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Homotopy invariance of 4-manifold decompositions: Connected sums

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ABSTRACT

Given any homotopy equivalence $f : M \to X_1 \# \cdots \# X_n$ of closed orientable 4-manifolds, where each fundamental group $\pi_1(X_i)$ satisfies Freedman's Null Disc Lemma, we show that M is topologically h-cobordant to a connected sum $M' = M'_1 \# \cdots \# M'_n$ such that f is h-bordant to some $f'_1 \# \cdots \# f'_n$ with each $f'_i : M'_i \to X_i$ a homotopy equivalence. Moreover, such a replacement M' of M is unique up to a connected sum of h-cobordisms. In summary, the existence and uniqueness, up to h-cobordism, of connected sum decompositions of such orientable 4-manifolds M is an invariant of homotopy equivalence. Also we establish that the Borel Conjecture is true in dimension 4, up to s-cobordism, if the fundamental group satisfies the Farrell–Jones Conjecture.

1. Introduction

1.1. Homotopy invariance of connected sums—stable version

For simplicity, we begin with the stable version of our main result (Theorem 1.7). This version follows easily from a recent algebraic calculation of UNil for free products of groups by F. Connolly and J. Davis [11] and from an earlier development of stable geometric topology by S. Cappell and J. Shaneson [8,7].

Theorem 1.1. *Let X be a compact connected orientable topological manifold of dimension 4. Suppose the fundamental group $\pi_1(X)$ is a free product of groups $\Gamma_1, \ldots, \Gamma_n$. Then there exist compact connected topological 4-manifolds $X_1, \ldots, X_n$ with each fundamental group $\pi_1(X_i)$ isomorphic to Γ_i such that there is a bijection between $(S^2 \times S^2)$-stable h-structure sets*:

$$\# : \prod_{i=1}^{n} \overline{\mathcal{S}}^h_{\mathrm{TOP}}(X_i) \to \overline{\mathcal{S}}^h_{\mathrm{TOP}}(X).$$

Moreover, these X_i are unique up to $(S^2 \times S^2)$-stabilization and re-ordering.

Proof. By the stable prime decomposition of Kreck, Lück and Teichner [31], there exist 4-manifolds X_i, unique up to stabilization and permutation, with fundamental groups Γ_i such that X is $(S^2 \times S^2)$-stably homeomorphic to $X_1 \# \cdots \# X_n$. By theorems of Waldhausen [42] and Connolly and Davis [11], the algebraic K- and L-theory splitting obstruction groups associated to each connecting 3-sphere vanish:

$$\widetilde{\mathrm{Nil}}_0 = 0 \quad \text{and} \quad \mathrm{UNil}^h_5 = 0.$$

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Therefore, by the equivalence of Theorem 3.2(2), using Cappell's high-dimensional splitting theorem [5,7], we obtain inductively that # is a bijection. □

1.2. Homotopy invariance of connected sums—unstable version

Our Main Theorem (Theorem 1.7) is phrased technically in terms of the classes *NDL* and SES^h_+, which we define below. The difficulty in the proof is in observing new extensions of the geometric topology developed by S. Cappell [7] and S. Weinberger [44].

Definition 1.2 *(Freedman).* A discrete group G is *NDL* (**or good**) if the π_1-Null Disc Lemma holds for it (see [21] for details). The class *NDL* is closed under the operations of forming subgroups, extensions, and filtered colimits.

This class contains subexponential and exponential growth [19,21,32].

Theorem 1.3 *(Freedman–Quinn, Freedman–Teichner, Krushkal–Quinn). The class NDL contains all virtually polycyclic groups and all groups of subexponential growth.*

Example 1.4. Here are some exotic examples in *NDL*. The semidirect product $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ with $\alpha = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is polycyclic but has exponential growth. For all integers $n \neq -1, 0, 1$, the Baumslag–Solitar group $BS(1,n) = \mathbb{Z}[1/n] \rtimes_n \mathbb{Z}$ is finitely presented and solvable but not polycyclic. Grigorchuk's infinite 2-group G is finitely generated but not finitely presented and has intermediate growth.

Recall that, unless specified in the notation, the structure sets $\mathcal{S}^h_{\mathrm{TOP}}$ and normal invariants $\mathcal{N}_{\mathrm{TOP}}$ are homeomorphisms on the boundary (that is, rel ∂) [24, §6.2].

Definition 1.5. Let Z be a non-empty compact connected topological 4-manifold. Denote the fundamental group $\pi := \pi_1(Z)$ and orientation character $\omega := w_1(\tau_Z)$. We declare that Z **has class** SES^h if there exists an exact sequence of based sets:

$$\mathcal{N}_{\mathrm{TOP}}(Z \times I) \xrightarrow{\sigma^h_5} L^h_5(\pi, \omega) \xrightarrow{\partial} \mathcal{S}^h_{\mathrm{TOP}}(Z) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(Z) \xrightarrow{\sigma^h_4} L^h_4(\pi, \omega).$$

The subclass SES^h_+ includes actions of groups in K- and L-theory (Definition 2.5).

This exact sequence has been proven for the above groups [19, Thm. 11.3A].

Theorem 1.6 *(Freedman–Quinn). Let X be a compact connected topological manifold of dimension 4. If $\pi_1(X)$ has class NDL, then X has class SES^h_+ and satisfies the s-cobordism conjecture (i.e., all s-cobordisms on X are homeomorphic to $X \times I$).*

Here is the Main Theorem of the paper. The existence and uniqueness question posed in the Title and Abstract, up to h-cobordism, is quantified in # of Part (2).

Theorem 1.7. *Let X be a compact connected topological manifold of dimension 4.*

1. *Suppose the fundamental group $\pi_1(X)$ is a free product of groups of class NDL. If X is non-orientable, assume $\pi_1(X)$ is 2-torsionfree. Then there exists $r \geqslant 0$ such that the r-th stabilization $X \# r(S^2 \times S^2)$ has class SES^h_+.*
2. *Suppose X has the homotopy type of a connected sum $X_1 \# \cdots \# X_n$ such that each X_i has class SES^h_+. If X is non-orientable, assume that $\pi_1(X)$ is 2-torsionfree. Then the homotopy connected sum X has class SES^h_+. Moreover, the following induced function is a bijection:*

$$\# : \prod_{i=1}^{n} \mathcal{S}^h_{\mathrm{TOP}}(X_i) \to \mathcal{S}^h_{\mathrm{TOP}}(X).$$

The proof of our theorem consists of two steps: first homology split along each essential 3-sphere [44], and then perform a neck exchange trick [19] to replace homology 3-spheres with genuine ones (cf. [30,25]). The first step is possible because the high-dimensional splitting obstruction group [5] has recently been shown to vanish [11]. No direct surgeries are performed—only cobordisms are attached. Our techniques do not show triviality of s-cobordisms.

Indeed, it turns out that a limited form of surgery does work for free groups.

Example 1.8. Suppose X is a closed connected topological 4-manifold with free fundamental group: $\pi_1(X) = F_n$. Then a fixed stabilization $X \# r(S^2 \times S^2)$ has a topological s-cobordism surgery sequence, for some $r \geqslant 0$ depending on X.

Here are other caveats, which place our Main Theorem into historical context.

Remark 1.9. A homotopy decomposition into a connected sum need not exist. A counterexample to the homotopy Kneser Conjecture with $\pi_1(X) = G_3 * G_5$ where $G_p := C_p \times C_p$ was constructed by M. Kreck, W. Lück, and P. Teichner [30].

Remark 1.10. Given a homotopy decomposition into a connected sum, a homeomorphism decomposition need not exist. There exist infinitely many examples of non-orientable closed topological 4-manifolds homotopy equivalent to a connected sum ($X = \mathbb{RP}^4 \# \mathbb{RP}^4$) that are not homeomorphic a non-trivial connected sum [25,4]. Hence # is not always a bijection in the case $\pi_1(X) = D_\infty \in NDL$.

Remark 1.11. For certain groups $\pi_1(X)$ unknown in *NDL*, such as poly-surface groups, results on exactness at $\mathcal{N}_{\mathrm{TOP}}(X)$ are found in [24,26,22,9].

