Erratum to: Dihedral manifold approximate fibrations over the circle

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Erratum to: Geom Dedicata (2010) 148:205–243
DOI 10.1007/s10711-009-9418-6

The original publication of the article has been published with an incorrect proof of Theorem 8.1 in [1]. Specifically, we know neither how to verify nor disprove the claim that the homeomorphism $J_1 : W \to W$ satisfies the dihedral relation.

Thus, we retract Theorem 8.1.

Consequently, the assertion that (3) implies (4) in Theorem 1.1 is unproven. The remaining implications in the proof of Theorem 1.1 hold. Theorems 1.2, 1.3, and 1.4 are independent of Theorem 8.1 and remain valid. In particular, the material in Sects. 2–7 holds (as well as Lemma 8.2).

Reference

1. Hughes, B., Khan, Q.: Dihedral manifold approximate fibrations over the circle. Geom. Dedicata **148**, 205–243 (2010)

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ORIGINAL PAPER

Dihedral manifold approximate fibrations over the circle

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Abstract   Consider the cyclic group C_2 of order two acting by complex-conjugation on the unit circle S^1. The main result is that a finitely dominated manifold W of dimension > 4 admits a cocompact, free, discontinuous action by the infinite dihedral group D_∞ if and only if W is the infinite cyclic cover of a free C_2-manifold M such that M admits a C_2-equivariant manifold approximate fibration to S^1. The novelty in this setting is the existence of codimension-one, invariant submanifolds of M and W. Along the way, we develop an equivariant sucking principle for orthogonal actions of finite groups on Euclidean space.

Keywords   Manifold approximate fibration · Equivariant sucking · Wrapping up

Mathematics Subject Classification (2000)   57N15 · 57S30

1 Introduction

We begin with a classical theorem, attributed to T.A. Chapman [4], on sucking and wrapping up manifolds over the real line. Recall that the infinite cyclic group C_∞ acts on the real line $\mathbb{R}$ by integer translations.

Theorem (Chapman) *Let W be a connected topological manifold of dimension > 4. The following statements are equivalent:*

1. *The space W is finitely dominated, and there exists a cocompact, free, discontinuous C_∞-action on W.*

Dedicated to Bruce Williams on the occasion of his 60th birthday.

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2. *There exists a proper bounded fibration $W \to \mathbb{R}$.*
3. *There exists a manifold approximate fibration $W \to \mathbb{R}$.*
4. *There exist a C_∞-action on W and C_∞-manifold approximate fibration $W \to \mathbb{R}$.*
5. *There exists a manifold approximate fibration $M \to S^1$ with $\overline{M}$ homeomorphic to W.*

This theorem can be viewed as an answer to the question:

When does a finitely dominated manifold W admit a cocompact, free, discontinuous action of the infinite cyclic group C_∞?

There are two essentially different answers. The first, spelled out by conditions (2) and (3), is that W admits a proper map to $\mathbb{R}$ with bounded or controlled versions of the homotopy lifting property (called a bounded or approximate fibration). The equivalence of the bounded and controlled versions is the main advance of Chapman's paper [4] (which was preceded by the Hilbert cube manifold case in [3]), and it is part of the phenomenon called *sucking*.

The second answer, formulated in conditions (4) and (5), is that the approximate fibration $W \to \mathbb{R}$ can be made equivariant with respect to some C_∞-action on W, or that $W \to \mathbb{R}$ can be *wrapped-up* to an approximate fibration $M \to S^1$. Chapman's wrapping-up construction is a variation of Siebenmann's twist-gluing construction (where the twist is the identity) used by him [28] in his formulation of Farrell's fibering theorem [12] (for more details see [16]). From this point of view, the question that is being raised is:

Given a discrete subgroup Γ of isometries on $\mathbb{R}$ and a manifold approximate fibration $W \to \mathbb{R}$, can the Γ-action on $\mathbb{R}$ be "approximately lifted" to a free, discontinuous Γ-action on W, so that there is a Γ-manifold approximate fibration $W \to \mathbb{R}$?

Chapman proves that this can always be done when $\Gamma = C_\infty$ and dim $W > 4$.

Since our formulation of Chapman's theorem does not appear explicitly in [4], we include a proof, which is actually a guide to finding the proof in the literature. Most aspects of this theorem were discovered independently by Ferry [13]. An analysis of many of the details appears in [16].

Proof The implication (1) $\implies$ (2) follows from Proposition 7.1 and then [16, Proposition 17.14]. The implication (2) $\implies$ (3) is the case $n = 1$ in Corollary 3.35, which is the ε-δ version of Chapman's sucking principle. The implication (3) $\implies$ (2) is clear from the definitions. The implication (3) $\implies$ (4) is [16, Lemma 17.8], which uses Hughes's approximate isotopy covering principle instead of an argument of Chapman (cf. [16, Remark 17.9]). The implication (4) $\implies$ (5) follows by taking the quotient of $W \to \mathbb{R}$ by the free C_∞-action. The implication (5) $\implies$ (3) follows from taking the infinite cyclic cover of $M \to S^1$. Finally, since W has the proper homotopy type of a locally finite simplicial complex [24, Essay III, Theorem 4.1.3], the implication (4) $\implies$ (1) follows from the elementary argument of [16, Proposition 17.12]. $\square$

The theme of this paper is to extend Chapman's results from the smallest infinite discrete group of isometries on $\mathbb{R}$, namely the infinite cyclic group C_∞, to the largest discrete group of isometries on $\mathbb{R}$, namely the infinite dihedral group D_∞. In particular, our main result can be viewed as an answer to the question:

When does a finitely dominated manifold W admit a cocompact, free, discontinuous action of the infinite dihedral group D_∞?

The main technical issues that arise involve the fact that D_∞ has torsion; it has a non-trivial finite subgroup, namely the cyclic group C_2 of order two, which acts by reflection on $\mathbb{R}$ fixing the origin. The lifted D_∞-action on W contains a C_2-action on W with a nonempty invariant subset. For notation, for any $N \geq 1$, consider the dihedral groups

$$D_\infty := C_\infty \rtimes_{-1} C_2$$
$$D_N := C_N \rtimes_{-1} C_2.$$

In particular, $D_1 = C_2$. There are short exact sequences

$$0 \longrightarrow C_\infty \longrightarrow D_\infty \longrightarrow C_2 \longrightarrow 0$$
$$0 \longrightarrow N \cdot C_\infty \longrightarrow D_\infty \longrightarrow D_N \longrightarrow 0.$$

Fix presentations:

$$C_\infty = \langle T \mid \rangle$$
$$C_2 = \langle R \mid R^2 = 1 \rangle$$
$$D_\infty = \langle R, T \mid R^2 = 1, RT = T^{-1}R \rangle.$$

The D_∞-action on $\mathbb{R}$ by isometries is given by $R(x) = -x$ and $T(x) = x + 1$.

Our main theorem, which we now state, contains analogues of all parts of Chapman's theorem.

Theorem 1.1 (Main Theorem) *Let W be a connected topological manifold of dimension >4. The following statements are equivalent:*

1. *The space W is finitely dominated, and there exists a cocompact, free, discontinuous D_∞-action on W.*
2. *There exist a free C_2-action on W and proper C_2-bounded fibration $W \to \mathbb{R}$.*
3. *There exist a free C_2-action on W and C_2-manifold approximate fibration $W \to \mathbb{R}$.*
4. *There exist a free D_∞-action on W and D_∞-manifold approximate fibration $W \to \mathbb{R}$.*
5. *For every $N \geq 1$, there exist a free D_N-action on a manifold M and D_N-manifold approximate fibration $M \to S^1$ with induced infinite cyclic cover $\overline{M}$ homeomorphic to W.*

The prototypical example of a manifold W satisfying the conditions of Theorem 1.1 is the product $W = S^n \times \mathbb{R}$, where the D_∞-action on W is given by $T(x, t) = (x, t + 1)$ and $R(x, t) = (-x, -t)$.

Here is a guide to finding the proof of the theorem in the rest of the paper.

Proof The implication (1) $\Longrightarrow$ (2) follows from Proposition 7.1 and then Theorem 7.6. The implication (2) $\Longrightarrow$ (3) is the special case of $G = C_2$ and $n = 1$ in Corollary 6.2, which requires freeness. The implication (3) $\Longrightarrow$ (2) is clear from the definitions. The implication (3) $\Longrightarrow$ (4) is Theorem 8.1. The implication (4) $\Longrightarrow$ (5) follows by taking the quotient by the action of the subgroup $N \cdot C_\infty$, where $N \geq 1$. The implication (5) $\Longrightarrow$ (3) follows from taking the infinite cyclic cover of $M \to S^1$ for the special case $D_1 = C_2$.

Finally, consider the implication (4) $\Longrightarrow$ (1). Restrict the D_∞-action on W and the D_∞-manifold approximate fibration $p : W \to \mathbb{R}$ to the finite-index subgroup C_∞. Therefore, by the implication (4) $\Longrightarrow$ (1) in Theorem 1, we obtain that W is finitely dominated and the C_∞-action (hence the D_∞-action) on W is cocompact, free, and discontinuous. $\qquad\square$

Most of the work in this paper applies to settings more general than the real line in the main theorem. We now discuss some of the highlights.

The first is a variation on the well-known property that close maps into an ANR B are closely homotopic. In the compact case, closeness is measured uniformly by using a metric on B. In the non-compact case, open covers of B are used (see Sect. 3.2). In this paper we are confronted with metrically close maps into a non-compact ANR (B, d) which we want to conclude are metrically closely homotopic. We isolate a condition, called *finite isometry type*, that allows us to do this. Here is the result.

Theorem 1.2 *Suppose (B, d) is a triangulated metric space with finite isometry type. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if X is any space and $f, g\colon X \to B$ are two close δ-close maps, then f and g are ε-homotopic rel*

$$X_{(f=g)} := \{x \in X \mid f(x) = g(x)\}.$$

See Sect. 3.1 for the definition of finite isometry type and Sect. 3.2 for a proof of the theorem (Proposition 3.16).

Likewise, we need a metric version of Chapman's manifold approximate fibration sucking theorem, which has an open cover formulation in Chapman's work. We again find the finite isometry type condition suitable for our purposes.

Theorem 1.3 (Metric MAF Sucking) *Suppose (B, d) is a triangulated metric space with finite isometry type, and let $m > 4$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if M is an m-dimensional manifold and $p\colon M \to B$ is a proper δ-fibration, then p is ε-homotopic to a manifold approximate fibration $p'\colon M \to B$.*

The proof of Theorem 1.3 is found in Sect. 3.3 (use Corollary 3.32, Proposition 3.16).

The following result is an equivariant version of Theorem 1.3, valid for many finite subgroups of $O(n)$. We will need it only for the simplest non-trivial case of $C_2 \leq O(1)$ when we perform dihedral wrapping up. However, we expect future applications in the theory of locally linear actions of finite groups on manifolds.

Theorem 1.4 (Orthogonal Sucking) *Suppose G is a finite subgroup of $O(n)$ acting freely on S^{n-1}, and let $m > 4$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if M is a free G-manifold of dimension m and $p\colon M \to \mathbb{R}^n$ is a proper G-δ-fibration, then p is G-ε-homotopic to a G-manifold approximate fibration $p'\colon M \to \mathbb{R}^n$.*

The proof of Theorem 1.4 is located in Sect. 6.

Let us consider an example that illustrates how one may enter into the situation of Theorem 1.1. Consider an equivariant version of Farrell's celebrated fibering theorem [11,12], which solves the problem of when a map $f\colon M \to S^1$ from a high-dimensional manifold M to the circle S^1 is homotopic to a fiber bundle projection. One usually inputs the necessary condition that $\overline{M}$ be finitely dominated. The first non-trivial equivariant version of this problem concerns the action of C_2 on S^1 by complex conjugation and a free action of C_2 on M. The prototypical example of such a C_2-fiber bundle is the projection $p\colon S^n \times S^1 \to S^1$, where the free C_2-action on $S^n \times S^1$ is given by $R(x, z) = (-x, \bar{z})$. Observe that the orbit space is a connected sum of projective spaces, $(S^n \times S^1)/C_2 = \mathbb{R}P^{n+1} \# \mathbb{R}P^{n+1}$. With this action of C_2 on $M = S^n \times S^1$, the infinite cyclic cover is $\overline{p}\colon \overline{M} = S^n \times \mathbb{R} \to \mathbb{R}$ with the D_∞-action described above. In the language of Sect. 7, (M, f, R) is a C_2-manifold band. It is shown in Theorem 7.6 that $\overline{f}\colon \overline{M} \to \mathbb{R}$ is a C_2-bounded fibration. Thus, Theorem 1.1 is a preliminary step in constructing a C_2-relaxation of a C_2-manifold band, following Siebenmann's [28] approach in the non-equivariant case. We expect to return to the C_2-fibering problem in a future paper.

