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Countable approximation of topological G-manifolds, II: linear Lie groups G

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Let G be a matrix group. Topological G-manifolds with Palais-proper action have the G-homotopy type of countable G-CW complexes (3.2). This generalizes Elfving's dissertation theorem for locally linear G-manifolds (1996). Also, we improve the Bredon–Floyd theorem from compact Lie groups G to arbitrary Lie groups G.

Keywords: Topological manifold; proper action; Lie group; CW complex.

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1. Equivariant Cohomology Manifolds

Definition 1.1. Let G be a topological group. Let X be a G-*space*, that is, a topological space equipped with a left G-action. For any $x \in X$, its *orbit type* (G_x) is the G-conjugacy class of its *isotropy group* $G_x := \{g \in G \mid gx = x\} \leq G$. The G-space X is G-*metrizable* if it has a G-invariant metric: $d(gx, gy) = d(x, y)$. The group G is *Lie* if G is a real-analytic manifold with $(a, b) \mapsto a^{-1}b$ analytic.

Proper in the sense of Bourbaki is [33, I:3.17] and of Palais is [30, 1.2.2].

Definition 1.2. Let G be a topological group. Let X be a G-space. Define a map

$$\theta : G \times X \to X \times X; \quad (g, x) \to (x, gx).$$

The G-space X is *Bourbaki-proper* if θ is proper in the sense of Bourbaki: the product $\theta \times \mathrm{id}_Z$ is a closed function for any topological space Z [11, 10:1.1].

Now, suppose G is locally compact and X is completely regular, both without any assumption of Hausdorff. The G-space X is *Palais*(*-proper*) if each point $x \in X$

has a neighborhood U such that any $y \in X$ has a neighborhood V with

$$\langle U, V\rangle_G := \{g \in G \mid U \cap gV \neq \varnothing\}$$

precompact [30, 1.2.2]. More generally, X is *Cartan(-proper)* if each point $x \in X$ has a neighborhood U such that $\langle U, U\rangle_G \subset G$ is precompact [30, 1.1.2].

We recall $\mathbb{Z}$-cohomology manifold [10, I:3.3], without separable or metrizable. Our description uses Čech cohomology for Alexander–Spanier cohomology [14].

Definition 1.3. (Borel) Let M be a locally compact, Hausdorff topological space. Let $n \in \mathbb{N}$. Then M is an *n-dimensional* $\mathbb{Z}$*-cohomology manifold* (n-$\mathrm{cm}_{\mathbb{Z}}$) if:

(1) $\dim_{\mathbb{Z}}(M) \leq n$, that is, $\check{H}^n(M;\mathbb{Z}) \to \check{H}^n(A;\mathbb{Z})$ is onto for all closed $A \subset M$
(2) $\forall\, x \in M$: local Betti numbers $\beta^i_{\mathbb{Z}}(M,x) = 0\ \forall\, i < n$ and $\beta^n_{\mathbb{Z}}(M,x) = 1$ in the sense of Borel [10, I:2.1] extending Aleksandrov (1935) and Čech (1934)
(3) $\forall\, x \in M$: a local $\mathbb{Z}$-orientation of M at x exists in the sense of [10, I:3.2].

We generalize the Bredon–Floyd theorem [10, VII:2.2] to noncompact groups, by adapting the circle of ideas within Floyd's initial argument for [10, VI:1.1].

Theorem 1.4. *Let G be a Lie group. Let M be a $\mathbb{Z}$-cohomology manifold with Bourbaki-proper G-action. Any compact set in M has only finitely many orbit types.*

Proof. Assume not. Then there exists an infinite sequence $\{x_i\}_{i=0}^{\infty}$ in some compact subset K of M such that no two of the isotropy groups G_{x_i} are conjugate in G. Since the action is proper, $C := \{g \in G \mid gK \cap K \neq \varnothing\}$ is compact [33, I:3.21]. In particular, since each G_{x_i} is a closed subset of C, each G_{x_i} is compact. Recall that the set $\mathrm{Cpt}(G)$ of non-empty compact subsets of a metric space (G,d) admits the Hausdorff–Pompeiu metric d_{HP}, which is compact if the ambient metric space is compact [29, 45:7]. Then the infinite sequence $\{G_{x_i}\}_{i=0}^{\infty}$ in the compact metric space $(\mathrm{Cpt}(C), d_{\mathrm{HP}})$ has a convergent subsequence, which we may reindex to be the original. By continuity of multiplication and inversion in G, the compact subset $H := \lim_{i\to\infty} G_{x_i}$ is a subgroup of G. Thus H is a Lie group [24, 20.10].