Remark 1.12. The modular group $PSL(2,\mathbb{Z}) \cong C_2 * C_3$ is an example of a free product of *NDL* groups. It has a discrete cofinite-area action on $\mathbb{H}^2$. However our theorem in the non-orientable case excludes it and $SL(2,\mathbb{Z}) \cong C_4 *_{C_2} C_6$. The group $PSL(2,\mathbb{Z})$ plays a key role in the orientable case of free products [11].

Let us conclude this subsection with an application to fibering of 5-manifolds. Partial results were obtained in [44,27]. The proof is located in Section 4.

Theorem 1.13. *Let M be a closed topological 5-manifold. Let X be a closed topological 4-manifold of class SES^h_+. Suppose $f: M \to S^1$ is a continuous map such that the induced infinite cyclic cover $\overline{M} = \mathrm{hofiber}(f)$ is homotopy equivalent to X. If the Farrell–Siebenmann fibering obstruction $\tau(f) \in \mathrm{Wh}_1(\pi_1 M)$ vanishes, then f is homotopic to a topological s-block bundle projection with pseudofiber X.*

Note we obtain a fiber bundle projection if X satisfies the s-cobordism conjecture.

1.3. Topological s-rigidity for 4-dimensional manifolds

The purpose of this final subsection is an elementary observation (Theorem 1.18) from which we conclude the Borel Conjecture is true in dimension 4 up to topological s-cobordism, given that the fundamental group satisfies the Farrell–Jones Conjecture (see Corollary 1.23).

Definition 1.14. A compact topological manifold Z is **topologically rigid** if, for all compact topological manifolds M, any homotopy equivalence $h: M \to Z$, with restriction $\partial h: \partial M \to \partial Z$ a homeomorphism, is homotopic to a homeomorphism.

Recall the Borel Conjecture is proven for certain good groups [19, Thm. 11.5].

Theorem 1.15 *(Freedman–Quinn). Suppose Z is an aspherical compact topological 4-manifold such that $\pi_1(Z)$ is virtually polycyclic. Then Z is topologically rigid.*

The following crystallographic examples include the 4-torus T^4. It turns out that there are only finitely many examples in any dimension (e.g., see [14, Thm. 21]).

Example 1.16. Suppose Γ is a Bieberbach group of rank 4, that is, a torsionfree lattice in the Lie group $\mathrm{Isom}(\mathbb{E}^4)$. Then $Z = \mathbb{R}^4/\Gamma$ is topologically rigid (cf. [16]).

Let us now turn our attention to a weaker form of rigidity for general groups.

Definition 1.17. A compact topological manifold Z is **topologically s-rigid** if, for all compact topological manifolds M, any homotopy equivalence $h: M \to Z$, with restriction $\partial h: \partial M \to \partial Z$ a homeomorphism, is itself topologically s-bordant rel ∂M to a homeomorphism. It suffices that the Whitehead group $\mathrm{Wh}_1(\pi_1 Z)$ vanishes and the topological s-cobordism structure set $\mathcal{S}^s_{\mathrm{TOP}}(Z)$ is a singleton.

The following important basic observation does not seem to have appeared in the literature. In particular, we do not assume that the fundamental group is *NDL*.

Theorem 1.18. *Let Z be a compact topological 4-manifold with fundamental group π and orientation character $\omega: \pi \to \{\pm 1\}$. Suppose the surgery obstruction map $\sigma^s_4: \mathcal{N}_{\mathrm{TOP}}(Z) \to L^s_4(\pi,\omega)$ is injective, and suppose the surgery obstruction map $\sigma^s_5: \mathcal{N}_{\mathrm{TOP}}(Z \times I) \to L^s_5(\pi,\omega)$ is surjective. If $\mathrm{Wh}_1(\pi) = 0$ then Z is topologically s-rigid. Also Z has class SES^s_+.*

We sharpen an observation of J. Hillman [24, Lem. 6.10] to include map data.

Corollary 1.19. *Let Z be a compact topological 4-manifold. Suppose the product $Z \times S^1$ is topologically rigid. If $\mathrm{Wh}_1(\pi_1 Z) = 0$ then Z is topologically s-rigid.*

This allows us to generalize a theorem of J. Hillman for surface bundles over surfaces [24, Thm. 6.15]. His conclusion was that the source and target are abstractly s-cobordant. Our new feature is s-rigidity of the homotopy equivalence.

Example 1.20. Suppose Z is a compact topological 4-manifold that is the total space of a topological fiber bundle of aspherical surfaces over an aspherical surface. Then Z is topologically s-rigid, as follows. By [24, Thm. 6.2], the group $\mathrm{Wh}_1(\pi_1 Z)$ vanishes. By the proof of [24, Thm. 6.15], the set $\mathcal{S}^s_{\mathrm{TOP}}(Z \times S^1)$ is a singleton. Now apply Corollary 1.19. Alternatively, we can use Corollary 1.23 and the recently established validity of FJ_L for poly-surface groups [2].

In the topology of high-dimensional manifolds, the following class of fundamental groups has been of intense interdisciplinary interest for at least the past two decades.

Definition 1.21. Denote FJ_L as the class of groups Γ that are K-flat and satisfy the Farrell–Jones Conjecture in L-theory [17]. That is, the elements Γ of FJ_L satisfy $\mathrm{Wh}_1(\Gamma \times \mathbb{Z}^n) = 0$ and $H^\Gamma_n(E_{\mathfrak{all}}\Gamma, E_{\mathfrak{vc}}\Gamma; \underline{\mathbb{L}}^{-\infty}_{\mathbb{Z}}) = 0$ for all $n \geqslant 0$ (see [12]).

We shall focus on the torsionfree case. This has nice subclasses [18,3,2].

Theorem 1.22 *(Farrell–Jones, Bartels–Lück, Bartels–Farrell–Lück). Let Γ be a discrete torsionfree group. Then Γ has class FJ_L if:*

- *Γ is the fundamental group of a complete A-regular Riemannian manifold with all sectional curvatures non-positive, or*
- *Γ is hyperbolic with respect to the word metric, or*
- *Γ admits a cocompact proper action by isometries on a complete finite-dimensional $\mathrm{CAT}(0)$ metric space, or*
- *Γ is a virtually polycyclic group (equivalently, a virtually poly-$\mathbb{Z}$ group), or*
- *Γ is a cocompact lattice in a virtually connected Lie group.*

We state our s-cobordism answer to the Borel Conjecture for exponential growth.

Corollary 1.23. *Suppose Z is an aspherical compact topological 4-manifold such that $\pi_1(Z)$ has class FJ_L. Then Z is topologically s-rigid. Also Z has class SES^h_+.*

Example 1.24. Topological s-rigidity occurs if $Z - \partial Z$ is complete finite-volume hyperbolic. That is, $Z - \partial Z = \mathbb{R}^4/\Gamma$ for some torsionfree lattice Γ in $\mathrm{Isom}(\mathbb{H}^4)$.

Example 1.25. A non-Riemannian example of topological s-rigidity is the closed 4-manifold Z of M. Davis [13]. The universal cover $\widetilde{Z}$ is a complete CAT(0) metric space. Most strikingly, $\widetilde{Z}$ is contractible but not homeomorphic to $\mathbb{R}^4$.

The next example involves multiple citations, so we give a formal proof later. Currently, due to Nil summands, it is unknown if its Whitehead group vanishes.

Corollary 1.26. *Suppose Z is the mapping torus of a homeomorphism of an aspherical closed 3-manifold K. If $\mathrm{Wh}_1(\pi_1 Z) = 0$ then Z is topologically s-rigid.*

Now, let us pass to connected sums, which fail to be aspherical if there is more than one factor. The next statement shall follow from Theorems 1.7 and 1.18. Below, we write $\mathrm{cdim}(G)$ for the cohomological dimension of any discrete group G.

Corollary 1.27. *Let $n > 0$. For each $1 \leqslant i \leqslant n$, let X_i be a compact oriented topological 4-manifold. If each fundamental group $\Gamma_i := \pi_1(X_i)$ is torsionfree of class FJ_L with $\mathrm{cdim}(\Gamma_i) \leqslant 4$, and each mod-two second homotopy group vanishes: $\pi_2(X_i) \otimes \mathbb{Z}_2 = 0$, then the connected sum $X := X_1 \# \cdots \# X_n$ is topologically s-rigid.*

Next, we illustrate the basic but important example of non-aspherical oriented factors $X_i = S^1 \times S^3$. Here, the connected sum X has free fundamental group F_n.