2 Preliminary notions

This section contains most of the standard terminology that we will use throughout the paper. In addition, there are statements of two known principles that we will need: relative sucking and approximate isotopy covering.

By *manifold* we mean a metrizable topological manifold without boundary. An action of a discrete group Γ on a topological space X is *discontinuous* if, for all $x \in X$, there exists a neighborhood U of x in X such that the set $\{g \in \Gamma \mid U \cap g(U) = \varnothing\}$ is finite. The action is *cocompact* if X/Γ is compact. In this paper, every *ANR* (absolute neighborhood retract) is understood to be locally compact, separable, and metrizable.

We denote the *natural numbers* by $\mathbb{N} := \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$, and the *unit interval* by $I := [0, 1]$. For each $k \in \mathbb{N}$ and $r > 0$, denote the *open* and *closed cubes*:

$$\mathring{\mathrm{B}}_r^k := (-r, r)^k \subseteq \mathrm{B}_r^k := [-r, r]^k \subseteq \mathbb{R}^k.$$

We denote the various *open cones* on a topological space X as follows:

$$\mathring{\mathrm{c}}(X) := (X \times [0, \infty))/(X \times \{0\})$$
$$\mathrm{c}_r(X) := (X \times [0, r])/(X \times \{0\})$$
$$\mathring{\mathrm{c}}_r(X) := (X \times [0, r))/(X \times \{0\}).$$

If (Y, d) is a metric space, $\varepsilon > 0$, and $H: X \times I \to Y$ is a homotopy, then H is an *ε-homotopy* if the diameter of $H(\{x\} \times I)$ is less than ε for all $x \in X$. The homotopy H is a *bounded homotopy* if it is an ε-homotopy for some $\varepsilon > 0$. Given two maps $u, v : W \to Y$, we say that *u is ε-close to v* if $d(u(w), v(w)) < \varepsilon$ for all $w \in W$. Furthermore, given $\mu > 0$ and $A \subseteq Y$, we say that *u is (ε, μ)-close to v with respect to A* if u is ε-close to v and $u|v^{-1}(A)$ is μ-close to $v|v^{-1}(A)$.

2.1 Lifting problems

The notion of approximate fibration is due to Coram–Duvall [8]. A convenient source for the following definitions is [16, §16].

Definition 2.1 Let (Y, d) be a metric space, and let $p: X \to Y$ be a map.

1. Let $\varepsilon > 0$. The map p is an *ε-fibration* if, for every space Z and commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\;f\;} & X \\
{\scriptstyle \times 0}\Big\downarrow & \nearrow{\scriptstyle \tilde{F}} & \Big\downarrow{\scriptstyle p} \\
Z \times I & \xrightarrow{\;F\;} & Y
\end{array}
$$

   there exists an *ε-solution*. That is, there exists a homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(-, 0) = f$ and $p\tilde{F}$ is ε-close to F. The above diagram is a **homotopy lifting problem**.
2. Let $\varepsilon > 0$ and $A \subseteq Y$. The map p is an *ε-fibration over A* if every homotopy lifting problem with $F(Z \times I) \subseteq A$ has an *ε-solution*. In particular, the map p is a ε-fibration if p is a ε-fibration over Y.
3. The map p is a **bounded fibration** if p is an ε-fibration for some $\varepsilon > 0$. The map p is a **manifold approximate fibration (MAF)** if X and Y are topological manifolds and p is a proper ε-fibration for every $\varepsilon > 0$.

Remark 2.2 By an ε-*fibration*, Chapman [4] means a proper map between ANRs, that is, an ε-fibration for the class of locally compact, separable, metric spaces. This restriction on the class of spaces is of no consequence in the present paper as can be seen in either of two ways. First, the reader will note that the only property we use of ε-fibrations is the ε-homotopy lifting property for compact, metric spaces. Second, the techniques of Coram–Duvall [6] can be adapted to show: for a proper map $E \to B$ between ANRs, the δ-homotopy lifting property for compact, metric spaces implies the ε-homotopy lifting property for all spaces (where $\delta > 0$ depends only on $\varepsilon > 0$ and the metric on B).

2.2 Equivariant lifting problems

The notion of (ε, ν)-fibration is due to Hughes [19]. See Prassidis [26] for a more complete treatment.

Let G be a group. A G-*space* is a topological space with a left G-action, and a G-*map* between G-spaces is a continuous, equivariant map. Throughout the paper, we assume that the unit interval I is a trivial G-space.

Definition 2.3 Let (Y, d) be a metric space, and let $p \colon X \to Y$ be a G-map.

1. Let $\varepsilon > 0$. The map p is a G-ε-**fibration** if, for every G-space Z and commutative diagram of G-maps:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & X \\
{\scriptstyle \times 0}\downarrow & {\scriptstyle \tilde{F}}\;\nearrow & \downarrow{\scriptstyle p} \\
Z \times I & \xrightarrow{\ F\ } & Y
\end{array}
$$

   there exists a G-ε-**solution**. That is, there exists a G-homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(-, 0) = f$ and $p\tilde{F}$ is ε-close to F. The diagram above is called a G-**homotopy lifting problem**.
2. Let $\varepsilon > 0$ and $A \subseteq Y$. The map p is a G-ε-**fibration over** A if every G-homotopy lifting problem with $F(Z \times I) \subseteq A$ has a G-ε-**solution**. In particular, the map p is a G-ε-**fibration** if p is a G-ε-fibration over Y.
3. The map p is a G-**bounded fibration** if p is a G-ε-fibration for some $\varepsilon > 0$. The map p is a G-**manifold approximate fibration** (G-**MAF**) if X and Y are topological manifolds and p is a proper G-ε-fibration for every $\varepsilon > 0$.
4. Let $\varepsilon, \mu > 0$ and $A \subseteq B \subseteq Y$. The map p is a G-(ε, μ)-**fibration over** (B, A) if every G-homotopy lifting problem with $F(Z \times I) \subseteq B$ has a G-(ε, μ)-**solution over** (B, A). That is, there exists a G-homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(-, 0) = f$ and $p\tilde{F}$ is (ε, μ)-close to F with respect to A.

2.3 Stratified lifting problems

See Hughes [14] for the definitions here.

Definition 2.4 Let X be a topological space.

1. A **stratification of** X is a locally finite partition $\{X_i\}_{i \in \mathcal{I}}$ for some set $\mathcal{I}$ such that each X_i, called the i-**stratum**, is a locally closed subspace of X. We call X a **stratified space**.
2. If X is a stratified space, then a map $f : Z \times A \to X$ is **stratum-preserving along** A if, for each $z \in Z$, the image $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a **stratum-preserving homotopy** if f is stratum-preserving along I.

Definition 2.5 Let (Y, d) be a metric space, and let $p : X \to Y$ be a map. Suppose X and Y are spaces with stratifications $\{X_i\}_{i \in \mathcal{I}}$ and $\{Y_j\}_{j \in \mathcal{J}}$.

1. The map p is a **stratified fibration** if, for every commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & X \\
{\scriptstyle \times 0} \big\downarrow & {\scriptstyle \tilde{F}} \nearrow & \big\downarrow {\scriptstyle p} \\
Z \times I & \xrightarrow{\ F\ } & Y
\end{array}
$$

   with F a stratum-preserving homotopy, there exists a **stratified solution**. That is, there exists a stratum-preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(-, 0) = f$ and $p\tilde{F} = F$. The diagram above is called a **stratified homotopy lifting problem**.

2. Let $\varepsilon > 0$. The map p is a **stratified ε-fibration** if, for every stratified homotopy lifting problem, there exists a **stratified ε-solution**. That is, there exists a stratum-preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(-, 0) = f$ and $p\tilde{F}$ is ε-close to F.

2.4 Relative sucking

In Sect. 8 we will need the following relative version of Chapman's MAF Sucking Theorem [4, Theorem 1]. We include a proof that shows it is a formal consequence of Chapman's work.

Proposition 2.6 *Suppose $a_1 < a_3 < b_3 < b_1$ are real numbers. For every $m > 4$ there exists $\varepsilon > 0$ such that: if W is an m-dimensional manifold, $p : W \to \mathbb{R}$ is a proper ε-fibration, and p is an approximate fibration over $(-\infty, a_3) \cup (b_3, \infty)$, then there exist a manifold approximate fibration $p' : W \to \mathbb{R}$ and a homotopy $p \simeq p'$ rel $p^{-1}((-\infty, a_1] \cup [b_1, \infty))$.*

Proof Let $m > 4$ be given. Let $\mathcal{U}$ be an open cover of (a_1, b_1) such that: if $p : X \to \mathbb{R}$ is a map and $p|p^{-1}(a_1, b_1)$ is $\mathcal{U}$-homotopic to a map $q : p^{-1}(a_1, b_1) \to (a_1, b_1)$ via a homotopy H, then H extends to a homotopy $\tilde{H} : p \simeq \tilde{q}$ rel $p^{-1}((-\infty, a_1] \cup [b_1, \infty))$. For example, this property is satisfied by the open cover

$$
\mathcal{U} := \{(x - r_x, x + r_x) | x \in (a_1, b_1)\} \text{ where } r_x := \tfrac{1}{2}\min\{x - a_1, b_1 - x\}.
$$

By Chapman's MAF Sucking Theorem [4, Theorem 1] (see Theorem 3.18 below), there exists an open cover $\mathcal{V}$ of (a_1, b_1) such that: if M is an m-dimensional manifold and $p : M \to (a_1, b_1)$ is a proper $\mathcal{V}$-fibration, then p is $\mathcal{U}$-homotopic to a MAF.

Next, select $a_2 \in (a_1, a_3)$ and $b_2 \in (b_3, b_1)$. Consider an open cover $\mathcal{O}$ of (a_1, b_1):

$$
\mathcal{O} := \{(a_1, (a_2 + a_3)/2), (a_2, b_2), ((b_3 + b_2)/2, b_1)\}.
$$

By a local-global result of Chapman [3, Proposition 2.2] [4, Proposition 2.2], there exists an open cover $\mathcal{W}$ of (a_1, b_1) such that any proper map of an ANR to (a_1, b_1) restricting to a $\mathcal{W}$-fibration over the closure of each member of $\mathcal{O}$ is a $\mathcal{V}$-fibration. Finally, choose $\varepsilon > 0$ such that the ε-ball about any point of $[a_2, b_2]$ is contained in some member of $\mathcal{W}$. One checks that ε satisfies the desired conditions. $\qquad\square$

2.5 Approximate isotopy covering

We begin with the 1-simplex version of the Approximation Theorem [20, Theorem 14.1].

Theorem 2.7 (Hughes) *Let* $p : M \times I \to B \times I$ *be a fiber-preserving MAF such that* $\dim M > 4$. *For every open cover* δ *of* B, *there exists a fiber-preserving homeomorphism* $H : M \times I \to M \times I$ *such that* $H_0 = \mathrm{id}_M$ *and* pH *is* δ-*close to* $p_0 \times \mathrm{id}_I$. ☐

In [18, Theorem 7.1], the proof of the above theorem is given only for $\varepsilon > 0$ and closed manifolds B that admit a handlebody decomposition (e.g. a compact, piecewise linear manifold without boundary). However, as observed in [20, §14], that proof (by induction on the index of handles) adapts to give a proof for any connected manifold B that admits a handlebody decomposition. Moreover, the arguments in [20, §13] can be adapted to prove the general case of B any topological manifold. In this paper, we only require the case of $B = \mathbb{R}$.

A well-known consequence of Hughes's Approximation Theorem is the Approximate Isotopy Covering Theorem for MAFs (see [21, §6] [16, Theorem 17.4]). The following metric version is an immediate consequence of Theorem 2.7.

Corollary 2.8 (Approximate Isotopy Covering) *Let* $p : M \to B$ *be a MAF with* $\dim M > 4$. *Suppose* $g : B \times I \to B$ *is a isotopy from the identity* id_B. *For every* $\varepsilon > 0$, *there exists an isotopy* $G : M \times I \to M$ *from the identity* id_M *such that* pG_s *is* ε-*close to* $g_s p$ *for all* $s \in I$.