Let U be a compact neighborhood of the neutral element in G. On the one hand, since H is a compact Lie group acting on the $\mathbb{Z}$-cohomology manifold M, by the Bredon–Floyd theorem [10, VII:2.2], the compact set $UK \subset M$ supports only finitely many H-orbit types. On the other hand, by the Montgomery–Zippin neighboring-subgroups theorem [30, 4.2], there is a neighborhood N of H in G so any subgroup of G contained in N is U-conjugate to a subgroup of H. Since H is a limit, there exists i_0 such that $G_{x_i} \subset N$ for all $i \geq i_0$. Re-index so that $i_0 = 0$. Then there exists $u_i \in U$ such that $G_{u_i x_i} = u_i G_{x_i} u_i^{-1} \subset H$ for each i. Note $\{u_i x_i\}_{i=0}^{\infty}$ is an infinite sequence in $U\,K$ such that no two $G_{u_i x_i}$ are G-conjugate hence not H-conjugate, contradicting that UK has only finitely many H-orbit types. □

2. Equivariant Absolute Neighborhood Retracts

Recall X is a *G-ANR for the class $\mathcal{C}$* ($\mathcal{C}$-absolute G-neighborhood retract) if X belongs to $\mathcal{C}$ and, for any closed G-embedding of X into a member of $\mathcal{C}$, there is a G-neighborhood of X with G-retraction to X. More generally, X is a *G-ANE for the class $\mathcal{C}$* ($\mathcal{C}$-absolute G-neighborhood extensor) if, for any member B of $\mathcal{C}$ and closed G-subset A of B and any G-map $A \to X$, there is a G-extension $U \to X$ from some G-neighborhood U of A in B. Notice a G-ANE need not belong to $\mathcal{C}$.

Not long ago, Antonyan [3, 5.7] made equivariant Hanner's open-union theorem (see [18, III:8.3]), providing a local-to-global principle for G-extensors.

Theorem 2.1. (Antonyan) *Let G be a locally compact Hausdorff group. Let $\mathcal{C}$ be a subclass of the class G-$\mathcal{P}$ of paracompact Palais G-spaces with paracompact orbit space. Any union of open G-subsets that are G-ANEs for $\mathcal{C}$ is also a G-ANE for $\mathcal{C}$.*

Equivariant CW structures were found over very general groups, using the nerves of locally finite coverings of neighborhoods in certain G-Banach spaces [7, 1.1]. Recall that Matumoto defined the notion of a G-CW complex [27, 1.2, 1.5].

Theorem 2.2. (Antonyan–Elfving) *Let G be a locally compact Hausdorff group. Suppose that X is a G-ANR for the class G-$\mathcal{M}$ of G-metrizable Palais G-spaces. Then X has the equivariant homotopy type of a G-CW complex with Palais action.*

Remark 2.3. Observe that the class G-$\mathcal{M}$ is a subclass of G-$\mathcal{P}$, as follows. Let X be a member of G-$\mathcal{M}$. Since X is G-metrizable, the orbit space X/G has an induced metric given by an infimum. Then, since both X and X/G are metrizable, by Stone's theorem [29, 41.4], we have that both X and X/G are paracompact.

As classes, observe $\mathcal{C} \cap G\text{-ANE}(\mathcal{C}) \subseteq G\text{-ANR}(\mathcal{C})$; a converse is [4, 6.3].

Theorem 2.4. (Antonyan–Antonyan–Martín-Peinador) *Let G be a locally compact Hausdorff group. Then $G\text{-ANR}(G\text{-}\mathcal{M}) = G\text{-}\mathcal{M} \cap G\text{-ANE}(G\text{-}\mathcal{M})$.*

The following technical notion over compact groups was introduced in [22]. We restate from [5, 2.2] the generalization over noncompact groups.