Example 1.28. Let $n > 0$. Recall $\mathrm{Wh}_1(\mathbb{Z}) = 0$. Then, by Corollary 1.27, the closed 4-manifold $X = \#n(S^1 \times S^3)$ has class SES^h_+ and is topologically s-rigid.

Finally, we specialize Corollary 1.27 to the setting of the Borel Conjecture.

Corollary 1.29. *Let $n > 0$. For each $1 \leqslant i \leqslant n$, suppose X_i is an aspherical compact oriented topological 4-manifold with fundamental group $\Gamma_i := \pi_1(X_i)$ of class FJ_L. Then the connected sum $X := X_1 \# \cdots \# X_n$ is topologically s-rigid.*

Here is an outline of the rest of the paper. Foundations are laid in Sections 2–3, where we expand work of Cappell and Weinberger in dimension four. Applications are made in Sections 4–5, where we prove the stated results of the Introduction. The reader may find most of our notation and terminology in Kirby and Taylor [29].

2. The language of structure sets

To start, the following equivalence relations play prominent roles in Section 3.

Definition 2.1. Let Z be a topological space. Let M, M' be compact topological manifolds. Let $h: M \to Z$ and $h': M' \to Z$ be continuous maps. A **bordism** $H: h \to h'$ **rel** ∂ is a compact topological cobordism $(W; M, M')$ rel ∂ and a continuous map $|H|: W \to Z \times I$ such that $H|_M = h$ and $H|_{M'} = h'$. We call $H: h \to h'$ a h**-bordism rel** ∂ (resp. s**-bordism rel** ∂) if $(W; M, M')$ is an h-cobordism (resp. s-cobordism).

Next, we relativize the surgical language in the Introduction (cf. [43]).

Definition 2.2. Let Z be a topological manifold such that the boundary ∂Z is collared. Let $\partial_0 Z$ be a union of components of ∂Z. The pair $(Z, \partial_0 Z)$ is called a TOP **manifold pair**. Write $\partial_1 Z := \partial Z - \partial_0 Z$. The induced triple $(Z; \partial_0 Z, \partial_1 Z)$ is an example of a TOP **manifold triad** (in other words, a cobordism).

Here is the precise definition of the relative structure set that we use in proofs.

Definition 2.3. Let $(Z, \partial_0 Z)$ be a compact TOP 4-manifold pair. Write $\Gamma_0 := \pi_1(\partial_0 Z)$, the fundamental groupoid of $\partial_0 Z$. The **structure set** $\mathcal{S}^h_{\mathrm{TOP}}(Z, \partial_0 Z)$ consists of $\sim$-equivalence classes of continuous maps $(h; \partial_0 h, \partial_1 h): (M; \partial_0 M, \partial_1 M) \to (Z; \partial_0 Z, \partial_1 Z)$ of compact TOP 4-manifolds triads such that:

- $h: M \to Z$ is a homotopy equivalence,
- $\partial_0 h: \partial_0 M \to \partial_0 Z$ is a $\mathbb{Z}[\Gamma_0]$-homology equivalence, and
- $\partial_1 h: \partial_1 M \to \partial_1 Z$ is a homeomorphism.

We call such $(h, \partial_0 h): (M, \partial_0 M) \to (Z, \partial_0 Z)$ a **homotopy–homology equivalence**. Here, $h \sim h'$ if there exists a TOP bordism $(H; \partial_0 H, \partial_1 H): (h; \partial_0 h, \partial_1 h) \to (h'; \partial_0 h', \partial_1 h')$ such that:

- $H: W \to Z \times I$ is a homotopy equivalence,
- $\partial_0 H: \partial_0 W \to \partial_0 Z \times I$ is a $\mathbb{Z}[\Gamma_0]$-homology equivalence, where $(\partial_0 W; \partial_0 M, \partial_0 M')$ is a cobordism, and
- $\partial_1 H: \partial_1 W \to \partial_1 Z \times I$ is a homeomorphism, where $(\partial_1 W; \partial_1 M, \partial_1 M')$ is a cobordism.

We call such $(H, \partial_0 H): (h, \partial_0 h) \to (h', \partial_0 h')$ a **homotopy–homology** h**-bordism**.

The 4-dimensional relative surgery sequence is defined carefully as follows. It is an h-version of Wall's sequence (middle of [43, p. 115]) with homotopy equivalences to $\partial_0 Z$ and with homeomorphisms to $\partial_1 Z$.

Definition 2.4. Let $(Z, \partial_0 Z)$ be a compact TOP 4-manifold pair. Denote the fundamental groupoids $\Gamma := \pi_1(Z)$ and $\Gamma_0 := \pi_1(\partial_0 Z)$. Denote the orientation character $\omega := w_1(\tau_Z): \Gamma \to \{\pm 1\}$. We declare that $(Z, \partial_0 Z)$ **has class** SES^h if there exists an exact sequence of based sets:

$$\mathcal{N}_{\mathrm{TOP}}(Z \times I, \partial_0 Z \times I) \xrightarrow{\sigma^h_5} L^h_5(\Gamma, \Gamma_0, \omega) \xrightarrow{\partial} \mathcal{S}^h_{\mathrm{TOP}}(Z, \partial_0 Z) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(Z, \partial_0 Z) \xrightarrow{\sigma^h_4} L^h_4(\Gamma, \Gamma_0, \omega).$$

Last is the enhancement to include actions of certain groups in K- and L-theory.

Definition 2.5. In addition, we declare that $(Z, \partial_0 Z)$ **has class** SES^h_+ if, for all elements $h \in \mathcal{S}^h_{\mathrm{TOP}}(Z, \partial_0 Z)$ and $t \in \mathrm{Wh}_1(\Gamma)$ and $x \in L^h_5(\Gamma, \Gamma_0, \omega)$, there exist:

- an action of the group $\mathrm{Wh}_1(\Gamma)$ on the set $\mathcal{S}^h_{\mathrm{TOP}}(Z, \partial_0 Z)$ such that:
  - there is an h-bordism $F: W \to Z \times I$ rel ∂ from $h: M \to Z$ to $t(h): M' \to Z$ with Whitehead torsion $\tau(W; M, M') = t$, and

- an action of the group $L_5^h(\Gamma, \Gamma_0, \omega)$ on the set $\mathcal{S}_{\mathrm{TOP}}^h(Z, \partial_0 Z)$ such that:
  - there exists a normal bordism F from h to $x(h)$ with $\sigma_5^h(F) = x$, and
  - the equation $\partial(x) = x(\mathrm{id}_Z)$ holds.

Before moving on, we consider the stable version of the above structure set.

Definition 2.6. Let $(Z, \partial_0 Z)$ be a compact TOP 4-manifold pair. The **stable structure set** $\overline{\mathcal{S}}_{\mathrm{TOP}}^h(Z, \partial_0 Z)$ consists of $\approx$-equivalence classes of homotopy–homology equivalences $h : (M, \partial_0 M) \to (Z \# r(S^2 \times S^2), \partial_0 Z)$ for any $r \geqslant 0$. Here, we define $h \approx h'$ if there exist $s, s' \geqslant 0$ and a homotopy–homology h-bordism $H : h \# \mathrm{id}_{s(S^2 \times S^2)} \to h' \# \mathrm{id}_{s'(S^2 \times S^2)}$.

The next theorem was proven by S. Cappell and J. Shaneson [8] (cf. [29]) and reformulated here.

Theorem 2.7 *(Cappell–Shaneson). Let $(Z, \partial_0 Z)$ be a compact* TOP *4-manifold pair. Denote the fundamental groupoids $\Gamma := \pi_1(Z)$ and $\Gamma_0 := \pi_1(\partial_0 Z)$ and orientation character $\omega : \Gamma \to \{\pm 1\}$. Then there is an exact sequence of based sets*:

$$\mathcal{N}_{\mathrm{TOP}}(Z \times I, \partial_0 Z \times I) \xrightarrow{\sigma_5^h} L_5^h(\Gamma, \Gamma_0, \omega) \xrightarrow{\partial} \overline{\mathcal{S}}_{\mathrm{TOP}}^h(Z, \partial_0 Z) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(Z, \partial_0 Z) \xrightarrow{\sigma_4^h} L_4^h(\Gamma, \Gamma_0, \omega).$$

The group $L_5^h(\Gamma, \Gamma_0, \omega)$ acts on the set $\overline{\mathcal{S}}_{\mathrm{TOP}}^h(Z, \partial_0 Z)$ in such a way that the above map ∂ is equivariant.