Proof Define $g' : B \times I \to B \times I$ by $g'(x, s) = (g_s(x), s)$. Then $g'(p \times \mathrm{id}_I) : M \times I \to B \times I$ is a fiber-preserving MAF. Given $\varepsilon > 0$, by Theorem 2.7, there exists a fiber-preserving homeomorphism $H : M \times I \to M \times I$ such that $H_0 = \mathrm{id}_M$ and $g'(p \times \mathrm{id}_I)H$ is ε-close to $g'(p \times \mathrm{id}_I)_0 \times \mathrm{id}_I = p \times \mathrm{id}_I$. It follows that $G : M \times I \to M$ defined by $G(x, s) = H_s^{-1}(x)$ is the desired isotopy. ☐

3 Finite isometry type and metric sucking

In this section we introduce a condition on a triangulated metric space, *finite isometry type*, which essentially says that the local geometry of the space has finite variation. This special condition allows us to phrase two controlled-topological results about non-compact triangulated spaces in terms of the metric rather than open covers. The two results concern close maps into an ANR and Chapman's MAF Sucking Theorem.

3.1 Finite isometry type

We begin by introducing terminology for the metric conditions to appear in the results of Sects. 3.2 and 3.3. This will allow for cleaner statements of our results on equivariant sucking in Sect. 6.

The following notion of shapes is used by Bridson–Haefliger [2, Chapter I.7] in a slightly more restrictive context.

Definition 3.1 A **simplicial isometry** between two metrized simplices is an isometry that takes each face onto a face. If (B, d) is a triangulated metric space, the **shapes of** B, denoted $\mathrm{Shapes}^{\triangle}(B)$, is the set of simplicial isometry classes of simplices of B.

Definition 3.2 If v is a vertex of a simplicial complex K, then K_v denotes the **closed star of** v **in** K. That is, K_v is the union of all simplices of K that contain v, triangulated by the

simplicial complex consisting of all simplices of K that are faces of simplices having v as a vertex.

We often abuse notation by identifying a simplicial complex with its underlying polyhedron.

Definition 3.3 Suppose (B, d) is a metric space with a triangulation $\phi : K \to B$ (i.e., K is a simplicial complex and ϕ is a homeomorphism). For vertices $v, w \in K$, the closed stars $\phi(K_v)$ and $\phi(K_w)$ in B are **simplicially isometric** if there exists an isometry $h : \phi(K_v) \to \phi(K_w)$ such that $\phi^{-1}h\phi : K_v \to K_w$ is a simplicial isomorphism. The **shapes of closed stars of** B, denoted $\text{Shapes}^{\boxtimes}(B)$, is the set of simplicial isometry classes of closed stars of B. More generally, if A is a closed sub-polyhedron of B, then $\text{Shapes}^{\boxtimes}(B, A)$ denotes the set of simplicial isometry classes of closed stars $\phi(K_v)$ of B such that $\phi(v) \in A$.

Remark 3.4 If $\phi : K \to B$ is a triangulation of (B, d) with respect to which $\text{Shapes}^{\boxtimes}(B)$ is finite, then $\text{Shapes}^{\triangle}(B)$ is also finite. Moreover, if K' is the first barycentric subdivision of K, then $\text{Shapes}^{\boxtimes}(B)$ is finite with respect to the induced triangulation $\phi : K' \to B$.

Definition 3.5 A complete metric space (B, d) with a triangulation $\phi : K \to B$ has **finite isometry type** if

1. K is a locally finite, simplicial complex,
2. there exists $d_0 > 0$ such that: if $v \neq w$ are vertices of B then $d(v, w) \geq d_0$,
3. for every $\alpha > 0$ and for every $n \geq 0$, there exists $\beta > 0$ such that: if x, y lie in distinct n-simplices of B and $d(x, y) < \beta$, then x and y are in the α-neighborhood of the $(n-1)$-skeleton of B, and
4. $\text{Shapes}^{\boxtimes}(B)$ is finite.

Remark 3.6 A finite triangulation has finite isometry type.

Remark 3.7 Let (R, e) be a metric space, and let $p : R \to M$ be a regular cover with M compact. Suppose the group D of covering transformations acts by isometries on (R, e). Observe that $p : (R, e) \to (M, e/D)$ is distance non-increasing and a local isometry. Also, p is regular implies that D acts transitively on the fibers. Therefore, if $\mathcal{S}$ is a finite triangulation of M, then the induced D-equivariant triangulation $p^*(\mathcal{S})$ of R has finite isometry type.

This finiteness condition arises in the following rigid situations, not pursued here.

Example 3.8 Let (B, d) be a triangulated, geodesic metric space such that each simplex of B is a geodesic subspace. It follows that the metric on each closed star in B is completely determined by the metric on the simplices in the star. Thus, if $\text{Shapes}^{\triangle}(B)$ is finite, then so is $\text{Shapes}^{\boxtimes}(B)$. It is also clear that for such B, the condition $\text{Shapes}^{\triangle}(B)$ is finite implies conditions (2) and (3) of Definition 3.5. In summary, if (B, d) is a complete, geodesic metric space triangulated by a locally finite simplicial complex, each simplex is a geodesic subspace, and $\text{Shapes}^{\triangle}(B)$ is finite, then (B, d) has finite isometry type.

Example 3.9 Let K be a connected M_κ-simplicial complex in the sense of Bridson–Haefliger [2]. If $\text{Shapes}^{\triangle}(K)$ is finite, then it follows from [2, Theorem 7.19, page 105] that (K, d) is a complete geodesic space, where d is the intrinsic metric. Thus, by applying Example 3.8, it follows that (K, d) has finite isometry type.

We show that split crystallographic groups induce triangulations of Euclidean space $(\mathbb{R}^n, e)$ with the standard metric with respect to which $(\mathbb{R}^n, e)$ has finite isometry type.

Definition 3.10 An action of a group Γ on a topological space R is **virtually free** if there is a finite-index subgroup $\Delta \leq \Gamma$ whose restricted action on R is free.

Proposition 3.11 *Let (R, e) be a metric space such that R is a smooth manifold. Let Γ be a discrete, cocompact subgroup of smooth isometries of (R, e). If the action of Γ on R is virtually free, then R admits a smooth, Γ-equivariant triangulation $\mathcal{T}$ of finite isometry type, which is unique up to Γ-combinatorial equivalence.*

Proof Since Γ has a virtually free action on R, there exists a finite-index subgroup Δ of Γ with a free action on R. Define a finite-index, normal subgroup Δ_0 of Γ by

$$\Delta_0 := \bigcap_{g \in \Gamma} g \Delta g^{-1}.$$

Then there is an exact sequence $1 \to \Delta_0 \to \Gamma \to G \to 1$, where $G := \Gamma / \Delta_0$ is a finite group. Since Δ_0 has a cocompact, free action on R, the quotient $M := R / \Delta_0$ is a compact smooth G-manifold. So, by a theorem of Illman [22], there exists a smooth, G-equivariant triangulation $\mathcal{S}$ of M which is unique up to G-combinatorial equivalence. Since $\Delta_0 \to R \to M$ is the sequence of a regular covering map, there is a unique, smooth, Γ-equivariant triangulation $\mathcal{T}$ of R covering $\mathcal{S}$. Therefore, by Remark 3.7, since $\mathcal{S}$ is finite, we conclude that the pullback $\mathcal{T}$ has finite isometry type. $\square$

Example 3.12 For any crystallographic group Γ of rank n, the Euclidean space $(\mathbb{R}^n, e)$, where e is the standard metric, admits a Γ-equivariant triangulation $\mathcal{T}$ of finite isometry type. Indeed, the action is virtually free, since there is an exact sequence $1 \to \mathbb{Z}^n \to \Gamma \to G \to 1$, where the subgroup $\mathbb{Z}^n$ acts freely on $\mathbb{R}^n$ by translations, and the finite group G is called the *point group* (see [10]). Suppose this short exact sequence splits, in which case we call Γ a **split** crystallographic group. Then G is a subgroup of the orthogonal group $O(n)$ and the triangulation $\mathcal{T}/G$ of $(\mathbb{R}^n/G, e/G)$ has finite isometry type.

More generally, we have an important geometric observation used in Sect. 6.

Proposition 3.13 *Let G be any finite subgroup of $O(n)$. Then $\mathbb{R}^n/G$ has finite isometry type.*

Proof Consider the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with the spherical metric s and induced isometric G-action. For any point $a \in S^{n-1}$, recall its *Dirichlet domain* $D(a)$ is the open neighborhood

$$D(a) := \{x \in S^{n-1} \mid \forall g \in G \ : \ a \neq ga \implies d(x, a) < d(x, ga)\}.$$

Then, by [27, Theorem 6.7.1], the closure $\overline{D(a)}$ is a convex polyhedron and a fundamental domain for the G-action on (S^{n-1}, s). Select a geodesic triangulation of $\overline{D(a)}$. This extends to a geodesic G-triangulation of S^{n-1}. That is, we obtain a G-homeomorphism $\phi : K \to S^{n-1}$, from a G-simplicial complex K, such that the image $\phi(\sigma)$ of each simplex σ of K is totally geodesic in (S^{n-1}, s). Upon replacing K with a (necessarily G-equivariant) barycentric subdivision, we may assume that $s(\phi v, \phi v') < 1$ for all vertices v, v' that share a simplex σ of K. Now define $\psi : K \to \mathbb{R}^n$ as the unique continuous G-map such that $\psi(v) := \phi(v)$ for each vertex v of K and that the image $\psi(\sigma)$ of each simplex σ of K is the convex hull in $(\mathbb{R}^n, e)$. It is clear that $0 \notin \psi(K)$ and $\psi(\sigma) \cap \psi(\sigma') \subseteq \psi(\partial\sigma) \cap \psi(\partial\sigma')$ for all simplices σ, σ' of K. Hence ψ is injective and extends to a G-homeomorphism

$$\Psi := \overset{\circ}{c}(\psi) : \overset{\circ}{c}(K) \longrightarrow \mathbb{R}^n \ ; \ (x, t) \longmapsto t\psi(x).$$

Let $\mathcal{T}$ be (the set of simplices of) a linear triangulation of $\mathbb{R}^n$ such that $\mathcal{T}$ is invariant under permutation of coordinates of $\mathbb{R}^n$ and under the standard action of $\mathbb{Z}^n$ on $\mathbb{R}^n$. For example, $\mathcal{T}$ can be taken to be the standard triangulation defined as follows. The set of vertices of $\mathcal{T}$ is $\mathbb{Z}^n$. There is a directed edge from a vertex $x = (x_1, x_2, \ldots x_n)$ to a vertex $y = (y_1, y_2, \ldots, y_n)$ if and only if $x \neq y$ and $x_i \leq y_i \leq x_i + 1$ for each $i = 1, 2, \ldots, n$. A finite set σ of vertices spans a simplex if and only if for any two vertices in σ, there is a directed edge from one to the other. It follows from the invariance properties that $\mathcal{T}$ is a triangulation of finite isometry type. Moreover, if $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is any non-singular linear transformation, then $A(\mathcal{T}) = \{A(\sigma) \mid \sigma \in \mathcal{T}\}$ is also a triangulation of finite isometry type.

Now, for each $(n-1)$-simplex σ of K, define a non-singular linear transformation $A_\sigma \colon \mathbb{R}^n \to \mathbb{R}^n$ as follows. Order the vertices $v_1, v_2, \ldots, v_n$ of $\Psi(\sigma)$, and define $A_\sigma(e_i) := v_i$ for each $i = 1, 2, \ldots, n$, where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. The triangulation $A_\sigma(\mathcal{T})$ of $\mathbb{R}^n$ restricts to a triangulation $\mathcal{T}_\sigma$ of $\Psi(\overset{\circ}{c}(\sigma))$ that is independent of the ordering of $v_1, v_2, \ldots, v_n$. Moreover, if σ and τ are two $(n-1)$-simplices of K, then $\mathcal{T}_\sigma$ and $\mathcal{T}_\tau$ agree on $\Psi(\overset{\circ}{c}(\sigma \cap \tau))$. It follow that $\mathcal{T}_\Psi := \bigcup_\sigma \mathcal{T}_\sigma$ is a G-equivariant triangulation of $\mathbb{R}^n$ of finite isometry type and induces a triangulation of $\mathbb{R}^n / G$ of finite isometry type. □

3.2 Metrically close maps into ANRs

For notation if $f, g \colon X \to Y$ are maps, then $X_{(f=g)} = \{x \in X \mid f(x) = g(x)\}$. Recall the following classic property of ANRs (cf. [25, p. 39]).

