifolds in the tangential homotopy type of $\mathbf{RP}^4 \# \mathbf{RP}^4$

Given a tangential homotopy equivalence to a certain TOP 4-manifold, the main goal of this section is to uniformly quantify the amount of topological stabilization sufficient for smoothing and for splitting along a two-sided 3-sphere. In particular, we sharpen a result of Jahren–Kwasik [15, Theorem 1(f)] on connected sum of real projective 4-spaces (3.5).

Let X be a compact connected DIFF 4-manifold, and write

$$(\pi, \omega) := (\pi_1(X), w_1(X)).$$

Suppose π is good [10]. Let $\vartheta \in L_5^s(\mathbf{Z}[\pi^\omega])$; represent it by a simple unitary automorphism of the orthogonal sum of r copies of the hyperbolic plane for some $r \geqslant 0$. Recall [10, §11] that there exists a unique homeomorphism class

$$(X_\vartheta, h_\vartheta) \in \mathcal{S}^s_{\mathrm{TOP}}(X)$$

as follows. It consists of a compact TOP 4-manifold X_ϑ and a simple homotopy equivalence $h_\vartheta : X_\vartheta \to X$ that restricts to a homeomorphism $h : \partial X_\vartheta \to \partial X$ on the boundary, such that there exists a normal bordism rel ∂X from h_ϑ to $\mathbf{1}_X$ with surgery obstruction ϑ. Such a homotopy equivalence is called *tangential;* equivalently, a homotopy equivalence $h : M \to X$ of TOP manifolds is **tangential** if the pullback microbundle $h^*(\tau_X)$ is isomorphic to τ_M.

Theorem 3.1. *The following r-stabilization admits a* DIFF *structure:*

$$X_\vartheta \# r(S^2 \times S^2).$$

Furthermore, there exists a TOP *normal bordism between h_ϑ and $\mathbf{1}_X$ with surgery obstruction $\vartheta \in L^s_5(\mathbf{Z}[\pi^\omega])$, such that it consists of exactly $2r$ many 2-handles and $2r$ many 3-handles. In particular X_ϑ is $2r$-stably homeomorphic to X.*

Proof. The existence and uniqueness of $(X_\vartheta, h_\vartheta)$ follow from [10, Theorems 11.3A, 11.1A, 7.1A]. But by [7, Theorem 3.1], there exists a DIFF s-bordism class of (X_α, h_α) uniquely determined as follows. Given a rank r representative α of the isometry class ϑ, this pair (X_α, h_α) consists of a compact DIFF 4-manifold X_α and a simple homotopy equivalence h_α that restricts to a diffeomorphism on the boundary:

$$\begin{aligned} h_\alpha \ &: \ (X_\alpha, \partial X_\alpha) \longrightarrow (X_r, \partial X) \\ X_r &:= X \# r(S^2 \times S^2). \end{aligned}$$

It is obtained from a DIFF normal bordism (W_α, H_α) rel ∂X from h_α to $\mathbf{1}_{X_r}$ with of surgery obstruction ϑ, constructed with exactly r 2-handles and r 3-handles, and clutched along a diffeomorphism which induces the simple unitary automorphism α on the surgery kernel

$$K_2(W_\alpha) = \mathscr{H}\left(\bigoplus_r \mathbf{Z}[\pi]\right).$$

This is rather the consequence, and not the construction[2] itself, of Wall realization [19, 6.5] in high odd dimensions.

By uniqueness in the simple TOP structure set, the simple homotopy equivalences $h_\vartheta \# \mathbf{1}_{r(S^2\times S^2)}$ and h_α are s-bordant. Hence they differ by pre-composition with a homeomorphism, by the s-cobordism theorem [10, Thm. 7.1A]. In particular, the domain $X_\vartheta \# r(S^2 \times S^2)$ is homeomorphic to X_α, inheriting its DIFF structure. Therefore, post-composition of H_α with the collapse map $X_r \to X$ yields a normal bordism between the simple homotopy equivalences h_ϑ and $\mathbf{1}_X$, obtained by attaching $r + r$ 2- and 3-handles. □

Next, we recall Hambleton–Kreck–Teichner classification of the homeomorphism types and simple homotopy types of closed 4-manifolds with fundamental group $\mathbf{C}_2^-$. Then, we shall give a partial classification of the simple homotopy types and stable homeomorphism types of their connected sums, which have fundamental group $\mathbf{D}_\infty^{-,-} = \mathbf{C}_2^- * \mathbf{C}_2^-$. The *star operation* $*$ [10, §10.4] flips the Kirby–Siebenmann invariant of some 4-manifolds.

[2] In the DIFF 4-dimensional case, via a self-diffeomorphism φ inducing α, embeddings are chosen within certain regular homotopy class of framed immersions of 2-spheres. Cappell and Shaneson [7, 1.5] cleverly construct φ using a circle isotopy theorem of Whitney.