3. A Weinberger-type homology splitting theorem

Now we are ready to improve the Λ-splitting theorem of S. Weinberger [44] by slightly modifying his proof. In essence Theorems 1.7 and 1.1 shall be its corollaries.

Definition 3.1. In the setting below, the homotopy equivalence $h : M \to X$ is $\mathbb{Z}[\Gamma_0]$**-split** if h is topologically transverse to X_0 and its restriction $h_0 : h^{-1}(X_0) \to X_0$ is a $\mathbb{Z}[\Gamma_0]$-homology equivalence (hence $h - h_0 : h^{-1}(X - X_0) \to X - X_0$ is also).

Theorem 3.2. *Let X be a non-empty compact connected topological 4-manifold. Let X_0 be a closed connected incompressible separating topological 3-submanifold of X. The decomposition of manifolds $X = X_1 \cup_{X_0} X_2$ induces the decomposition of fundamental groups $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$. Define a closed simply-connected 8-manifold*

$$Q := \mathbb{CP}^4 \# (S^3 \times S^5) \# (S^3 \times S^5).$$

Let M be a compact topological 4-manifold. Suppose $h : M \to X$ is a homotopy equivalence such that the restriction $\partial h : \partial M \to \partial X$ is a homeomorphism.

1. *Assume $(*)$: the group Γ_0 has class NDL and the 4-manifold pairs (X_1, X_0) and (X_2, X_0) have class SES_+^h. Then h is topologically s-bordant rel ∂M to a homotopy equivalence $h''' : M''' \to X$ $\mathbb{Z}[\Gamma_0]$-split along X_0 if and only if $h \times \mathrm{id}_Q$ is homotopic rel $\partial M \times Q$ to a homotopy equivalence split along $X_0 \times Q$.*
2. *Do not assume Hypothesis $(*)$. Then, for some $r \geqslant 0$, the r-th stabilization $h \# \mathrm{id}_{r(S^2 \times S^2)}$ is homotopic rel ∂M to a homotopy equivalence $h''' : M''' \to X \# r(S^2 \times S^2)$ $\mathbb{Z}[\Gamma_0]$-split along X_0 if and only if $h \times \mathrm{id}_Q$ is homotopic rel $\partial M \times Q$ to a homotopy equivalence split along $X_0 \times Q$.*

Moreover, there is an analogous statement if X_0 is two-sided and non-separating.

Note the map $\Gamma_0 \to \Gamma$ is injective, but the amalgam Γ need not have class *NDL*. Observe the 8-manifold Q has both Euler characteristic and signature equal to one.

Corollary 3.3 *(Weinberger). In the previous theorem, instead of $(*)$, assume $(**)$: ∂X is empty and the fundamental group Γ has class NDL. Then h is homotopic to a $\mathbb{Z}[\Gamma_0]$-split homotopy equivalence along X_0 if and only if $h \times \mathrm{id}_Q$ is homotopic to a split homotopy equivalence along $X_0 \times Q$.*

Proof. Since Γ has class *NDL*, the subgroups $\Gamma_0, \Gamma_1, \Gamma_2$ have class *NDL*. Then, since $\Gamma_0, \Gamma_1, \Gamma_2$ have class *NDL*, by [21,32], the 4-manifold pairs (X_i, X_0) have class SES_+^h. Hence Hypothesis $(**)$ implies Hypothesis $(*)$. Now, since $\Gamma \in NDL$, by [21,32], the TOP s-cobordism of Theorem 3.2(1) is a product. □

Remark 3.4. Weinberger's theorem (Corollary 3.3) [44, Thm. 1] was stated in a limited form. The only applicable situations were injective amalgamated products $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_0 = \Gamma_1$ and $\Gamma = C_2 * C_2 = D_\infty$ in class *NDL*. (The second case was applied in [25,4].) We effectively delete the last phrase in his proof. Earlier, there was a homology splitting result of M. Freedman and L. Taylor [20] which required that $\Gamma = \Gamma_0 *_{\Gamma_0} \Gamma_0 = \Gamma_0$ but did not require that Γ have class *NDL*.

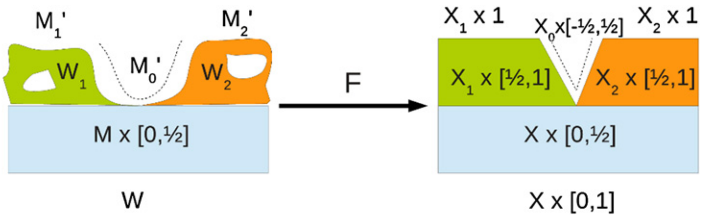

Fig. 1. Relabeled version of Weinberger's [44, Fig. 1].

Next, we modify Weinberger's clever cobordism argument, adding a few details. We suppress the orientation characters ω needed in the non-orientable case.

Proof of Theorem 3.2(1). For brevity, we sometimes denote ^{Q} for either $\times Q$ or $\times \mathrm{id}_Q$.

($\Rightarrow$) Since $\dim(X^Q) = 12 > 4$, this follows from two high-dimensional facts. By the TOP s-cobordism theorem [39], the product of any 5-dimensional s-cobordism with Q is homeomorphic to a product. By the handlebody version of Quillen's plus construction (for example, see [19, §11.2]; the high-dimensional TOP version can be extracted from [28, Annex 3, §6–§9]), the product of any 4-dimensional $\mathbb{Z}[\Gamma_0]$-split equivalence with id_Q can be exchanged along 2- and 3-handles in $M''' \times Q$ to become a split homotopy equivalence.

($\Leftarrow$) Suppose h^Q is homotopic to a homotopy equivalence split along X_0^Q. By TOP transversality [19], we may assume, up to homotopy rel ∂M, that $h : M \to X$ is TOP transverse to X_0. There is an induced decomposition of compact manifolds $M = M_1 \cup_{M_0} M_2$, where for all $j = 0, 1, 2$ the restrictions $h_j := h|M_j : M_j \to X_j$ are degree-one TOP normal maps and ∂h_j are homeomorphisms.

Since Q has signature equal to one, by the periodicity and product formulas [35, §8], for each $i = 1, 2$, the following relative surgery obstruction vanishes:

$$\sigma_*(h_i, h_0) \cong \sigma_*(h_i, h_0) \otimes \sigma^*(Q) = \sigma_*\big(h_i^Q, h_0^Q\big) = 0 \in L_{12}^h(\Gamma, \Gamma_0).$$

Then, by exactness at $\mathcal{N}_{\mathrm{TOP}}$ in Hypothesis ($*$), for each $i = 1, 2$, there exists a TOP normal bordism $(F_i, \partial_0 F_i) : (W_i, \partial W_i) \to (X_i, X_0)$ from $(h_i, h_0) : (M_i, M_0) \to (X_i, X_0)$ to a homotopy–homology equivalence $(h_i', \partial h_i') : (M_i', \partial M_i') \to (X_i, X_0)$. Note that the 3-manifolds $\partial M_1'$ and $\partial M_2'$ may not be homeomorphic.

We take three steps to construct an s-cobordism from h to an h'''. Fig. 1 illustrates the first step.

The precise, set-theoretic definitions are as follows:

$$F := F_1 \cup_{h_1} (h \times \mathrm{id}_{[0,\frac{1}{2}]}) \cup_{h_2} F_2, \qquad X' := (X_1 \times 1) \sqcup (X \times 0) \sqcup (X_2 \times 1),$$
$$W := W_1 \cup_{M_1} \left(M \times \left[0, \frac{1}{2}\right]\right) \cup_{M_2} W_2, \qquad M_0' := \partial_0 W_1 \cup_{M_0} \partial_0 W_2, \qquad M' := M_1' \sqcup (M \times 0) \sqcup M_2'.$$

Observe that $(F, \partial_0 F) : (W, M_0') \to (X \times [0,1], X_0 \times [-\frac{1}{2}, \frac{1}{2}])$ is a TOP normal map of manifold pairs, and that the restriction $\partial_1 F : M' \to X'$ is a homotopy equivalence.