Definition 2.5. (Jaworowski) Let G be a Lie group. A Palais G-space X has *finite structure* if it has only finitely many orbit types and, for each orbit type (H), the quotient map $X_{(H)} \to X_{(H)}/G$ is a G/H-bundle with only finitely many local trivializations. Here (H) is the conjugacy class of H in G, $X_{(H)} := \{x \in X \mid (G_x) = (H)\}$ is the (H)-stratum, $G_x := \{g \in G \mid gx = x\}$ is an isotropy group.

Remark 2.6. Any compact Lie group is *linear*: it has an isomorphic topological embedding into $GL_n(\mathbb{R})$ for some n. This is a case of the following consequence of the Peter–Weyl theorem: any compact topological group G embeds into a product of unitary groups; if G has no small subgroups this product is finite [23, 4.1].

Recall $X^H := \{x \in X \mid \forall g \in H : gx = x\}$ denotes the *H-fixed subspace of X*. In the following recent theorem [5, 6.1], the Jaworowski–Lashof criterion for G-ANRs [22] is generalized from compact Lie groups G to all linear ones.

Theorem 2.7. (Antonyan–Antonyan–Mata-Romero–Vargas-Betancourt) *Let G be a linear Lie group. Let X be a G-metrizable Palais G-space with finite structure. Then X is a G-ANR for the class of G-metrizable Palais G-spaces, if and only if X^H is an ANR for the class of metrizable spaces for each compact subgroup $H < G$.*

3. Equivariant Topological Manifolds

Theorem 3.1. *Let G be a linear Lie group. Let M be a cohomology manifold over $\mathbb{Z}$ that is both separable and metrizable. Suppose M has Palais G-action and the fixed set M^H is ANR for the class of metrizable spaces for each compact subgroup H of G. Then M is G-homotopy equivalent to a countable proper G-CW complex.*

Proof. Let M be a $\mathbb{Z}$-cohomology manifold. Since M is separable and locally compact, there exists an increasing infinite sequence $\{M_i\}_{i=0}^{\infty}$ of open sets in M whose union is M and whose closures $\overline{M_i}$ in M are compact. By Theorem 1.4, the compact set $\overline{M_i}$, hence M_i, has only finitely many conjugacy classes of isotropy group. The G-saturation $GM_i = \bigcup_{g \in G} gM_i$ is also open [33, I:3.1(i)] and has only finitely many G-orbit types. Since $(GM_i)^H = GM_i \cap M^H$ is open in the ANR M^H, we have that $(GM_i)^H$ is also an ANR by Hanner's global-to-local principle [18, III:7.9].

Since G is a Lie group and $G\overline{M_i}$ is a Palais G-space, by Palais' slice theorem [30, 2.3.1, 2.1.2], $G\overline{M_i}$ has a covering $\mathcal{T}_i$ by G-tubes of varying orbit types. Furthermore, since $(G\overline{M_i})/G = \overline{M_i}/G$ is compact, $\mathcal{T}_i$ can be assumed finite. The stratum $(GM_i)_{(H)}$ of $GM_i \subset G\overline{M_i}$ has a single orbit type, so restriction of $\mathcal{T}_i$ to it gives a finite covering by local trivializations of a G/H-fiber bundle with structure group G. So, the Palais G-space GM_i has finite structure. By Palais' metrization theorem [30, 4.3.4], the separable metrizable M, hence GM_i, is G-metrizable. Since G is linear, GM_i is a G-ANR for G-$\mathcal{M}$ (2.7), hence is a G-ANE for G-$\mathcal{M}$ (2.4).

Thus, by Remark 2.3 and Theorem 2.1, $M = \bigcup_{i \in I} GM_i$ is a G-ANE for G-$\mathcal{M}$. Then, since M is also member of G-$\mathcal{M}$, M is a G-ANR for G-$\mathcal{M}$. Therefore, by Theorem 2.2, we conclude M has the G-homotopy type of a proper G-CW complex.