Proposition 3.14 *Let B be an ANR. For every open cover $\mathcal{U}$ of B there exists an open cover $\mathcal{V}$ of B such that: if X is a space and $f, g \colon X \to B$ are $\mathcal{V}$-close maps, then f is $\mathcal{U}$-homotopic to g rel $X_{(f=g)}$.*

We will need a metric version of Proposition 3.14 that is only valid for certain triangulated metric spaces B. For the proof of the metric version, we will need the following relative version of Proposition 3.14 in the compact case. It is a fairly routine application of Proposition 3.14 and the Estimated Homotopy Extension Theorem of Chapman–Ferry [5], but we include a proof for completeness.

Lemma 3.15 *Let (Y, d) be a compact, metric ANR. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if X is a space with a closed subspace X_0 and $f, g \colon X \to Y$ are δ-close maps for which there is a δ-homotopy $H \colon f|X_0 \simeq g|X_0$, then there exists an ε-homotopy $\widetilde{H} \colon f \simeq g$ such that $\widetilde{H}|X_0 \times I = H$.*

Proof Let $\varepsilon > 0$ be given. By Proposition 3.14 there exists $\mu > 0$ such that $\mu < \varepsilon$ and if X is any space, $f, g \colon X \to Y$ are μ-close maps, then there exists an $\varepsilon/2$-homotopy $f \simeq g$ rel $X_{(f=g)}$. Choose $\delta > 0$ such that $\delta < \mu/2$. Now suppose X is a space with a closed subspace X_0 and $f, g \colon X \to Y$ are δ-close maps for which there is a δ-homotopy $H \colon f|X_0 \simeq g|X_0$. By the Estimated Homotopy Extension Theorem [5, Proposition 2.1] there exists a map $\hat{g} \colon X \to Y$ and a δ-homotopy $\hat{H} \colon f \simeq \hat{g}$ such that $\hat{H}|(X_0 \times I) = H$. Thus, $\hat{g} = \hat{H}_1$ is δ-close to f, which in turn is δ-close to g. Thus, $\hat{g}$ is μ-close to g. By the choice of μ there is an $\varepsilon/2$-homotopy $G \colon \hat{g} \simeq g$ rel $X_{(\hat{g}=g)}$. Note $X_0 \subseteq X_{(\hat{g}=g)}$.

The concatenation

$$\hat{H} * G \colon X \times [0, 2] \longrightarrow Y; \quad (x, t) \longmapsto \begin{cases} \hat{H}(x, t) & \text{if } 0 \leq t \leq 1 \\ G(x, t-1) & \text{if } 1 \leq t \leq 2 \end{cases}$$

is an ε-homotopy, which can be re-parameterized to give the homotopy $\widetilde{H}$ as follows. Let $u \colon X \to [0, 1]$ be a map such that $u^{-1}(0) = X_0$. Define

$$q \colon X \times [0, 2] \longrightarrow X \times [0, 1]; \quad (x, t) \longmapsto \begin{cases} \left(x, t\left(1 - \frac{1}{2}u(x)\right)\right) & \text{if } 0 \le t \le 1 \\ \left(x, \frac{1}{2}u(x)t + 1 - u(x)\right) & \text{if } 1 \le t \le 2. \end{cases}$$

Then q is a quotient map with the property that each interval $\{x\} \times [0, 1]$ is taken linearly onto $\{x\} \times [0, 1 - \frac{1}{2}u(x)]$ and each interval $\{x\} \times [1, 2]$ is taken linearly onto $\{x\} \times [1 - \frac{1}{2}u(x), 1]$. The only non-degenerate point inverses of q are for $(x, 1)$ with $x \in X_0$, in which case $q^{-1}(x, 1) = \{x\} \times [1, 2]$. It follows that $\widetilde{H} := (\hat{H} * G) \circ q^{-1}$ is the desired homotopy. $\qquad\square$

The following result is our metric version of Proposition 3.14. It is used in the proof of Lemma 6.6.

Proposition 3.16 *Suppose (B, d) is a metric space triangulated by a locally finite, finite dimensional simplicial complex and A is a closed sub-polyhedron of B such that:*

1. *There exists $d_0 > 0$ such that: if v and w are distinct vertices of A, then $d(v, w) \ge d_0$.*
2. *For every $\alpha > 0$ and for every $n \ge 0$, there exists $\beta > 0$ such that: if x and y are in distinct n-simplices of A and $d(x, y) < \beta$, then x and y are in the α-neighborhood of the $(n - 1)$-skeleton of A.*
3. *Shapes$^{\triangle}(A)$ is finite.*

For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if X is a space and $f, g \colon X \to B$ are δ-close maps such that $f = g$ over $B \backslash A$, then f is ε-homotopic to g rel $X_{(f=g)}$.

Proof The proof is by induction on dim B. If dim $B = 0$, then choose $0 < \delta < d_0$ (which is independent of ε in this case). It follows that if $f, g \colon X \to B$ are δ-close and $f = g$ over $B \backslash A$, then $f = g$.

Next assume that dim $B = n > 0$ and the proposition is true in lower dimensions.

We will now define a particular strong deformation retraction of a neighborhood of the $(n - 1)$-skeleton of A to the $(n - 1)$-skeleton. Let A^{n-1} denote the $(n - 1)$-skeleton of A and let S_n denote the set of n-simplices of A. Since Shapes$^{\triangle}(A)$ is finite we can choose a finite subset T_n of S_n such that each member of S_n is simplicially isometric to a member of T_n. For each $\tau \in T_n$, choose $b_\tau \in \tau \backslash \partial\tau$ and let $r_\tau \colon \tau \backslash \{b_\tau\} \times I \to \tau \backslash \{b_\tau\}$ be a strong deformation retraction onto $\partial\tau$. Using the finiteness of Shapes$^{\triangle}(A)$, we can extend the selection of the points b_τ to a selection of points $b_\sigma \in \sigma \backslash \partial\sigma$ for every $\sigma \in S_n$, and we can extend the strong deformation retractions r_τ to a strong deformation retraction of $B \backslash \{b_\sigma \mid \sigma \in S_n\}$ onto $(B \backslash A) \cup A^{n-1}$ so that the following is true. There is a homotopy

$$r \colon B \backslash \{b_\sigma \mid \sigma \in S_n\} \times I \to B \backslash \{b_\sigma \mid \sigma \in S_n\}$$

such that:

1. $r_0 = \text{id}$
2. $r_t | (B \backslash A) \cup A^{n-1} = \text{incl}$ for all $t \in I$
3. The image of r_1 is $(B \backslash A) \cup A^{n-1}$
4. $r_t(\sigma \backslash \{b_\sigma\}) \subseteq \sigma \backslash \{b_\sigma\}$ for all $t \in I$ and $\sigma \in S_n$

5. **(Finiteness)** For each $\sigma \in S_n$ there exists $\tau \in T_n$ and a simplicial isometry $h : \sigma \to \tau$ such that $h(b_\sigma) = b_\tau$ and the following diagram commutes for all $t \in I$:

$$
\begin{array}{ccc}
\sigma \backslash \{b_\sigma\} & \xrightarrow{\;h|\;} & \tau \backslash \{b_\tau\} \\[4pt]
{\scriptstyle r_t|} \Big\downarrow & & \Big\downarrow {\scriptstyle (r_\tau)_t} \\[4pt]
\sigma \backslash \{b_\sigma\} & \xrightarrow{\;h|\;} & \tau \backslash \{b_\tau\}
\end{array}
$$

It follows that $(*)$ for every $\gamma > 0$ there exists $\rho > 0$ such that:

1. If $\sigma \in S_n$, then $b_\sigma \notin \bar{N}_\rho(A^{n-1})$, the closed ρ-neighborhood about A^{n-1} in B.
2. If $x, y \in \bar{N}_\rho(A^{n-1})$ and $d(x, y) < \rho$, then $d(r_1(x), r_1(y)) < \gamma$.
3. For every $x \in \bar{N}_\rho(A^{n-1})$, the track $r(\{x\} \times I)$ has diameter $< \gamma$.

Let $\varepsilon > 0$ be given. Use Lemma 3.15 and the assumption that $\mathrm{Shapes}^\triangle(A)$ is finite to choose $\mu > 0$ with the following property: if Δ is any simplex of A, X is any space with a closed subspace X_0 and $f, g : X \to \Delta$ are μ-close maps for which there is a μ-homotopy $H : f|X_0 \simeq g|X_0$, then there exists an $\varepsilon/3$-homotopy $\widetilde{H} : f \simeq g$ such that $\widetilde{H}|X_0 \times I = H$.

Let B^{n-1} denote the $(n-1)$-skeleton of B and use the inductive hypothesis to choose $\delta_1 > 0$ with the following property: if X is a space and $f, g : X \to B^{n-1}$ are δ_1-close maps such that $f = g$ over $B^{n-1} \backslash A$, then f is μ-homotopic to g rel $X_{(f=g)}$. It follows that if X is a space and $f, g : X \to B$ are δ_1-close maps such that $f^{-1}(B \backslash B^{n-1}) \cup g^{-1}(B \backslash B^{n-1}) \subseteq X_{(f=g)}$, and $f = g$ over $B \backslash A$, then f is μ-homotopic to g rel $X_{(f=g)}$.

Let $\rho = \rho(\gamma)$ be given by $(*)$ above where $\gamma = \min\{\delta_1, \varepsilon/3\}$. Let $\beta > 0$ be given by hypothesis (2) in the proposition for $\alpha = \rho$. Choose $\delta > 0$ such that $\delta < \min\{\delta_1, \rho, \beta, \mu\}$.

Now suppose given a space X and δ-close maps $f, g : X \to B$ such that $f = g$ over $B \backslash A$. We must show that f is ε-homotopic to g rel $X_{(f=g)}$. For each $\sigma \in S_n$, define subspaces

$$
X^{n-1} = X_{(f=g)} \cup \left(f^{-1}(A^{n-1}) \cap g^{-1}(A^{n-1})\right) \quad Y = \bigcup_{\sigma \in S_n} X_\sigma
$$

$$
X_\sigma = f^{-1}(\sigma) \cap g^{-1}(\sigma) \quad Z = X \backslash (Y \cup X^{n-1}).
$$

If $x \in Z$, then $f(x), g(x)$ are in distinct n-simplices of A and $d(f(x), g(x)) < \delta$. The choice of $\delta < \beta$ implies that $f(x), g(x) \in \bar{N}_\rho(A^{n-1})$. Define maps

$$
f^{n-1} : X^{n-1} \cup Z \longrightarrow B; \quad f^{n-1} = \begin{cases} f & \text{on } X^{n-1} \\ r_1 f & \text{on } Z \end{cases}
$$

and

$$
g^{n-1} : X^{n-1} \cup Z \longrightarrow B; \quad g^{n-1} = \begin{cases} g & \text{on } X^{n-1} \\ r_1 g & \text{on } Z. \end{cases}
$$

The choice of ρ implies that there are $\varepsilon/3$-homotopies $E : f| \simeq f^{n-1}$ rel X^{n-1} and $F : g| \simeq g^{n-1}$ rel X^{n-1}. Moreover, the choice of ρ implies that f^{n-1} and g^{n-1} are δ_1-close. Define

$$
f' : X \longrightarrow B; \quad f' = \begin{cases} f^{n-1} & \text{on } X^{n-1} \cup Z \\ f & \text{on } Y \end{cases} = \begin{cases} f & \text{on } X^{n-1} \cup Y \\ r_1 f & \text{on } Z \end{cases}
$$

and

$$
g' : X \longrightarrow B; \quad g' = \begin{cases} g^{n-1} & \text{on } X^{n-1} \cup Z \\ g & \text{on } Y \end{cases} = \begin{cases} g & \text{on } X^{n-1} \cup Y \\ r_1 g & \text{on } Z. \end{cases}
$$

The homotopies E and F can be extended to $\varepsilon/3$-homotopies $E'\colon f \simeq f'$ rel $X^{n-1} \cup Y$ and $F'\colon g \simeq g'$ rel $X^{n-1} \cup Y$. By the inductive assumption, there exists a μ-homotopy

$$H^{n-1}\colon f^{n-1} \simeq g^{n-1} \text{ rel } X_{(f=g)}.$$

The choice of μ implies that H^{n-1} can be extended to an $\varepsilon/3$-homotopy $H\colon f' \simeq g'$. Clearly, H is rel $X_{(f=g)}$. Finally, we concatenate the three $\varepsilon/3$-homotopies E', H, and F' to get an ε-homotopy $f \simeq f' \simeq g' \simeq g$ rel $X_{(f=g)}$. $\qquad\square$

Corollary 3.17 *Suppose (B, d) is a triangulated, metric space of finite isometry type. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if X is a space and $f, g\colon X \to B$ are δ-close maps, then f is ε-homotopic to g rel $X_{(f=g)}$.*

Proof Apply Proposition 3.16, Definition 3.5, and Remark 3.4. $\qquad\square$

3.3 Metric version of Chapman's MAF sucking theorem

A fundamental result concerning approximate fibrations defined on manifolds is the following theorem of Chapman [4].