Next, the second step is to leech off surgery obstructions of the two halves of F by attaching cobordisms. Select a homotopy $H : M^Q \times [-1, 0] \to X^Q$ to h^Q from a homotopy equivalence $g = g_1 \cup_{g_0} g_2$ split along X_0^Q. By TOP transversality [19], assume H and F^Q are transverse to X_0^Q. Define TOP normal maps

$$G_0 := H_0 \cup_{(h_0^Q \times 0)} \left(h_0^Q \times \left[0, \frac{1}{2}\right]\right), \qquad G_i := H_i \cup_{(h_i^Q \times 0)} \left(h_i^Q \times \left[0, \frac{1}{2}\right]\right) \cup_{(h_i^Q \times \frac{1}{2})} F_i^Q.$$

Note $H \cup_h F^Q = G_1 \cup_{G_0} G_2$. Observe the restriction $\partial_1 G_i = g_i \sqcup h_i'$ is a homotopy equivalence and the complement $\partial_0 G_i = G_0 \cup_{(h_0^Q \times \frac{1}{2})} \partial_0 F_i^Q$ is a normal map. So there are defined surgery obstructions

$$x := \sigma_*(F, \partial_0 F) \in L_5^h(\Gamma, \Gamma_0), \qquad x_i := \sigma_*(G_i, \partial_0 G_i) \in L_{13}^h(\Gamma_i, \Gamma_0).$$

Denote the inclusion-induced homomorphism $j_i : L_5^h(\Gamma_i, \Gamma_0) \to L_5^h(\Gamma, \Gamma_0)$. Then, by periodicity with Q, the cobordism invariance of surgery obstructions, and Wall's π–π theorem [43] (here, $L_*^h(\Gamma_0, \Gamma_0) = 0$), we obtain:

$$x \cong \sigma_*(F, \partial_0 F) \otimes \sigma^*(Q) = \sigma_*\big(F^Q, \partial_0 F^Q\big) = \sigma_*\big(H \cup_h F^Q, \partial_0 F^Q\big) = j_1(x_1) + j_2(x_2).$$

In particular, since Q has Euler characteristic equal to one, we obtain that $x \in L_5^h(\Gamma, \Gamma_0)$ is the image of a surgery obstruction $x^B \in L_5^B(\Gamma, \Gamma_0)$ uniquely determined by F^Q, where the decoration subgroup is

$$B := j_1 \mathrm{Wh}_1(\Gamma_1) + j_2 \mathrm{Wh}_1(\Gamma_2) \subseteq \mathrm{Wh}_1(\Gamma).$$

By existence of an L^h_5-action in Hypothesis $(*)$, for each $i = 1, 2$, there exists a TOP normal bordism $(F'_i, \partial_0 F'_i) : (W'_i, \partial_0 W'_i) \to (X_i, X_0)$ from $(h'_i, \partial h'_i)$ to $(h''_i, \partial h''_i)$ with surgery obstruction $\sigma_*(F'_i, \partial F'_i) = -x_i$ such that $(h''_i, \partial h''_i) : (M''_i, \partial M''_i) \to (X_i, X_0)$ is a homotopy–homology equivalence. Define:

$$F' := F'_1 \cup_{h'_1} F \cup_{h'_2} F'_2, \qquad M''_0 := \partial_0 W'_1 \cup_{\partial M'_1} M'_0 \cup_{\partial M'_2} \partial_0 W'_2, \qquad M'' := M''_1 \sqcup (M \times 0) \sqcup M''_2.$$

Observe $(F', \partial_0 F') : (W', M''_0) \to (X \times [0, 1], X_0 \times [-\frac{1}{2}, \frac{1}{2}])$ is a TOP normal map of pairs, and the complement $\partial_1 F' : M'' \to X'$ is a homotopy equivalence. So there is defined a surgery obstruction which vanishes:

$$\sigma_*\bigl(F', \partial_0 F'\bigr) = j_1(-x_1) + x + j_2(-x_2) = 0 \in L^B_5(\Gamma, \Gamma_0).$$

Since the Null Disc Lemma holds for Γ_0, by 5-dimensional relative surgery [43,19], there is a normal bordism G rel M'' to a B-torsion homotopy equivalence of pairs:

$$\bigl(F'', \partial_0 F''\bigr) : \bigl(W'', M'''_0\bigr) \to \left(X \times [0, 1], X_0 \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right).$$

In particular, $\partial_1 F'' = \partial_1 F'$ restricts to a $\mathbb{Z}[\Gamma_0]$-homology equivalence $\partial h'_1 : \partial M'_1 \to X_0$. Hence F'' is a B-torsion TOP h-bordism from h to a homotopy equivalence $\partial_+ F'' : \partial_+ W'' \to X$ $\mathbb{Z}[\Gamma_0]$-split along X_0.

Finally, the third step is to leech off the torsion obstructions of the two halves of the h-bordism F''. Consider its Whitehead torsion

$$t := \tau\bigl(M \hookrightarrow W''\bigr) \in B.$$

Then there exist $t_i \in \mathrm{Wh}(\Gamma_i)$ such that $t = j_1(t_1) + j_2(t_2)$. By existence of a Wh_1-action in Hypothesis $(*)$, for all $i = 1, 2$, there exist h-bordisms F''_i rel ∂ such that torsion of the domain h-cobordism is $j_i(-t_i)$. Therefore, by the sum formula, attaching these h-cobordisms to the top of W'' produces a TOP s-bordism $F''' := F''_1 \cup F'' \cup F''_2$ rel ∂M from h to a homotopy equivalence $h''' := \partial_+ F'''$ $\mathbb{Z}[\Gamma_0]$-split along X_0. □

Proof of Theorem 3.2(2). The argument in the stable case, Part (2), is similar to the unstable case, Part (1). The places where we used the hypothesis that (X_1, X_0) and (X_2, X_0) have class SES^h_+ can be replaced with the use of Theorem 2.7. Moreover, the places where we used the hypothesis that Γ_0 has class *NDL* had target $X_0 \times I$ for surgery problems, and so can be replaced with the use of Theorem 2.7.

Realization of elements of the Whitehead group by h-cobordisms on any given compact 4-manifold is the same as in high dimensions [37, p. 90]. Finally, by [19, Thm. 9.1], any TOP s-cobordism W''' on a compact 4-manifold admits a TOP handlebody structure. Then we proceed as in the proof of the high-dimensional s-cobordism theorem (e.g., see [37, Thm. 6.19]), except we resolve double-point singularities of immersed Whitney 2-discs via Norman tricks [34, Lem. 1]. We conclude, for some $r \geqslant 0$, that the sum stabilization $W''' \natural r(S^2 \times S^2 \times I)$ (defined on [19, p. 107]) is homeomorphic to the product $(X \# r(S^2 \times S^2)) \times I$. □

4. Proofs for the surgery sequence

Again, we suppress the orientation characters ω used in the non-orientable case. We start with a puncturing lemma. Section 3 contains the terminology for pairs.

Lemma 4.1. *Let Z be a non-empty compact connected topological 4-manifold. Write $pZ := Z - \operatorname{int} D^4$. If Z has class SES^h_+, then (pZ, S^3) has class SES^h_+.*

Proof. Denote the fundamental group $\Gamma := \pi_1(Z)$. First, let $(M, \partial_0 M)$ be a compact topological 4-manifold pair, and let $(f, \partial_0 f) : (M, \partial_0 M) \to (pZ, S^3)$ be a degree-one TOP normal map of pairs that restricts to a homeomorphism $\partial_1 f : \partial_1 M \to \partial Z$. Suppose the relative surgery obstruction vanishes: $\sigma^h_4(f) = 0 \in L^h_4(\Gamma, 1)$. Recall the geometric exact sequence of C.T.C. Wall [43, Cor. 3.1.1]:

$$\mathbb{Z} = L^h_4(1) \xrightarrow{\varepsilon} L^h_4(\Gamma) \to L^h_4(\Gamma, 1) \to L^h_3(1) = 0.$$

Then $\partial_0 f : \partial_0 M \to S^3$ is DIFF normally bordant to a $\mathbb{Z}$-homology equivalence $g : \Sigma \to S^3$. Since any closed oriented 3-manifold Σ is parallelizable, by a theorem of M. Freedman [19, Cor. 9.3C], it follows there exists a TOP normal null-bordism of g over D^4. Thus $(f, \partial_0 f)$ is TOP normally bordant, as a pair relative to $\partial_1 M$, to a degree-one map $f' : M' \to Z$ such that $\partial f' : \partial_1 M \to \partial Z$ is a homeomorphism. Moreover, by connecting sum with copies of the TOP E_8-manifold or its reverse, we may assume that the absolute surgery obstruction vanishes: $\sigma^h_4(f') = 0 \in L^h_4(\Gamma)$. By hypothesis, f' is TOP normally bordant to a homotopy equivalence $h : M'' \to Z$. We may assume that h is transverse to a point $z \in Z$ and that $h^{-1}\{z\}$ is a singleton. Thus $(f, \partial_0 f)$ is normally bordant to a homotopy equivalence $(ph, \mathrm{id}) : (pM'', S^3) \to (pZ, S^3)$. Therefore we obtain exactness at the normal invariants $\mathcal{N}_{\mathrm{TOP}}(pZ, S^3)$.