We now make some remarks on how to guarantee only countably many G-cells. The proof of Theorem 2.2 starts in [6, 5.2], with a closed G-embedding of X into a G-normed linear space L with Palais action on some G-neighborhood. Specifically, those authors take $L = E \times N$ [6, 3.10], which is valid for any G-metrizable Palais G-space X. Since our $X = M$ is locally compact, *alternatively use* the simpler and more classical G-Banach space $L = C_0(X)$, where

$$C_0(X) := \{f \in C(X) \mid \forall \varepsilon > 0, \exists \text{ compact } K \subset X, \forall x \in X - K : |f(x)| < \varepsilon\}$$

$$\|f\| := \sup\{|f(x)| \mid x \in X\}, \text{ which is well-defined.}$$

Indeed, Elfving in [17, Propositions 2,3] showed the existence of a Kurotowski-like G-embedding of X into $C_0(X) - \{0\}$ on which the continuous G-action is Palais.

Since X is separable, there exists a countable dense subset $\Delta \subset X$. Since X is locally compact, the Alexandroff one-point compactification X^* exists. Since X is second-countable, so is X^*, hence X^* admits a metric d by the Urysohn metrization theorem [29, 34.1]. Consider the countable collection $\Delta_d \subset C(X^*)$ defined by

$$\Delta_d := \{1\} \cup \{d(-,p) \in C(X^*) \,|\, p \in \Delta\}.$$

Since Δ_d contains a nonzero constant function and separates points because Δ is dense in X^*, by the Stone–Weierstrass theorem [32, Corollary 3, p. 174], the countable subring $\mathbb{Q}\langle\Delta_d\rangle$, consisting of rational polynomials in the elements of Δ_d:

$$\mathbb{Q}\langle\Delta_d\rangle := \mathrm{Im}(\mathbb{Q}[\Delta_d] \to C(X^*))$$

is dense in $C(X^*)$. Hence $C_0(X) \subset C(X^*)$ is separable.

Then the G-neighborhood U of X in $C_0(X) - \{0\}$, on which the G-retraction $U \to X$ is defined, is Lindelöf, as it is separable and metrizable. So in the proof of [7, Proposition 5.2], the rich G-normal cover $\mathcal{U}$ with index set $G \times \mathcal{M}$ can be assumed to have $\mathcal{M}$ a countable set. The geometric G-nerve $K(\mathcal{U})$ is indexed [7, p. 166] by certain finite subsets of $\mathcal{M}$. Thus the semisimplicial G-space $K(\mathcal{U})$ has only countably many G-cells, according to the proof of [7, Theorem 5.3], which relies on Illman [20] and this in turn involves only countably many G-cells for a smooth G-manifold. Finally, since [7, Proposition 5.2] states that $K(\mathcal{U})$ G-dominates X, by a G-version of Mather's trick (see second paragraph of [23, Proof 2.5]), the G-CW complex for $X = M$ has only countably many G-cells.

For the convenience of the reader, we detail the conclusion of this last sentence. Since the G-CW complex $K(\mathcal{U})$ G-dominates X, there are G-maps $u : X \to K(\mathcal{U})$, $d : K(\mathcal{U}) \to X$, and G-homotopy $h : X \times [0,1] \to X$ from $h_1 = d \circ u$ to $h_0 = \mathrm{id}_X$. By G-cellular approximation, there exists a cellular G-map $\alpha : K(\mathcal{U}) \to K(\mathcal{U})$ that is G-homotopic to $u \circ d$ [33, II:2.1]. On the one hand, the mapping torus

$$\mathrm{Torus}(\alpha) := \frac{K(\mathcal{U}) \times [0,1]}{(x,1) \sim (\alpha(x),0)}$$

is a G-CW complex [33, I:1.11]. On the other hand, it is G-homotopy equivalent to

$$\mathrm{Torus}(u \circ d) \simeq_G \mathrm{Torus}(d \circ u) \simeq_G \mathrm{Torus}(\mathrm{id}_X) = X \times S^1.$$

Thus $X \simeq X \times \mathbb{R}$ is G-homotopy equivalent to the infinite cyclic cover of Torus(α), namely the bi-infinite mapping telescope of α — a countable G-CW complex. □

Finally, we generalize [23, 2.5] from G being compact. Note that the manifold must be noncompact if G is noncompact in order for the action to be Cartan-proper.