Theorem 3.18 (Chapman's MAF Sucking Theorem) *Suppose B is a locally compact, separable, metrizable, locally polyhedral space. For each integer $m > 4$ and each open cover α of B, there exists an open cover β of B such that: if M is an m-dimensional manifold and $p\colon M \to B$ is a proper β-fibration, then p is α-close to a proper approximate fibration $p'\colon M \to B$.*

It is referred to as "sucking" because it says that a map that is nearly an approximate fibration can be deformed, or sucked, into the space of approximate fibrations. Chapman had earlier proved a Hilbert cube manifold sucking theorem [3].

The purpose of this section is to establish a metric version of Chapman's result in which B is given a fixed metric and the open covers α and β of B are replaced by numbers $\varepsilon > 0$ and $\delta > 0$, respectively. See Corollary 3.32 below. In fact, we establish a relative result in Corollary 3.31, A special case of Corollary 3.31, namely Corollary 3.34, is the key result that will be applied in Lemma 6.6 in the course of proving an equivariant version of sucking in Sect. 6.

Of course, the numbers ε and δ correspond to open covers of the metric space B by balls of radius ε and δ, respectively. Thus, the metric result applies to fewer situations (because not all open covers consist of balls of fixed radius), but has a stronger conclusion than Chapman's Theorem 3.18. Such a variation is not true in general without further restrictions on the metric space B. For example, let $B = \bigsqcup_{i=1}^{\infty} S_i^1$ be the disjoint union of circles metrized so that each circle is a subspace of B and $\lim_{i\to\infty} \delta_i = 0$, where $\delta_i = \mathrm{diam}(S_i^1)$. If $M = S^k$ is a single k-sphere, where $k > 1$, then there exists no approximate fibration $M \to B$. On the other hand, if for each $i = 1, 2, \ldots$, $p_i\colon M \to B$ is a map such that $p_i(M) \subseteq S_i^1$, then p_i is a proper δ_i-fibration.

That there are metric versions of Chapman's theorem is not a new observation. It was pointed out by Hughes [19, Remark 7.4] that such a metric version holds for $B = \mathbb{R}^n$ with the standard Euclidean metric. Hughes–Prassidis [15, Footnote, p. 10] assert the metric result for "non-compact manifolds with sufficiently homogeneous metrics." Hughes–Ranicki [16, Theorem 16.9] assert and use Corollary 3.35 in the case $n = 1$. In each of these three references it is claimed that these variations can be proved by closely examining Chapman's proof. This is indeed the case; however, detailed explanations have not heretofore appeared

in the literature. Since we require a yet more general result, we provide a detailed outline of proof.

The proof of our metric result follows Chapman's papers [3] [4] as well as Hughes [17]. The heart of Chapman's proof consists of his Handle Lemmas (quoted below as Lemmas 3.21 and 3.22), which we can use without change. Chapman proves those lemmas by engulfing and torus geometry (also known as torus tricks). His methods require high dimensions. What we have written here is just a careful repackaging of that part of Chapman's proof that comes after his Handle Lemmas.

We begin by discussing a limit result (Lemma 3.20) implicit in Chapman's work [3] [4]. The proof requires the following result, the proof of which is based on Coram and Duvall [8, Proposition 1.1].

Lemma 3.19 *Suppose (B, d) is a metric ANR and U is an open subset of B. For every $\mu > 0$, for every compact metric space Z, and for every homotopy $F : Z \times I \to U$, there exists $\nu > 0$ such that the following holds: if $\varepsilon > 0$, E is an ANR, $p : E \to B$ is a proper ε-fibration over U, and $f : Z \to E$ is a map with pf ν-close to F_0, then there is a homotopy $\widetilde{F} : Z \times I \to E$ such that $\widetilde{F}_0 = f$ and $p\widetilde{F}$ is $(\varepsilon + \mu)$-close to F.*

Proof Given $\mu > 0$, a compact metric space Z, and a homotopy $F : Z \times I \to U$, choose a compact neighborhood K of $F(Z \times \{0\})$ with $K \subseteq U$. Choose $d_0 > 0$ such that the d_0-neighborhood of $F(Z \times \{0\})$ is contained in K. Let $\delta = \min\{d_0, \mu/2\}$. Choose $\nu > 0$ such that any two ν-close maps into K are δ-homotopic in U (see Proposition 3.14).

Now suppose given $\varepsilon > 0$, an ANR E, a map $p : E \to B$ that is a proper ε-fibration over U, and a map $f : Z \to E$ such that pf is ν-close to F_0. Let $: Z \times [-1, 0] \to U$ be a δ-homotopy such that $J_{-1} = f$ and $J_0 = F_0$. Thus, the image of J is contained in $K \subseteq U$. Define

$$\Phi : Z \times [-1, 1] \longrightarrow U; \quad (z, t) \longmapsto \begin{cases} J(z, t) & \text{if } -1 \le t \le 0 \\ F(z, t) & \text{if } 0 \le t \le 1. \end{cases}$$

It follows that there exists a lift $\widetilde{\Phi} : Z \times [-1, 1] \to E$ such that $\widetilde{\Phi}_0 = f$ and $p\widetilde{\Phi}$ is ε-close to Φ. Since Z is compact, there exists $q \in (0, 1)$ such that $F(\{z\} \times [0, q)])$ has diameter less than $\mu/2$ for every $z \in Z$. Finally, define

$$\widetilde{F} : Z \times I \longrightarrow E; \quad (z, t) \longmapsto \begin{cases} \widetilde{\Phi}(z, 2t/q - 1) & \text{if } 0 \le t \le q/2 \\ \widetilde{\Phi}(z, 2t - q) & \text{if } q/2 \le t \le q \\ \widetilde{\Phi}(z, t) & \text{if } q \le t \le 1 \end{cases}$$

One may check that $\widetilde{F}_0 = f$ and that $p\widetilde{F}$ is $(\varepsilon + \mu)$-close to f. $\qquad\qquad\square$

Lemma 3.20 (Limit Lemma) *Suppose E and B are ANRs, $\mathcal{U}$ is a collection of open subsets of B, $\{\varepsilon_i\}_{i=1}^{\infty}$ is a sequence of positive numbers with $\lim_{i \to \infty} \varepsilon_i = 0$, and there are proper maps $q, q_i : E \to B$ for $i = 1, 2, 3, \ldots$ with $\lim_{i \to \infty} q_i = q$ (uniformly). If q_i is an ε_i-fibration over U for each $U \in \mathcal{U}$, then q is an approximate fibration over $\cup\mathcal{U}$.*

Proof It suffices to show that given $U \in \mathcal{U}$, $q| : q^{-1}(U) \to U$ is an approximate fibration for the class of compact metric spaces. For then results of Coram and Duvall [6, Theorem 2.6], [7, Uniformization, page 43] imply that $q| : q^{-1}(\cup\mathcal{U}) \to \cup\mathcal{U}$ is an approximate fibration.

Thus, suppose given $\varepsilon > 0$, a compact metric space Z, and a homotopy lifting problem

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & q^{-1}(U) \\
{\scriptstyle \times 0}\Big\downarrow & \overset{\tilde{F}}{\nearrow} & \Big\downarrow {\scriptstyle q|} \\
Z \times I & \xrightarrow{\ F\ } & U
\end{array}
$$

For $\mu = \varepsilon/3$, let $\nu = \nu(U, \mu, Z, F)$ be given by Lemma 3.19. Now choose $i \in \mathbb{N}$ such that q_i is ν-close to q and q_i is an $\varepsilon/3$-fibration over U. It follows that $q_i f$ is ν-close to $qf = F_0$. Thus, Lemma 3.19 implies there exists $\tilde{F} \colon Z \times I \to E$ such that $q_i \tilde{F}$ is $2\varepsilon/3$-close to F. It follows that $q\tilde{F}$ is ε-close to F. $\qquad\square$

We next quote the two handle lemmas of Chapman [4, Lemma 5.1, Theorem 5.2].

Lemma 3.21 (Chapman's First Handle Lemma) *Suppose k is a positive integer and $\mathbb{R}^k \hookrightarrow B$ is an open embedding, where B is an ANR. For every $m > 4$ and $\varepsilon > 0$ there is exists a $\delta > 0$ such that: if $\mu > 0$, M is an m-manifold, and $p \colon M \to B$ is a proper map that is a δ-fibration over B_3^k, then there is a proper map $p' \colon M \to B$ such that*

1. *p' is a μ-fibration over B_1^k,*
2. *p' is ε-close to p,*
3. *$p = p'$ on $M \backslash p^{-1}(\overset{\circ}{\mathrm{B}}{}_3^k)$.* $\qquad\square$

Lemma 3.22 (Chapman's Second Handle Lemma) *Suppose k is a nonnegative integer and $\overset{\circ}{\mathrm{c}}(X) \times \mathbb{R}^k \hookrightarrow B$ is an open embedding, where B is an ANR and X is a compact ANR. For every $m > 4$ and $\varepsilon > 0$ there is exists a $\delta > 0$ such that: if $\mu > 0$ there exists $\nu > 0$ so that the following statement is true:*

if M is a m-manifold and $p \colon M \to B$ is a proper map that is a δ-fibration over $c_3(X) \times \mathrm{B}_3^k$ and a ν-fibration over $[c_3(X) \backslash \overset{\circ}{\mathrm{c}}_{1/3}(X)] \times \mathrm{B}_3^k$, then there is a proper map $p' \colon M \to B$ such that

1. *p' is a μ-fibration over $c_1(X) \times \mathrm{B}_1^k$,*
2. *p' is ε-close to p,*
3. *$p = p'$ on $M \backslash p^{-1}(\overset{\circ}{\mathrm{c}}_{2/3}(X) \times \overset{\circ}{\mathrm{B}}{}_3^k)$.* $\qquad\square$

Remark 3.23 These two handle lemmas are not an exact quote of Chapman [4, Lemma 5.1, Theorem 5.2], but the difference is insignificant. Chapman considers maps directly to $\mathbb{R}^k$ and $\overset{\circ}{\mathrm{c}}(X) \times \mathbb{R}^k$, rather than to manifolds in which these spaces are embedded. The lemmas above are formal, immediate consequences of Chapman's lemmas.

Remark 3.24 It is important to note that both of these Handle Lemmas are independent of the metric on B. That is, the various constants δ and ν depend on the metric, but their existence is independent of the metric. This is because they depend only on the metric on a compact portion of B.

Hypothesis 3.25 The following list of technical hypotheses are used later in this section.

1. Suppose B be a locally finite polyhedron and let A be a closed sub-polyhedron of dimension n.

2. Fix a locally finite triangulation of B with respect to which A is triangulated as a closed subcomplex. We will abuse notation and make no distinction between a simplicial complex and its underlying polyhedron.

3. Suppose d is a metric for B compatible with the topology on B.

4. Let $\mathcal{B}$ be the set of barycenters of simplices in A. For each $0 \leq k \leq n$, let $\mathcal{B}_k = \{b \in \mathcal{B} \mid b$ is the barycenter of a simplex of A of dimension $k\}$. For each $b \in \mathcal{B}$, let σ_b denote the simplex of A of which b is the barycenter.

5. For each $b \in \mathcal{B}$, fix an open neighborhood V_b of b in B, a compact polyhedron X_b and a homeomorphism $\phi_b \colon \overset{\circ}{c}(X_b) \times \mathbb{R}^k \to V_b$, where $b \in \mathcal{B}_k$. If $k = n$, then $X_b = \varnothing$ and $\overset{\circ}{c}(X_b)$ is a single point.