Next, define an appropriate action of $L^h_5(\Gamma, 1)$ on $\mathcal{S}^h_{\mathrm{TOP}}(pZ, S^3)$ as follows. By puncturing at a transversal singleton $\{z\} \subset Z$ with connected preimage, we obtain a function $p : \mathcal{S}^h_{\mathrm{TOP}}(Z) \to \mathcal{S}^h_{\mathrm{TOP}}(pZ, S^3)$. By the existence of 1-connected TOP h-cobordism from a homology 3-sphere Σ to the genuine one [19, Cor. 9.3C], it follows that p is surjective. By the topological plus construction [19, Thm. 11.1A], applied to any homology h-cobordism of S^3 to itself, it follows that p is injective. By hypothesis, there is an appropriate action of $L^h_5(\Gamma)$ on $\mathcal{S}^h_{\mathrm{TOP}}(Z)$. This extends, via the bijection p, to an action of $L^h_5(\Gamma)$ on $\mathcal{S}^h_{\mathrm{TOP}}(pZ, S^3)$. For any orientation character ω, there is a unique $k \geqslant 0$ such that Wall's exact sequence becomes

$$0 \to L^h_5(\Gamma) \xrightarrow{\iota} L^h_5(\Gamma, 1) \to k\mathbb{Z} = \mathrm{Ker}(\varepsilon) \to 0.$$

(Here $k = 0$ if and only if $\omega = 1$, equivalently, Z is orientable.) Since these groups are abelian, we obtain a non-canonical isomorphism

$$\varphi : L^h_5(\Gamma, 1) \to L^h_5(\Gamma) \oplus k\mathbb{Z}.$$

The relevant action of $L^h_4(1)$ on the homology structure set $\mathcal{S}^{h\mathbb{Z}}_{\mathrm{TOP}}(S^3)$ via twice-punctured E_8-manifolds restricts/extends to an action of $k\mathbb{Z}$ on $\mathcal{S}^h_{\mathrm{TOP}}(pZ, S^3)$. Thus, via the isomorphism φ, we obtain an appropriate action of $L^h_5(\Gamma, 1)$, given by concatenation of the actions. Therefore, we obtain (pZ, S^3) has class SES^h_+. □

At last, we are ready to establish our Main Theorem using homology splitting. For any non-empty compact connected 4-manifold Z, we use the following notation:

$$pZ := Z - \mathrm{int}\, D^4, \qquad \widetilde{\mathcal{N}}_{\mathrm{TOP}}(Z) := \mathrm{Ker}\big(\mathcal{N}_{\mathrm{TOP}}(Z) \to L^h_4(1)\big), \qquad \widetilde{L}^h_4(\pi_1 Z) := \mathrm{Cok}\big(L^h_4(1) \to L^h_4(\pi_1 Z)\big).$$

Proof of Theorem 1.7. Since $\Gamma := \pi_1(X)$ is isomorphic to a free product $\Gamma_1 * \cdots * \Gamma_n$, by an existence theorem of J. Hillman [23] (cf. [31,33]), there exist $r \geqslant 0$ and closed topological 4-manifolds $X_1, \ldots, X_n$ with each $\pi_1(X_i)$ isomorphic to Γ_i such that $X \# r(S^2 \times S^2)$ is homeomorphic to $X_1 \# \cdots \# X_n$. For Part (1), since each Γ_i has class *NDL*, by Theorem 1.6, we obtain that each X_i has class SES^h_+. For Part (2), this is assumed of the X_i, and the SES^h_+ property only depends on the homotopy type of X. Therefore we may assume that $X = X_1 \# \cdots \# X_n$ with each X_i of class SES^h_+. Write $\Gamma_i := \pi_1(X_i)$ for each fundamental group.

We induct on $n > 0$. Assume for some $n \geqslant 1$ that the $(n-1)$-fold connected sum of all compact connected topological 4-manifolds of class SES^h_+ has class SES^h_+, where in the non-orientable case we assume 2-torsionfree fundamental group. Write

$$X' := X_1 \# \cdots \# X_{n-1}, \qquad \Gamma' := \Gamma_1 * \cdots * \Gamma_{n-1}.$$

Hence $X = X' \# X_n$ and $\Gamma = \Gamma' * \Gamma_n$. By hypothesis, both X' and X_n have class SES^h_+. Then, by Lemma 4.1, the pairs (pX', S^3) and (pX_n, S^3) have class SES^h_+. Next, we show our original 4-manifold has class SES^h_+:

$$X = pX' \cup_{S^3} pX_n.$$

First, the K-theory splitting obstruction group vanishes [42], and, by a recent vanishing result [10,3,11], so do the L-theory obstruction groups[1]:

$$\widetilde{\mathrm{Nil}}_0\big(\mathbb{Z}; \mathbb{Z}\big[\Gamma' - 1\big], \mathbb{Z}[\Gamma_n - 1]\big) = 0, \qquad \mathrm{UNil}^h_4\big(\mathbb{Z}; \mathbb{Z}\big[\Gamma' - 1\big], \mathbb{Z}[\Gamma_n - 1]\big) = 0,$$
$$\mathrm{UNil}^h_5\big(\mathbb{Z}; \mathbb{Z}\big[\Gamma' - 1\big], \mathbb{Z}[\Gamma_n - 1]\big) = 0.$$

So observe, by Stalling's theorem for Whitehead groups of free products [40] and the Mayer–Vietoris type exact sequence for L-theory groups [6], that:

$$\mathrm{Wh}_1(\Gamma) = \mathrm{Wh}_1\big(\Gamma'\big) \oplus \mathrm{Wh}_1(\Gamma_n), \qquad \widetilde{L}^h_4(\Gamma) = \widetilde{L}^h_4\big(\Gamma'\big) \oplus \widetilde{L}^h_4(\Gamma_n), \qquad \widetilde{L}^h_5(\Gamma) = \widetilde{L}^h_5\big(\Gamma'\big) \oplus \widetilde{L}^h_5(\Gamma_n).$$

Here, from the Mayer–Vietoris sequence for any free product $G = G_1 * G_2$, we write

$$\widetilde{L}^h_5(G) := \mathrm{Ker}\big(\partial : L^h_5(G) \to L^h_4(1)\big).$$

Second, since $\mathcal{N}_{\mathrm{TOP}}(S^3)$ and $\widetilde{\mathcal{N}}_{\mathrm{TOP}}(S^3 \times I)$ are singletons, by TOP transversality [19] and by attaching thickened normal bordisms, we obtain:

$$\widetilde{\mathcal{N}}_{\mathrm{TOP}}(X) = \widetilde{\mathcal{N}}_{\mathrm{TOP}}\big(X'\big) \times \widetilde{\mathcal{N}}_{\mathrm{TOP}}(X_n).$$

[1] If Γ is 2-torsionfree, then $\mathrm{UNil}^h_* = 0$ by Cappell's earlier result [6], [7, Lem. II.10]. Furthermore, we require Γ to be 2-torsionfree in the non-orientable case, due to the example of non-vanishing of these two UNil-groups for $\mathbb{RP}^4 \# \mathbb{RP}^4$.