Corollary 3.2. *Let G be a linear Lie group. Any topological G-manifold with Palais action has the equivariant homotopy type of a countable proper G-CW complex.*

Here, by *topological G-manifold* [23, 2.2], we mean the H-fixed subspace is a topological (C^0) manifold for each closed subgroup H of a topological group G. Herein, a topological manifold shall be separable, metrizable, and locally euclidean.

Proof. Let M be a topological G-manifold with Palais action. By Hanner's local-to-global principle [18, III:8.3], each manifold M^H is an ANR for the class of metrizable spaces. Also M is separable, metrizable, and a $\mathbb{Z}$-cohomology manifold. Therefore, we are done by Theorem 3.1. □

Thus more tractible are its Davis–Lück G-spectral homology groups [13, 3.7, 4.3], since we conclude countability of the G-CW complex that left-approximates.

Corollary 3.3. *Let G be a linear Lie group. Let $f : M \to N$ be a G-map between topological G-manifolds with Palais actions. Then f is a G-homotopy equivalence if and only if $f^H : M^H \to N^H$ is a homotopy equivalence for each closed H of G.*

Proof. This is immediate from Corollary 3.2 and the corresponding theorem for G-CW complexes [33, II:2.7], which is proven using G-obstruction theory. □

In particular, we generalize the main result of Elfving's thesis [16, 4.20]. The definition of *locally linear*, along with some discussion, is found in [23, 3.6, 3.7]. Note any smoothable action is locally linear, but not vice versa; see [12, VI:9.6].

Corollary 3.4. (Elfving) *Let G be a linear Lie group. Let M be a locally linear G-manifold with Palais action. If M has only finitely many orbit types, then M has the equivariant homotopy type of a G-CW complex.*

Proof. This special case now follows immediately from Corollary 3.2. □

4. Examples That Are Not Locally Linear

We continue the three families of uncountable examples of [23, 3.1, 3.2, 3.3]. The purpose here is to show there do exist topological G-manifolds that are not locally linear when G is a noncompact linear Lie group with torsion. (All principal bundles are trivial if G is connected torsionfree, such as $G = \mathbb{R}$ for complete flows.)

Their common trick is that the diagonal action will become Palais [30, 1.3.3], even though it is not on the first factor, using a homogeneous space G/H with H compact for the second factor. These G/H are exactly those with transitive Palais G-action. The transitivity on the second factor guarantees the same quotient space as the first's. Any C^1 Palais action by a Lie group is C^ω [19, 21]; ours are C^0.

Indeed, there is no contradiction to Palais' slice theorem [30, 2.3.1, 2.1.2]. There does exist a G_x-slice for each point x of the Palais G-manifolds, but *not all the slices are euclidean*, and this is why in particular these slices are not G_x-linear.

Example 4.1. (Bing) Consider the double $D := E \cup_A E$ of the non-simply connected side E in S^3 of the Alexander horned sphere $A \approx S^2$, whose embedding is *not* *locally flat*. This double has obvious involution r_B that interchanges the two pieces and leaves the horned sphere fixed pointwise. Bing showed D is homeomorphic to S^3 [8]. Thus r_B minus a fixed point (so on $\mathbb{R}^3$) negatively answers a question of Montgomery [15, 39b], asking if the action is conjugate to an isometric one.

Consider the Lie group $G = \mathrm{Isom}(\mathbb{R}) = \mathbb{R} \rtimes_{-1} O_1$, a closed subgroup of $GL_2(\mathbb{R})$:

$$\left\langle \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\rangle.$$

Define a non-Cartan action of G on S^3 by epimorphism to $O_1 \cong \langle r_B \rangle \leq \mathrm{Homeo}(S^3)$. As noted above, the diagonal action of G on the product of S^3 and the homogeneous space $\mathbb{R} = G/O_1$ is Palais. Then Corollary 3.2 applies to the topological G-manifold $S^3 \times \mathbb{R}$. In the orbit space $(S^3 \times \mathbb{R})/G = S^3/r_B = E$, the stratum A is not locally cofibrant, so the C^0 action of G on the 4-manifold $S^3 \times \mathbb{R}$ cannot be locally linear.