6. Assume that $V_{b_1} \cap V_{b_2} = \varnothing$ whenever $0 \leq k \leq n$ and $b_1, b_2 \in \mathcal{B}_k$.

7. For each $b \in \mathcal{B}$, let C_b denote the closed star neighborhood of b in the second barycentric subdivision of B. Thus, $A \subseteq \bigcup\{C_b \mid b \in \mathcal{B}\}$.

8. For each $0 \leq k \leq n$ and $b \in \mathcal{B}_k$, assume $C_b = \phi_b\left(c_1(X_b) \times \mathrm{B}_1^k\right)$ and that $\phi_b\left(\{v\} \times \mathbb{R}^k\right)$ is a neighborhood of b in σ_b, where v is the cone point of $\overset{\circ}{c}(X_b)$.

9. Let $V = \bigcup\{V_b \mid b \in \mathcal{B}\}$.

10. The metric d restricts to a complete metric on the closure of V.

11. For each $0 \leq k \leq n$ and $b \in \mathcal{B}_k$, let $W_b = \phi_b\left(c_{1.1}(X_b) \times \mathrm{B}_{1.1}^k\right)$.

12. Choose numbers $1.2 < r_0 < r_1 < \cdots < r_n = 1.3$ and assume that for each $0 \leq k < n$ and $b \in \mathcal{B}_k$, we have

    (a) $\phi_b\left(\left[c_3(X_b)\backslash\overset{\circ}{c}_{1/3}(X_b)\right] \times B_3^k\right) \subseteq \bigcup\{\phi_a\left(c_{r_{k+1}}(X_a) \times B_{r_{k+1}}^\ell\right) \mid k + 1 \leq \ell \leq n, a \in \mathcal{B}_\ell\}$,

    (b) $\phi_b\left(c_{2/3}(X_b) \times B_3^k\right)$ misses $\bigcup\{\phi_a\left(c_{r_k}(X_a) \times B_{r_k}^\ell\right) \mid k + 1 \leq \ell \leq n, a \in \mathcal{B}_\ell\}$.

13. **(Finiteness)** For each $0 \leq k \leq n$ and $b \in \mathcal{B}_k$, let $d_b = \phi_b^* d$ be the metric on $\overset{\circ}{c}(X_b) \times \mathbb{R}^k$ obtained by pulling back d along ϕ_b. For each $0 \leq k \leq n$, assume that $\{d_b \mid b \in \mathcal{B}_k\}$ is finite.

Remark 3.26 The Finiteness condition above says in the first place that the set $\{X_b \mid b \in \mathcal{B}\}$ of non-isomorphic polyhedra that are links of barycenters is finite. In the second place, it says that for any given polyhedron X occurring as a link and any $0 \leq k \leq n$, even though there might be infinitely many different open embeddings given of $\overset{\circ}{c}(X) \times \mathbb{R}^k$ into B, there are only finitely many different induced metrics on $\overset{\circ}{c}(X) \times \mathbb{R}^k$.

Remark 3.27 Note that given condition (1) in Hypothesis 3.25, conditions (2) through (12) may always be achieved. They are listed to fix notation. Thus, condition (13) is the only extra assumption.

The proof of the next result is based on Hughes [17, Lemma 10.1], which in turn is based on Chapman [3, Section 6].

Proposition 3.28 *Assume Hypothesis 3.25. For each integer $m > 4$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration over V_b for each $b \in \mathcal{B}$, then p is ε-close to a proper map $p' \colon M \to B$ such that $p = p'$ on $M \backslash p^{-1}(V)$ and p' is a μ-fibration over W_b for each $b \in \mathcal{B}$.*

Proof Let $\varepsilon > 0$ be given. Inductively define small positive numbers

$$\varepsilon_0, \delta_0, \varepsilon_1, \delta_1, \ldots, \varepsilon_n, \delta_n$$

with the following properties:

1. $0 < \varepsilon_0 < \varepsilon/(n+1)$,
2. $\delta_k < \delta(\varepsilon_k)$, where $\delta(\varepsilon_k)$ is given by the Handle Lemma 3.21 (if $k = n$) or 3.22 (if $k < n$) for the open embeddings $\phi_b \colon \mathbb{R}^n \hookrightarrow B$ or $\phi_b \colon \overset{\circ}{c}(X_b) \times \mathbb{R}^k \hookrightarrow B$ for each $b \in \mathcal{B}_k$ (The handle lemmas are applied independently for each $b \in \mathcal{B}_k$. Since $\mathcal{B}_k$ may be infinite, the Finiteness condition of Hypothesis 3.25 is crucial at this step.),
3. $\delta_k < \delta_{k-1}/2$,
4. $\varepsilon_k < \varepsilon/(n+1)$,
5. For each $b \in \mathcal{B}_k$, any map to B that is ε_k-close to a $(\delta_{k-1}/2)$-fibration over $\phi_b (\mathbb{R}^n)$ or $\phi_b \left(c_3(X_b) \times \mathrm{B}_3^k\right)$ is itself a δ_{k-1}-fibration over $\phi_b (\mathbb{R}^n)$ or $\phi_b \left(c_3(X_b) \times \mathrm{B}_3^k\right)$, respectively. (The Finiteness condition is again being used here.)

Set $\delta = \delta_m$ and let $\mu > 0$ be given. let $p \colon M \to B$ be given as in the hypothesis. We will produce a map $p' \colon M \to B$ that is a μ-fibration over each W_b. It suffices to construct a sequence of maps $p = p^{n+1}, p^n, \ldots, p^1, p^0 = p'$ such that p^k is ε_k-close to p^{k+1} and p^k is a μ-fibration over $\phi_b \left(c_{r_k}(X_b) \times \mathrm{B}_{r_k}^\ell\right)$ for $k \leq \ell \leq n$ and $b \in \mathcal{B}_\ell$. First, inductively define small positive numbers $\nu_{-1}, \nu_0, \ldots, \nu_n$ be setting $\nu_{-1} = \mu$ and for $k = 0, \ldots, n-1$, choosing $\nu_k < \mu$ such that $\nu_k < \nu(\nu_{k-1})$, where $\nu(\nu_{k-1})$ is given by the Handle Lemma 3.22 for the open embeddings $\phi_b \colon \overset{\circ}{c}(X_b) \times \mathbb{R}^k \hookrightarrow B$ (The Finiteness condition of Hypothesis 3.25 is used here to apply the handle lemma independently for each $b \in \mathcal{B}_k$.)

Using the appropriate Handle Lemma, we inductively produce the maps p^k (starting with $k = n$) so that

1. p^k is a ν_{k-1}-fibration over $\phi_b \left(c_{r_k}(X_b) \times \mathrm{B}_{r_k}^\ell\right)$ for each $b \in \mathcal{B}_\ell$,
2. p^k is ε_k-close to p^{k+1},
3. $p^k = p^{k+1}$ over $B \setminus \bigcup_{b \in \mathcal{B}_k} \left[\overset{\circ}{c}_{2/3}(X_b) \times \overset{\circ}{\mathrm{B}}_3^k \right]$.

In order to apply the Handle Lemma inductively simply observe that p^k is a δ_{k-1}-fibration over $\phi_b (c_3(X_b) \times \mathrm{B}_3^k)$. Also observe that p^k is a μ-fibration over $\phi_b \left(c_{r_k}(X_b) \times \mathrm{B}_{r_k}^\ell\right)$ for each $k \leq \ell \leq n$ and $b \in \mathcal{B}_\ell$. □

The next corollary is essentially a renaming of some of the sets in Proposition 3.28. As such, it can be viewed as a corollary to the proof of Proposition 3.28. However, a more formal derivation is also given. We begin by introducing some more notation.

Notation Assume Hypothesis 3.25. For each $R > 1$, $0 \leq k \leq n$, and $b \in \mathcal{B}_k$, define

$$V_b^R := \phi_b \left(\overset{\circ}{c}_R(X_b) \times \overset{\circ}{\mathrm{B}}_R^k \right) \subseteq B.$$

Corollary 3.29 *Assume Hypothesis 3.25. For each integer $m > 4$, each $1 < R_2 < R_1$, and each $\varepsilon > 0$ there exists a $\delta = \delta(m, \varepsilon, R_1, R_2) > 0$ such that for every $\mu > 0$ if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration over $V_b^{R_1}$ for each $b \in \mathcal{B}$, then p is ε-close to a proper map $p' \colon M \to B$ such that $p = p'$ on $M \setminus \bigcup_{b \in \mathcal{B}} p^{-1}(V_b^{R_1})$ and p' is a μ-fibration over $V_b^{R_2}$ for each $b \in \mathcal{B}$.*

Proof Fix a homeomorphism $h \colon [0, R_1) \to [0, \infty)$ such that $h(t) = t$ for all $0 \leq t \leq 1$ and $h(R_2) = 1.1$. For each $0 \leq k \leq n$ and $b \in \mathcal{B}_k$, define a homeomorphism $h_b \colon \overset{\circ}{c}_{R_1}(X_b) \times \overset{\circ}{\mathrm{B}}_{R_1}^k \to \overset{\circ}{c}(X_b) \times \mathbb{R}^k$ by $h_b([x, t], y) = ([x, h(t)], (h(y_1), h(y_2), \ldots, h(y_k)))$ for all $x \in X_b$, $t \in [0, R_1)$, and $y = (y_1, y_2, \ldots, y_k) \in \overset{\circ}{\mathrm{B}}_{R_1}^k$. Apply Proposition 3.28 to the open embeddings $\phi_b \circ h_b^{-1}$. □

Note that in the statement of Corollary 3.29, $\delta = \delta(m, \varepsilon, R_1, R_2)$ also depends on the set-up in Hypothesis 3.25, but we suppress that dependence in the notation.

Theorem 3.30 *Assume Hypothesis 3.25. For each integer $m > 4$ and $\varepsilon > 0$ there exists $\delta > 0$ such that: if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration over V, then p is ε-close to a proper map $p' \colon M \to B$ such that $p = p'$ on $M \backslash p^{-1}(V)$ and p' is an approximate fibration over A.*

Proof For each $i = 1, 2, 3, \ldots$, let $R_i = 2 + \frac{1}{i}$. Let $m > 4$ and $\varepsilon > 0$ be given. For each $i = 1, 2, 3, \ldots$ let $\delta_i = \delta(m, \varepsilon/2^i, R_i, R_{i+1}) > 0$ be given by Corollary 3.29. We may also assume that $\delta_i < 1/i$ so that $\lim_{i \to \infty} \delta_i = 0$. Let $\delta = \delta_1$ and suppose M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration over V. Use Corollary 3.29 to define inductively a sequence of proper maps $q_i \colon M \to B$, $i = 1, 2, 3, \ldots$, such that:

1. $q_1 = p$,
2. q_i is $(\varepsilon/2^i)$-close to q_{i+1},
3. q_i is a δ_i-fibration over $V_b^{R_i} \supseteq V_b^2$ for each $b \in \mathcal{B}$,
4. $q_i = q_{i+1}$ on $M \backslash \bigcup_{b \in \mathcal{B}} p^{-1} \left(V_b^{R_i} \right)$.

The completeness of the metric on the closure of V implies that the uniform limit $p' = \lim_{i \to \infty} q_i$ exists. Clearly, p' is ε-close to p and $p = p'$ on $M \backslash \bigcup_{b \in \mathcal{B}} p^{-1} \left(V_b^{R_1} \right) \supseteq M \backslash p^{-1}(V)$. By Lemma 3.20, p' is proper and an approximate fibration over $\bigcup_{b \in \mathcal{B}} V_b^2 \subseteq A$. $\qquad\square$

Corollary 3.31 *Suppose (B, d) is a metric space triangulated by a locally finite simplicial complex, A is a closed sub-polyhedron of B, $\mathrm{Shapes}^{\boxtimes}(B, A)$ is finite, U is an open subset of B containing A, and the metric d is complete on the closure of U. For every integer $m > 4$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration over U, then p is ε-close to a proper map $p' \colon M \to B$ such that $p = p'$ on $M \backslash p^{-1}(U)$ and p' is an approximate fibration over A.*

Proof This follows directly from Theorem 3.30. $\qquad\square$

The following corollary follows immediately from Corollary 3.31 by taking $A = B$. It is the metric version of Chapman's MAF Sucking Theorem 3.18.