So, since the surgery sequence for both X' and X_n is exact at $\mathcal{N}_{\mathrm{TOP}}$, the surgery sequence for the connected sum X is exact at $\mathcal{N}_{\mathrm{TOP}}$.

Third, since (pX', S^3) and (pX_n, S^3) have class SES^h_+ and the splitting obstruction groups vanish, by Theorem 3.2(1), any homotopy equivalence to X is TOP s-bordant rel ∂M to a $\mathbb{Z}$-homology split map along S^3. That is, the top part of the s-bordism is a homotopy equivalence whose preimage of S^3 is a $\mathbb{Z}$-homology 3-sphere Σ. Thus the following inclusion is an equality (compare [5, Thm. 3]):

$$\subseteq : \mathcal{S}^{\mathbb{Z}\text{-split}}_{\mathrm{TOP}}\big(X; S^3\big) \to \mathcal{S}^h_{\mathrm{TOP}}(X).$$

By [19, Cor. 9.3C], there exists a TOP $\mathbb{Z}$-homology h-cobordism $(W; \Sigma, S^3)$ such that W is 1-connected. Furthermore, there exists an extension of the degree-one normal map $\Sigma \to S^3$ to a degree-one normal map $W \to S^3 \times I$. Thus, by attaching the thickened normal bordism, the following inclusion is an equality:

$$\subseteq : \mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}\big(X; S^3\big) \to \mathcal{S}^{\mathbb{Z}\text{-split}}_{\mathrm{TOP}}\big(X; S^3\big).$$

(The process of this last equality is called *neck exchange*, cf. [30,25].) Therefore the following map #, given by interior connected sum, is surjective:

$$\# : \mathcal{S}^h_{\mathrm{TOP}}\big(X'\big) \times \mathcal{S}^h_{\mathrm{TOP}}(X_n) \to \mathcal{S}^h_{\mathrm{TOP}}(X).$$

In order to show that # is injective, suppose $h_1 \# h_2$ is TOP h-bordant to $h'_1 \# h'_2$, say by a map $H : W \to X \times I$. Since $S^3 \times I$ is a 1-connected 4-manifold [19], and ∂H is split along $S^3 \times \partial I$, by the relative 5-dimensional form of Cappell's nilpotent normal cobordism construction [5,7], there exists a TOP normal bordism rel ∂W from H to an h-bordism $H' : W' \to X \times I$ split along $S^3 \times I$. So $H' = H'_1 \# H'_2$. Therefore # is injective. Now $\mathrm{Wh}_1(\Gamma)$ and $\widetilde{L}^h_5(\Gamma)$ can be given product actions on $\mathcal{S}^h_{\mathrm{TOP}}(X)$. The latter extends to an action of $L^h_5(\Gamma)$ by attaching a thickened multiple of a twice-punctured E_8 manifold along S^3. Hence the surgery sequence for X is exact at $\mathcal{S}^h_{\mathrm{TOP}}$ and L^h_5. This completes the induction. Therefore arbitrary connected sums $X = X_1 \# \cdots \# X_n$ have class SES^h_+. □

The following argument is partly based on Farrell's 1970 ICM address [15].

Proof of Theorem 1.13. One repeats the mapping torus argument of the proof of [27, Thm. 5.6], constructing a homotopy equivalence $h : X \to X$ using f. Since the achieved homotopy equivalence $g : M \to X \rtimes_h S^1$ has Whitehead torsion $\tau(g) = \tau(f) = 0$, there are no splitting obstructions. Since X has class SES^h_+, the proof of splitting g along X holds [27, Thm. 5.4]; one no longer requires that M and X be DIFF manifolds. Therefore the argument of [27, Thm. 5.6] shows that $f : M \to S^1$ is homotopic to a TOP s-block bundle projection. □

5. Proofs for topological rigidity

The following elementary argument is similar to J. Hillman's [24, Cor. 6.7.2].

Proof of Theorem 1.18. First, we show that the s-cobordism structure set $\mathcal{S}^s_{\mathrm{TOP}}(Z)$ is a singleton. Let M be a compact topological 4-manifold, and let $h : M \to Z$ be a simple homotopy equivalence that restricts to a homeomorphism $\partial h : \partial M \to \partial Z$. Then the surgery obstruction $\sigma^s_4(\eta(h)) \in L^s_4(\pi, \omega)$ vanishes. Since σ^s_4 is injective, there exists a TOP normal bordism $F : W \to Z \times I$ to $\eta(h)$ from the identity id_Z. Since σ^s_5 is surjective, there exists a TOP normal bordism $F' : W' \to Z \times I$ to id_Z from id_Z with opposite surgery obstruction: $\sigma^s_5(F') = -\sigma^s_5(F)$. Hence the union

$$F'' := F' \cup_{\mathrm{id}_Z} F : W' \cup_Z W \to Z \times I$$

is a TOP normal bordism to $\eta(h)$ from id_Z with vanishing surgery obstruction: $\sigma^s_5(F'') = 0$. Therefore, by 5-dimensional TOP surgery theory [43,28], we obtain that F'' is TOP normally bordant rel ∂ to a simple homotopy equivalence $F''' : (W'''; Z, M) \to (Z \times I; Z \times 0, Z \times 1)$ of manifold triads. Therefore we have found a TOP s-bordism to h from id_Z. That is, $\mathcal{S}^s_{\mathrm{TOP}}(Z)$ is a singleton $\{*\}$.

Next, observe that trivially we obtain an exact sequence of based sets:

$$\mathcal{N}_{\mathrm{TOP}}(Z \times I) \xrightarrow{\sigma^s_5} L^s_5(\pi, \omega) \xrightarrow{\partial} \{*\} \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(Z) \xrightarrow{\sigma^s_4} L^s_4(\pi, \omega).$$

We declare the action of $L^s_5(\pi, \omega)$ on $\mathcal{S}^s_{\mathrm{TOP}}(Z)$ to be trivial. Finally, if $\mathrm{Wh}_1(\pi) = 0$, then homotopy equivalences to Z are simple, and so Z is topologically s-rigid. □

We employ a case of a lemma of Hillman [24, Lem. 6.8], providing its details.

Proof of Corollary 1.19. Let $k \geqslant 0$. By the Mayer–Vietoris sequence in homology, the Shaneson sequence in L-theory [38], and the Ranicki assembly map [36, p. 148], the following diagram commutes with right-split exact rows:

$$\begin{array}{ccccc}
H_{5+k}(Z;\mathbb{L}_0) & \xrightarrow{i_*} & H_{5+k}(Z\times S^1;\mathbb{L}_0) & \xrightarrow{\partial} & H_{4+k}(Z;\mathbb{L}_0) \\
\downarrow{\scriptstyle A^s_{5+k}(Z)} & & \downarrow{\scriptstyle A^s_{5+k}(Z\times S^1)} & & \downarrow{\scriptstyle A^h_{4+k}(Z)} \\
L^s_{5+k}(Z) & \xrightarrow{i_*} & L^s_{5+k}(Z\times S^1) & \xrightarrow{\partial} & L^h_{4+k}(Z).
\end{array}$$

Moreover, the algebraic right-splitting is given by multiplying local or global quadratic complexes by the symmetric complex of the circle. This choice of splitting commutes with the connective assembly maps $A^s_{5+k}(Z\times S^1)$ and $A^h_{4+k}(Z)$.