For each $n \geq 3$, Lininger [26, 9, 10] applies [9] to produce uncountably many inequivalent involutions on S^n with fixed set an $(n-1)$-sphere and quotient not a manifold-with-boundary, so none is equivalent to a locally linear action. They arise from uncountably inequivalent embeddings in S^{n-1} of Cantor's space $2^{\mathbb{N}}$; in the form of multiparameter Antoine necklaces, the $n = 4$ case is due to Sher [31].

Example 4.2. (Montgomery–Zippin) Adaptation of Bing's 1952 idea produces an involution r_{MZ} of S^3 whose fixed set is an *embedded circle* K that is not locally flat [28, §2]. In Example 4.1, replacing r_B and A with r_{MZ} and K works verbatim. Note r_{MZ} preserves orientation and was first to negatively answer the C^0 version of a question of Paul A Smith [15, 36], asking if the fixed circle is unknotted.

Alford gave uncountably many inequivalent involutions fixing a wild circle [1].

Higher *codimension-two* examples are provided by Lininger. He uses rotation of the Alexander horned sphere A in 4-space to obtain a semifree U_1-action on S^4 with fixed set a 2-sphere [26, 7]. More generally, using Bing's later techniques [9], he obtains uncountably many inequivalent semifree U_1-actions on S^n whose fixed set is an $(n-2)$-sphere and quotient not a manifold-with-boundary [26, 8, 10].

Example 4.3. (Lininger) For each $k \geq 3$, there are uncountably many inequivalent *free* U_1-actions on S^{2k-1} whose quotients are not C^0 manifolds [25, Remark 2]. At the root of Lininger's work are Andrews–Curtis decomposition spaces [2]: non-Euclidean quotients Q by a wild arc, any of whose product with $\mathbb{R}$ is euclidean.

Consider the Lie group $G = \mathrm{Isom}^+(\mathbb{C}) = \mathbb{C} \rtimes U_1$, a closed subgroup of $GL_2(\mathbb{C})$:

$$\left\langle \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C}, u \in U_1 \right\rangle.$$

Define a non-Cartan action of G on S^{2k-1} by epimorphism to U_1 then use Lininger. The diagonal action of G on the product of S^{2k-1} and homogeneous space $\mathbb{C} = G/U_1$ is Palais, as well as free. The orbit space $(S^{2k-1} \times \mathbb{C})/G = S^{2k-1}/U_1$ is not a topological manifold, though the projection from $S^{2k-1} \times \mathbb{C}$ is a principal G-bundle. In particular, none in this uncountable family of free G-actions can be locally linear.

The same holds for $G = U_1 \times G'$ with G' a linear Lie group and $M = S^{2k-1} \times G'$.

We end with a family of examples whose linear Lie group G is *arbitrarily large.*

Example 4.4. (Lininger) For each $n > k+1 \geq 3$, there are uncountably many inequivalent semifree SO_k-actions on S^n whose fixed set is a wild $(n-k-1)$-sphere [26, 11]. Again, the construction arises from the quotient by any wild arc [2].

The Lie group $G = \mathrm{Isom}^+(\mathbb{R}^k) = \mathbb{R}^k \rtimes SO_k$ is a closed subgroup of $GL_{2k}(\mathbb{R})$:

$$\left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R}^k, r \in SO_k \right\rangle.$$

Define a non-Cartan action of G on S^n by epimorphism to SO_k then use Lininger. The diagonal action of G on the product of S^n and homogeneous space $\mathbb{R}^k = G/SO_k$ is Palais. The orbit space $(S^n \times \mathbb{R}^k)/G = S^n/SO_k$ minus the singular set is not a manifold, so none in this uncountable family of semifree G-actions is locally linear.

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