Theorem 3.2 *([13, Theorem 3]). Every closed nonorientable topological 4-manifold with fundamental group order two is homeomorphic to exactly one manifold in the following list of so-called w_2-types.*

- *(I) The connected sum of $*\mathbf{CP}^2$ with $\mathbf{RP}^4$ or its star. The connected sum of $k \geqslant 1$ copies of $\mathbf{CP}^2$ with $\mathbf{RP}^4$ or $\mathbf{RP}^2 \times S^2$ or their stars.*
- *(II) The connected sum of $k \geqslant 0$ copies of $S^2 \times S^2$ with $\mathbf{RP}^2 \times S^2$ or its star.*
- *(III) The connected sum of $k \geqslant 0$ copies of $S^2 \times S^2$ with $S(\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1)$ or $\#_{S^1} r\mathbf{RP}^4$ or their stars, for unique $1 \leqslant r \leqslant 4$.*

We explain the terms in the above theorem. Firstly,

$$\mathbf{R} \longrightarrow \gamma^1 \longrightarrow \mathbf{RP}^2$$

denotes the canonical line bundle, and

$$\varepsilon^1 := \mathbf{R} \times \mathbf{RP}^2$$

denotes the trivial line bundle. Secondly,

$$S^2 \longrightarrow S(\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1) \longrightarrow \mathbf{RP}^2$$

is the sphere bundle of the Whitney sum. Finally, the **circular sum**

$$M \#_{S^1} N := M \setminus E \bigcup_{\partial E} N \setminus E$$

is defined by codimension zero embeddings of E in M and N that are not null-homotopic, where E is the nontrivial bundle:

$$D^3 \longrightarrow E \longrightarrow S^1.$$

Corollary 3.3 *([13, Corollary 1]). Let M and M' be closed nonorientable topological 4-manifolds with fundamental group of order two. Then M and M' are (simple) homotopy equivalent if and only if*

1. *M and M' have the same w_2-type,*
2. *M and M' have the same Euler characteristic, and*
3. *M and M' have the same Stiefel–Whitney number: $w_1^4[M] = w_1^4[M']$ mod 2;*
4. *M and M' have $\pm$ the same Brown–Arf invariant mod 8, in case of w_2-type III.*

The following theorem is the main focus of this section. The pieces M and M' are classified by Hambleton–Kreck–Teichner [13], and the UNil-group is computed by Connolly–Davis [8]. Since $\mathbf{Z}$ is a regular coherent ring, by Waldhausen's vanishing theorem [18, Theorems 1, 2, 4], $\widetilde{\mathrm{Nil}}_0(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-) = 0$. Hence $\mathrm{UNil}_5^s = \mathrm{UNil}_5^h$ [6].

Theorem 3.4. *Let M and M' be closed nonorientable topological 4-manifolds with fundamental group of order two. Write $X = M \# M'$, and denote S as the 3-sphere defining the connected sum. Let $\vartheta \in \mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-)$.*

1. *There exists a unique homeomorphism class $(X_\vartheta, h_\vartheta)$, consisting of a closed* TOP *4-manifold X_ϑ and a tangential homotopy equivalence $h_\vartheta : X_\vartheta \to X$, such that it has splitting obstruction*

$$\mathrm{split}_L(h_\vartheta; S) = \vartheta.$$

The function which assigns ϑ to such a $(X_\vartheta, h_\vartheta)$ is a bijection.

2. *Furthermore,*

$$X_\vartheta \# 3(S^2 \times S^2) \quad \textit{is homeomorphic to} \quad X \# 3(S^2 \times S^2).$$

It admits a DIFF *structure if and only if X does. There exists a* TOP *normal bordism between h_ϑ and $\mathbf{1}_X$, with surgery obstruction $\vartheta \in L_5^h(\mathbf{D}_\infty^{-,-})$, such that it is composed of exactly six 2-handles and six 3-handles.*

Proof. Recall that the forgetful map

$$L_5^s(\mathbf{D}_\infty^{-,-}) \longrightarrow L_5^h(\mathbf{D}_\infty^{-,-})$$

is an isomorphism, since the Whitehead group $\mathrm{Wh}(\mathbf{D}_\infty)$ vanishes. Then the existence and uniqueness of $(X_\vartheta, h_\vartheta)$ and its handle description follow from Theorem 3.1, using $r = d + 1 = 3$ from Proposition 2.2 and Proof 2.1. By [6, Theorem 6], the following composite function is the identity on $\mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-)$:

$$\vartheta \longmapsto (X_\vartheta, h_\vartheta) \longmapsto \mathrm{split}_L(h_\vartheta; S).$$

In order to show that the other composite is the identity, note that two tangential homotopy equivalences $(X_\vartheta, h_\vartheta)$ and $(X'_\vartheta, h'_\vartheta)$ with the same splitting obstruction ϑ must be homeomorphic, by freeness of the UNil_5^h action on the structure set $\mathcal{S}_{\mathrm{TOP}}^h(X)$. Finally, since the 4-manifolds X_ϑ and X are 6-stably homeomorphic via the TOP normal bordism between h_ϑ and $\mathbf{1}_X$, we conclude that they are in fact 3-stably homeomorphic by Corollary 2.3. □

The six 2-handles are needed for map data and only three are needed to relate domains.

Corollary 3.5. *The above theorem is true for $X = \mathbf{RP}^4 \# \mathbf{RP}^4$, with $\mathbf{RP}^4$ of w_2-type III.* □

Remark 3.6. We comment on a specific aspect of the topology of X. Every homotopy automorphism of $\mathbf{RP}^4 \# \mathbf{RP}^4$ is homotopic to a homeomorphism [15, Lemma 1]. Then any automorphism of the group $\mathbf{D}_\infty$ can be realized [15, Claim]. The homeomorphism classes of closed topological 4-manifolds X' in the (not necessarily tangential) homotopy type of X has been computed in [5, Theorem 2]. The classification involves the study [5, Theorem 1] of the effect of transposition of the bimodules $\mathbf{Z}^-$ and $\mathbf{Z}^-$ in the abelian group $\mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-)$. As promised in the introduction, Corollary 3.5 provides a *uniform upper bound* on the number of $S^2 \times S^2$ connected-summands sufficient for [15, Theorem 1(f)], and on the number of 2- and 3-handles sufficient for [15, Proof 1(f)].

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