Assume $Z\times S^1$ is topologically rigid. Then $\mathcal{S}^s_{\mathrm{TOP}}(Z\times S^1)=\{*\}$. So, by Wall's surgery exact sequence [43, §10] and Ranicki's identification of the surgery obstruction map with the assembly map [36, Prop. 18.3(1)] via topological transversality [19], we obtain that $A^s_5(Z\times S^1)$ is injective and $A^s_6(Z\times S^1)$ is surjective. Hence, using $k=0$ in the above diagram and the right-splitting, $\sigma^h_4 = A^h_4(Z)$ is injective. Also, using $k=1$ in the above diagram, $\sigma^h_5 = A^h_5(Z)$ is surjective. Therefore, by Theorem 1.18, we obtain that $\mathcal{S}^h_{\mathrm{TOP}}(Z)=\{*\}$. Hence, since $\mathrm{Wh}_1(\pi_1 Z)=0$ by hypothesis, we conclude that Z is topologically s-rigid. □

Proof of Corollary 1.23. Denote $\Gamma := \pi_1(Z)$. Via topological transversality [19], there are commutative squares with bijective left vertical maps [36, Prop. 18.3(1)]:

$$\begin{array}{ccc}
\mathcal{N}_{\mathrm{TOP}}(Z) & \xrightarrow{\sigma^s_4} & L^s_4(\Gamma) \\
\downarrow{\scriptstyle \cap[Z]_{\mathbb{L}^0}} & & \downarrow{\scriptstyle A^\Gamma_4} \\
H_4(Z;\mathbb{L}_0) & \xrightarrow{u_4} & H_4(B\Gamma;\mathbb{L}_0)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{N}_{\mathrm{TOP}}(Z\times I) & \xrightarrow{\sigma^s_5} & L^s_5(\Gamma) \\
\downarrow{\scriptstyle \cap[Z]_{\mathbb{L}^0}} & & \downarrow{\scriptstyle A^\Gamma_5} \\
H_5(Z;\mathbb{L}_0) & \xrightarrow{u_5} & H_5(B\Gamma;\mathbb{L}_0).
\end{array}$$

Here, we are using the identification $\mathcal{N}_{\mathrm{TOP}}(Z)=[Z/\partial Z, G/TOP]_+$. Since Z is aspherical, the bottom horizontal maps are isomorphisms. Since Γ is torsionfree with $\mathrm{cdim}(\Gamma)=4$ and has class FJ_L, $\mathrm{Wh}_1(\Gamma)=0$, the map A^Γ_4 is a monomorphism, and A^Γ_5 is an isomorphism. Hence σ^s_4 is injective and σ^s_4 is surjective. Therefore, by Theorem 1.18, we obtain that Z is topologically s-rigid and has class SES^h_+. □

Proof of Corollary 1.26. Let $\alpha : K \to K$ be the homeomorphism. It follows from the homotopy sequence of a fibration that $Z = K \rtimes_\alpha S^1$ is aspherical. By a recent theorem[2] of Bartels, Farrell and Lück [2], we obtain that $\Gamma_0 := \pi_1(K)$ has class FJ_L.

Write $\Gamma := \pi_1(Z)$. Then $\Gamma = \Gamma_0 \rtimes_\alpha \mathbb{Z}$. By the excessive Wang sequence and the Shaneson Wang-type sequence, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H_n(B\Gamma_0;\mathbb{L}) & \xrightarrow{1-\alpha_*} & H_n(B\Gamma_0;\mathbb{L}) & \xrightarrow{i_*} & H_n(B\Gamma;\mathbb{L}) & \xrightarrow{\partial} & H_{n-1}(B\Gamma_0;\mathbb{L}) \\
\downarrow{\scriptstyle A^{\Gamma_0}_n} & & \downarrow{\scriptstyle A^{\Gamma_0}_n} & & \downarrow{\scriptstyle A^{\Gamma}_n} & & \downarrow{\scriptstyle A^{\Gamma_0}_{n-1}} \\
L^{s=-\infty}_n(\Gamma_0) & \xrightarrow{1-\alpha_*} & L^s_n(\Gamma_0) & \xrightarrow{i_*} & L^s_n(\Gamma) & \xrightarrow{\partial} & L^{h=s}_{n-1}(\Gamma_0).
\end{array}$$

Since Γ_0 is torsionfree and has class FJ_L, the non-connective assembly maps $A^{\Gamma_0}_*$ are isomorphisms. Hence, by the five lemma, the non-connective assembly maps A^Γ_* are isomorphisms. Using topological transversality and Poincaré duality, similar to the proof of Corollary 1.23, by Theorem 1.18, we obtain that $\mathcal{S}^s_{\mathrm{TOP}}(Z)=\{*\}$. Hence, since $\mathrm{Wh}_1(\pi_1 Z)=0$, we conclude that Z is topologically s-rigid. □

Proof of Corollary 1.27. Since each X_i is orientable and has class SES^h_+, by Theorem 1.7, we obtain that X has class SES^h_+ and the following function is a bijection:

$$\# : \prod_{i=1}^{n} \mathcal{S}^h_{\mathrm{TOP}}(X_i) \to \mathcal{S}^h_{\mathrm{TOP}}(X).$$

Next, let $1 \leqslant i \leqslant n$. Consider the connective assembly map components [41]:

$$A_4 = (\,I_0 \quad \kappa_2\,) : H_4(B\Gamma_i;\mathbb{L}_0) = H_0(B\Gamma_i;\mathbb{Z}) \oplus H_2(B\Gamma_i;\mathbb{Z}_2) \to L^h_4(\Gamma_i),$$
$$A_5 = (\,I_1 \quad \kappa_3\,) : H_5(B\Gamma_i;\mathbb{L}_0) = H_1(B\Gamma_i;\mathbb{Z}) \oplus H_3(B\Gamma_i;\mathbb{Z}_2) \to L^h_5(\Gamma_i).$$

[2] Their proof depends on G. Perelman's affirmation of Thurston's Geometrization Conjecture (cf. [1]). It also depends on individual casework of S. Roushon and P. Kühl.

Assume Γ_i is torsionfree and $\pi_2(X_i) \otimes \mathbb{Z}_2 = 0$. Since Γ_i has class FJ_L and $\operatorname{cdim}(\Gamma_i) \leqslant 4$, we obtain that A_4 is a monomorphism and A_5 is an isomorphism. Recall the universal covering $\widetilde{X_i} \to X_i$ is classified by a unique homotopy class of map $u : X_i \to B\Gamma_i$, which induces an isomorphism on fundamental groups. Since X_i is a closed oriented topological manifold, using topological transversality [19], the Quinn–Ranicki H-space structure on G/TOP, and Poincaré duality with respect to the $\mathbb{L}^0$-orientation [36], we obtain induced homomorphisms

$$u'_4 : \mathcal{N}_{\mathrm{TOP}}(X_i) \cong \big[(X_i)_+, G/TOP\big]_+ \cong H_4(X_i; \mathbb{L}_0) \xrightarrow{u_*} H_4(B\Gamma_i; \mathbb{L}_0),$$
$$u'_5 : \mathcal{N}_{\mathrm{TOP}}(X_i \times I) \cong \big[(X_i)_+ \wedge S_1, G/TOP\big]_+ \cong H_5(X_i; \mathbb{L}_0) \xrightarrow{u_*} H_5(B\Gamma_i; \mathbb{L}_0)$$

such that the surgery obstruction map factors: $\sigma_4^h = A_4 \circ u'_4$ and $\sigma_5^h = A_5 \circ u'_5$. Recall the Hopf sequence, which is obtained from the Leray–Serre spectral sequence:

$$H_3(X_i; \mathbb{Z}_2) \xrightarrow{u_3} H_3(B\Gamma_i; \mathbb{Z}_2) \to H_2(\widetilde{X}; \mathbb{Z}_2) \to H_2(X; \mathbb{Z}_2) \xrightarrow{u_2} H_2(B\Gamma_i; \mathbb{Z}_2) \to 0.$$

Since $H_2(\widetilde{X}; \mathbb{Z}_2) = \pi_2(X_i) \otimes \mathbb{Z}_2 = 0$, we have $\operatorname{Ker}(u_2) = 0$ and $\operatorname{Cok}(u_3) = 0$. Hence

$$\operatorname{Ker}\big(\sigma_4^h\big) = \operatorname{Ker}\big(u'_4\big) = \operatorname{Ker}(u_0) \oplus \operatorname{Ker}(u_2) = 0,$$
$$\operatorname{Cok}\big(\sigma_5^h\big) = \operatorname{Cok}\big(u'_5\big) = \operatorname{Cok}(u_1) \oplus \operatorname{Cok}(u_3) = 0.$$

Therefore, since X_i has class SES_+^h and $\mathrm{Wh}_1(\Gamma_i) = 0$, we obtain that $\mathcal{S}^s_{\mathrm{TOP}}(X_i)$ is a singleton. Thus, since # is a bijection, the Whitehead group $\mathrm{Wh}_1(\Gamma)$ and s-cobordism structure set $\mathcal{S}^s_{\mathrm{TOP}}(X)$ of $X = X_1 \# \cdots \# X_n$ are singletons also. □

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