Corollary 3.32 (Metric MAF Sucking) *Suppose (B, d) is a complete metric space triangulated by a locally finite simplicial complex such that $\mathrm{Shapes}^{\boxtimes}(B)$ is finite. For every integer $m > 4$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration, then p is ε-close to a manifold approximate fibration $p' \colon M \to B$.* $\qquad\square$

Corollary 3.33 *Suppose (B, d) is a triangulated metric space of finite isometry type. For every integer $m > 4$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if M is an m-dimensional manifold and $p \colon M \to B$ is a proper δ-fibration, then p is ε-homotopic to a manifold approximate fibration $p' \colon M \to B$.*

Proof Apply Corollary 3.32, Corollary 3.17, and Definition 3.5. $\qquad\square$

The following corollary is the exact form of metric MAF sucking that we will use in Lemma 6.6 below in the course of proving an equivariant version of MAF sucking.

Corollary 3.34 *Suppose Y is a compact metric space and d is a complete metric for the open cone $\overset{\circ}{c}(Y)$. Restrict the metric to the subset $B := Y \times (0, \infty)$. Select $0 < r_1 < r_2 < r_3$ and an integer $m > 4$. Suppose B is triangulated by a locally finite simplicial complex such that there is a closed polyhedral neighborhood A of $Y \times [r_3, \infty)$ contained in $Y \times (r_2, \infty)$ with $\mathrm{Shapes}^{\boxtimes}(B, A)$ finite. For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if P is an m-dimensional manifold and $p \colon P \to B$ is a proper δ-fibration over $Y \times (r_1, \infty)$, then p is ε-close to a map p' that is a MAF over $Y \times (r_3, \infty)$ such that $p = p'$ on $p^{-1}(Y \times (0, r_2])$.*

Proof Apply Corollary 3.31 with $B = Y \times (0, \infty)$ and $U = Y \times (r_2, \infty)$. □

As mentioned above, the following result has been used in the literature (e.g., [16, Theorem 16.9]), but a detailed derivation has heretofore not appeared.

Corollary 3.35 *Suppose $\mathbb{R}^n$ is given its standard metric. For every $m > 4$ and every $\varepsilon > 0$ there exists $\delta = \delta(n, m, \varepsilon) > 0$ such that: if M is an m-manifold and $p \colon M \to \mathbb{R}^n$ is a proper δ-fibration, then p is ε-homotopic to a MAF. Consequently, any proper bounded fibration $p \colon M \to \mathbb{R}^n$ is boundedly homotopic to a MAF.*

Proof For the first statement, apply Corollary 3.31 with $\mathbb{R}^n = A = U = B$ and triangulate $\mathbb{R}^n$ so that $\mathrm{Shapes}^{\boxtimes}(\mathbb{R}^n)$ is finite. For the second statement, apply a standard scaling procedure; see Hughes–Ranicki [16, Corollary 16.10] and the proof of Corollary 6.2 below. □

4 Orthogonal actions

In this section we establish various lifting properties for maps associated to actions of certain finite subgroups G of the orthogonal group $O(n)$. For our Main Theorem 1.1, we only need the case of C_2 acting by reflection on $\mathbb{R}$. However, for the proof of the more widely applicable Theorem 1.4 (Orthogonal Sucking), we need to work in a more general context.

We shall assume the following notation throughout the remainder of the paper.

Notation Suppose G is a finite subgroup of the orthogonal group $O(n)$. We assume that $\mathbb{R}^n$ has the Euclidean metric. Write $X := S^{n-1}/G$. Since G acts on $\mathbb{R}^n$ by isometries, we may endow the open cone $\overset{\circ}{c}(X) = \mathbb{R}^n/G$ with the quotient metric. Then note that the quotient map $q_{\mathbb{R}^n} \colon \mathbb{R}^n \to \overset{\circ}{c}(X)$ is distance non-increasing.

At certain specified times, we shall assume a freeness hypothesis, as follows.

Hypothesis 4.1 Suppose G is a finite subgroup of $O(n)$. Furthermore:

1. Assume that G acts freely on S^{n-1}. Then X is a closed, smooth manifold of dimension $n - 1$. Suppose M is a G-space. The quotient map is denoted by $q_M \colon M \to N := M/G$.
2. Consider N to be a stratified space with exactly one stratum (itself). Consider $\overset{\circ}{c}(X)$ to be a stratified space with exactly two strata: the cone point $\{v\}$ and the complement $\overset{\circ}{c}(X) \backslash \{v\} = X \times (0, \infty)$.

For example, we assume Hypothesis 4.1 for the remainder of Sect. 4.

In the following lemma, we consider $\mathbb{R}^n$ to be a stratified space with exactly one stratum. However, the lemma holds equally well—with no change in the proof—if $\mathbb{R}^n$ has exactly two strata: the origin $\{0\}$ and $\mathbb{R}^n \backslash \{0\}$. The lemma is a special case of a more general theorem of A. Beshears [1, Theorem 4.6], but it is quite elementary in the case at hand; therefore, we include a proof.

Lemma 4.2 *The quotient map $q_{\mathbb{R}^n} \colon \mathbb{R}^n \to \overset{\circ}{c}(X)$ is a stratified fibration.*

Proof Consider a stratified homotopy lifting problem:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & \mathbb{R}^n \\
{\scriptstyle \times 0}\Big\downarrow & \overset{\tilde{F}}{\nearrow} & \Big\downarrow{\scriptstyle q_{\mathbb{R}^n}} \\
Z \times I & \xrightarrow{\ F\ } & \overset{\circ}{c}(X)
\end{array}
$$

The stratified condition is equivalent to saying: if $(z, t) \in Z \times I$, then $F(z, t) = v$ if and only if $F(z, 0) = v$ if and only if $f(z) = 0$. Let $Z_0 = f^{-1}(0)$. Then the stratified homotopy lifting problem above restricts to the following homotopy lifting problem:

$$
\begin{array}{ccc}
Z \backslash Z_0 & \xrightarrow{\ f|\ } & \mathbb{R}^n \backslash \{0\} \\
{\scriptstyle \times 0}\Big\downarrow & & \Big\downarrow{\scriptstyle q_{\mathbb{R}^n}|} \\
(Z \backslash Z_0) \times I & \xrightarrow{\ F|\ } & \overset{\circ}{c}(X) \backslash \{v\}
\end{array}
$$

Since $q_{\mathbb{R}^n}| \colon \mathbb{R}^n \backslash \{0\} \to \overset{\circ}{c}(X) \backslash \{v\}$ is a fibration (in fact, a covering map), this later problem has a solution. That is, there is a map $\hat{F} \colon (Z \backslash Z_0) \times I \to \mathbb{R}^n \backslash \{0\}$ such that $\hat{F}(z, 0) = f(z)$ and $q_{\mathbb{R}^n} \hat{F}(z, t) = F(z, t)$ for all $(z, t) \in (Z \backslash Z_0) \times I$. Define

$$
\tilde{F} \colon Z \times I \longrightarrow \mathbb{R}^n; \quad (z, t) \longmapsto \begin{cases} \hat{F}(z, t) & \text{if } z \in Z \backslash Z_0 \\ 0 & \text{if } z \in Z_0. \end{cases}
$$

Clearly, $\tilde{F}$ is the required stratified solution if $\tilde{F}$ is continuous. To verify continuity it suffices to consider a ball B centered at the origin in $\mathbb{R}^n$ and observe that $\tilde{F}^{-1}(B)$ is open because $q_{\mathbb{R}^n} B$ is open in $\overset{\circ}{c}(X)$ and $\tilde{F}^{-1}(B) = F^{-1}(q_{\mathbb{R}^n} B)$ (using the fact that $q_{\mathbb{R}^n}^{-1} q_{\mathbb{R}^n} B = B$). $\qquad \square$

Proposition 4.3 *For every $\delta > 0$ and every proper G-δ-fibration $p \colon M \to \mathbb{R}^n$, the induced map $p/G \colon N \to \overset{\circ}{c}(X)$ is a proper stratified δ-fibration.*

Proof For reference, we note that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\ q_M\ } & N \\
{\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle p/G} \\
\mathbb{R}^n & \xrightarrow{\ q_{\mathbb{R}^n}\ } & \overset{\circ}{c}(X)
\end{array}
$$

Consider a stratified homotopy lifting problem:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & N \\
{\scriptstyle \times 0}\Big\downarrow & \overset{\tilde{F}}{\nearrow} & \Big\downarrow{\scriptstyle p/G} \\
Z \times I & \xrightarrow{\ F\ } & \overset{\circ}{c}(X)
\end{array}
$$

Form the pull-back diagram

$$
\begin{array}{ccc}
\hat{Z} & \xrightarrow{\hat{f}} & M \\
\hat{q} \downarrow & & \downarrow q_M \\
Z & \xrightarrow{f} & N
\end{array}
$$

There is a natural G-action on $\hat{Z}$ ($g(z, x) = (z, gx)$) and $\hat{f}$ is a G-map. Note that there is a stratified homotopy lifting problem:

$$
\begin{array}{ccc}
\hat{Z} & \xrightarrow{p\hat{f}} & \mathbb{R}^n \\
\times 0 \downarrow & \overset{H}{\cdots\cdots\cdots} & \downarrow q_{\mathbb{R}^n} \\
Z \times I & \xrightarrow{F(\hat{q} \times \mathrm{id}_I)} & \overset{\circ}{c}(X)
\end{array}
$$

Lemma 4.2 implies that there is a stratified solution $H \colon \hat{Z} \times I \to \mathbb{R}^n$. Now we claim that H is a G-homotopy. This is essentially true because the action of G on $\mathbb{R}^n$ is free away from the origin and constant on the origin (thus, we are using the fact that the action of G on S^{n-1} is free). In more detail, first note that $F(\hat{q} \times \mathrm{id}_I)(\hat{z}, t) = F(\hat{q} \times \mathrm{id}_I)(g\hat{z}, t)$ for all $(\hat{z}, t) \in \hat{Z} \times I$ and $g \in G$. Then let $\hat{Z}_0 = (p\hat{f})^{-1}(0)$. Since $q_{\mathbb{R}}^n| \colon \mathbb{R}^n \backslash \{0\} \to \overset{\circ}{c}(X) \backslash \{v\}$ is a covering map and H_0 is a G-map, it follows that $H| \colon (\hat{Z} \backslash \hat{Z}_0) \times I \to \mathbb{R}^n \backslash \{0\}$ is a G-map. Finally, note that $H(\hat{Z}_0 \times I) = \{0\}$ and $\hat{Z}_0$ is G-invariant. Together these observations imply that H is a G-homotopy.

It follows that H fits into a G-homotopy lifting problem:

$$
\begin{array}{ccc}
\hat{Z} & \xrightarrow{\hat{f}} & M \\
\times 0 \downarrow & \overset{\tilde{H}}{\cdots\cdots} & \downarrow p \\
\hat{Z} \times I & \xrightarrow{H} & \mathbb{R}^n
\end{array}
$$

By hypothesis, there is a G-homotopy $\tilde{H} \colon \hat{Z} \times I \to M$ such that $\tilde{H}_0 = \hat{f}$ and $p\tilde{H}$ is δ-close to H. It follows that there is a unique map $\tilde{F} \colon Z \times I \to N$ making the following diagram commute:

$$
\begin{array}{ccc}
\hat{Z} \times I & \xrightarrow{\tilde{H}} & M \\
\hat{q} \times \mathrm{id}_I \downarrow & & \downarrow q_M \\
Z \times I & \xrightarrow{\tilde{F}} & N
\end{array}
$$

It follows that $\tilde{F}$ is a δ-solution of the original stratified problem. For this we use the assumption that $q_{\mathbb{R}^n} \colon \mathbb{R}^n \to \overset{\circ}{c}(X)$ is distance non-increasing together with the following diagram

$$
\begin{array}{ccccccc}
Z \times I & \xleftarrow{\hat{q} \times \mathrm{id}_I} & \hat{Z} \times I & \xrightarrow{=} & \hat{Z} \times I & \xrightarrow{\hat{q} \times \mathrm{id}_I} & Z \times I \\
F \downarrow & & H \downarrow & & \downarrow p\tilde{H} & & \downarrow (p/G) \circ \tilde{F} \\
\overset{\circ}{c}(X) & \xleftarrow{q_{\mathbb{R}^n}} & \mathbb{R}^n & \xrightarrow{=} & \mathbb{R}^n & \xrightarrow{q_{\mathbb{R}^n}} & \overset{\circ}{c}(X)
\end{array}
$$

where the outer two squares are commutative and the middle square commutes up to δ. $\qquad\square$

We set up two basic lemmas on lifting homotopies across $q_{\mathbb{R}^n} : \mathbb{R}^n \to \overset{\circ}{\mathrm{c}}(X)$, which will be used in Sect. 6. Recall that G is a finite subgroup of $O(n)$.

Lemma 4.4 *Let Z be a G-space, and denote the quotient map $p : Z \to Z/G$. Suppose $f : Z \to \mathbb{R}^n$ is a G-map and $F : Z/G \times I \to \overset{\circ}{\mathrm{c}}(X)$ is a stratum-preserving homotopy such that $q_{\mathbb{R}^n} \circ f = F(-, 0) \circ p$. Then there is a unique G-homotopy $\tilde{F} : Z \times I \to \mathbb{R}^n$ such that $\tilde{F}(-, 0) = f$ and $q_{\mathbb{R}^n} \circ \tilde{F} = F \circ (p \times \mathrm{id}_I)$.*

Proof Since $F \circ (p \times \mathrm{id}_I) : Z \times I \to \overset{\circ}{\mathrm{c}}(X)$ is a stratum-preserving homotopy, by Lemma 4.2, there exists a homotopy $\tilde{F} : Z \times I \to \mathbb{R}^n$ such that $\tilde{F}(-, 0) = f$ and $q_{\mathbb{R}^n} \circ \tilde{F} = F \circ (p \times \mathrm{id}_I)$. Observe, since $\tilde{F}$ is stratum-preserving and the restriction $q_{\mathbb{R}^n}| : \mathbb{R}^n \setminus \{0\} \to \overset{\circ}{\mathrm{c}}(X) \setminus \{v\}$ is a covering map, that $\tilde{F}$ is uniquely determined.

Define $Z_0 := Z \setminus f^{-1}\{0\}$. Note $\tilde{F}$ restricts to a homotopy $\hat{F} : Z_0 \times I \to \mathbb{R}^n \setminus \{0\}$. Let $g \in G$ and $z \in Z_0$. Consider the paths

$$\alpha := \hat{F}(g \cdot z, -), \ \beta := g \cdot \hat{F}(z, -) : I \longrightarrow \mathbb{R}^n \setminus \{0\}.$$

Note, since f is a G-equivariant, that

$$\alpha(0) = f(g \cdot z) = g \cdot f(z) = \beta(0).$$

Note, since p and $q_{\mathbb{R}^n}$ are G-invariant, that

$$q_{\mathbb{R}^n} \circ \alpha = F(p(g \cdot z), -) = F(p(z), -) = q_{\mathbb{R}^n} \circ \hat{F}(z, -) = q_{\mathbb{R}^n} \circ \beta.$$

Therefore, by the Path Lifting Property, we obtain that $\alpha = \beta$. Thus $\hat{F} : Z_0 \times I \to \mathbb{R}^n \setminus \{0\}$ is a G-homotopy. Hence $\tilde{F} : Z \times I \to \mathbb{R}^n$ is a G-homotopy. □

Recall that the open cone $\overset{\circ}{\mathrm{c}}(X) = \mathbb{R}^n/G$ has the quotient metric.

Lemma 4.5 *Let Z be a topological space, and let $r > 0$. There exists $\varepsilon_0 > 0$ such that: if $F : Z \times I \to \mathbb{R}^n \setminus \mathrm{B}_r^n$ is a homotopy and $q_{\mathbb{R}^n} \circ F : Z \times I \to \overset{\circ}{\mathrm{c}}(X) \setminus \mathrm{c}_r(X)$ is an ε_0-homotopy, then F is an ε_0-homotopy.*

Proof Since X is compact, there exists a finite cover $\mathcal{U}$ by non-empty open subsets $U \subseteq X$ such that the covering map $q : S^{n-1} \to X$ evenly covers each $U \in \mathcal{U}$. Then the induced cover on the metric subspace $X \times \{r\} \subset \overset{\circ}{\mathrm{c}}(X)$ has a Lebesgue number $\varepsilon_0 > 0$. So the restriction of the induced cover $\overset{\circ}{\mathrm{c}}(\mathcal{U}) := \{\overset{\circ}{\mathrm{c}}(U) \mid U \in \mathcal{U}\}$ to the frustum $\overset{\circ}{\mathrm{c}}(X) \setminus \mathrm{c}_r(X)$ has the same Lebesgue number $\varepsilon_0 > 0$.

Let $z \in Z$. Then, since the track $q_{\mathbb{R}^n} F(\{z\} \times I) \subset \overset{\circ}{\mathrm{c}}(X) \setminus \mathrm{c}_r(X)$ has diameter $< \varepsilon_0$, there exists $U \in \mathcal{U}$ such that $q_{\mathbb{R}^n} F(\{z\} \times I) \subset U \times (r, \infty)$. Let $V \subset q^{-1}(U)$ be the path component of the point $F(z, 0)$. Then, since $q|_V : V \to U$ is an isometry, the track $F(\{z\} \times I) \subset V$ has diameter $< \varepsilon_0$. Thus F is an ε_0-homotopy. □

5 Piecing together bounded fibrations

Throughout Sect. 5, we assume that G is a finite subgroup of $O(n)$.

Our goal here is to adapt [19, Proposition 2.6] and give a detailed proof. This result will be used in Sect. 6.

Theorem 5.1 *Let $K \subset V \subset C \subset U$ be G-subsets of $\mathbb{R}^n$ such that:*

1. *U, V are open subsets of $\mathbb{R}^n$,*
2. *C, K are closed subsets of $\mathbb{R}^n$, and*
3. *U (resp. V) contains a metric neighborhood of C (resp. K).*

For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mu > 0$ there exists $\nu > 0$ satisfying: If $p : E \to \mathbb{R}^n$ is a G-δ-fibration over U and a G-ν-fibration over V, then p is a G-(ε, μ)-fibration over (C, K) for the class of compact, metric G-spaces.

The proof of the theorem is located at the end of this section. The following corollary is required in Proposition 6.7 below.

Corollary 5.2 *Let $0 < r_1 < r_2$ be given. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mu > 0$ there exists $\nu > 0$ satisfying: if $p : E \to \mathbb{R}^n$ is a G-δ-fibration over $\mathbb{R}^n$ and a G-ν-fibration over $\mathbb{R}^n \backslash \mathrm{B}^n_{r_1}$, then p is a G-(ε, μ)-fibration over $(\mathbb{R}^n, \mathbb{R}^n \backslash \mathrm{B}^n_{r_2})$ for the class of compact, metric G-spaces.*

Proof This follows immediately from Theorem 5.1 by taking the G-subsets

$$K = \mathbb{R}^n \backslash \mathrm{B}^n_{r_2} \subset V = \mathbb{R}^n \backslash \mathrm{B}^n_{r_1} \subset C = U = \mathbb{R}^n.$$

$\square$

5.1 Homotopy extension

First, we construct a certain strong deformation retraction. A G-space Z is *normal* if Z/G is normal and the quotient map $Z \to Z/G$ is a closed map.

Lemma 5.3 *Let B be an open G-subset of a normal G-space Z. Let $A \subset B$ be a closed G-subset of Z. There is a G-homotopy $R : (Z \times I) \times I \to Z \times I$ such that:*

1. *$R(-, 0)$ has image in $Z \times \{0\} \cup B \times I$, and*
2. *$R(y, s) = y$ if $s = 1$ or $y \in Z \times \{0\} \cup A \times I$.*

Proof We may assume $Z \backslash B$ and A are non-empty. Since Z/G is a normal space, by the Urysohn lemma, there exists a G-map $\upsilon : Z \to I$ such that $\upsilon(Z \backslash B) = \{0\}$ and $\upsilon(A) = \{1\}$. Define a G-homotopy

$$R : (Z \times I) \times I \longrightarrow Z \times I; \quad ((z, t), s) \longmapsto (z, st + (1 - s)t\upsilon(z)).$$

This function satisfies the required properties. $\square$

We adapt the Estimated Homotopy Extension Property [5, Proposition 2.1].

Lemma 5.4 *Let Y_0 be a closed G-subset of a finite-dimensional, locally compact, metric, separable G-space Y. For every $\lambda > 0$: if $h : Y \to \mathbb{R}^n$ is a G-map and $H_0 : Y_0 \times I \to \mathbb{R}^n$ is a G-λ-homotopy such that $h|Y_0 = H_0(-, 0)$, then there exists a G-λ-homotopy $H : Y \times I \to \mathbb{R}^n$ extending H_0 such that $h = H(-, 0)$.*

Proof We may assume Y_0 is non-empty. Note that there are finitely many fixed-point sets $(\mathbb{R}^n)^H$, each of which is a vector subspace of $\mathbb{R}^n$, hence each $(\mathbb{R}^n)^H$ is an ANR. Also note that $Y_1 := Y \times \{0\} \cup Y_0 \times I$ is a closed G-subset of $Y \times I$. Since Y is a finite dimensional, locally compact, metric, separable G-space, by a theorem of J. Jaworowski [23, Theorem 2.2], there

exists an open G-neighborhood U_1 of Y_1 in Y and a G-map $H_U : U_1 \to \mathbb{R}^n$ extending the G-map $h \cup H_0 : Y_1 \to \mathbb{R}^n$. By the tube lemma and uniting open G-sets, there exists an open G-neighborhood U_0' of Y_0 in Y such that $U_0' \times I \subseteq U_1$. Since the tracks of $H_U | Y_0 \times I = H_0$ have diameter $< \lambda$, by continuity, there exists an open G-neighborhood $U_0 \subseteq U_0'$ of Y_0 in Y such that the tracks of $H_U | U_0 \times I$ have diameter $< \lambda$.

We may assume $Y \setminus U_0$ is non-empty. Since Y/G is a normal space, by the Urysohn lemma, there exists a G-map $\upsilon : Y \to I$ such that $\upsilon(Y \setminus U_0) = \{0\}$ and $\upsilon(Y_0) = \{1\}$. Then, since $Y_1 \subseteq Y \times \{0\} \cup U_0 \times I \subseteq U_1$, we can define a G-homotopy

$$H : Y \times I \longrightarrow \mathbb{R}^n; \quad (y, s) \longmapsto H_U(y, s\upsilon(y)).$$

Note that H extends $h \cup H_0$. Also note $\operatorname{diam} H(\{y\} \times I) \le \operatorname{diam} H_U(\{y\} \times I) < \lambda$ for all $y \in Y$. This completes the proof. □

We need an often-used corollary for close homotopies.

Corollary 5.5 *Let X_0 be a G-subset of a finite-dimensional, locally compact, metric, separable G-space X. For every $\lambda > 0$: if $f : X \to \mathbb{R}^n$ is a G-map and $F : X \times I \to \mathbb{R}^n$ is a G-homotopy, and if $F_0' : X_0 \times I \to \mathbb{R}^n$ is a G-homotopy such that $f | X_0 = F_0'(-, 0)$ and F_0' is λ-close to $F | X_0 \times I$, then there exists a G-homotopy $F' : X \times I \to \mathbb{R}^n$ extending F_0' such that $f = F'(-, 0)$ and F' is λ-close to F.*

Proof Define $Y_0 := X \times \{0\} \cup X_0 \times I$ and $Y := X \times I$ and $h := F : Y \to \mathbb{R}^n$. Observe that the straight-line homotopy

$$H_0 : Y_0 \times I \longrightarrow \mathbb{R}^n; \quad ((x, t), s) \longmapsto \begin{cases} (1-s)F(x, 0) + sf(x) & \text{if } t = 0 \\ (1-s)F(x, t) + sF_0'(x, t) & \text{if } x \in X_0 \end{cases}$$

is a G-λ-homotopy such that $h | Y_0 = H_0(-, 0)$. Then, by Lemma 5.4, there exists a G-λ-homotopy $H : Y \times I \to \mathbb{R}^n$ extending H_0 such that $h = H(-, 0)$. Define a G-homotopy

$$F' := H(-, 1) : X \times I \longrightarrow \mathbb{R}^n.$$

Note $F' | X_0 \times I = F_0'$ and $F'(-, 0) = H((-, 0), 1) = f$. Also note $\| F'(y) - F(y) \| = \| H(y, 1) - H(y, 0) \| < \lambda$ for all $y \in X \times I$. This completes the proof. □

5.2 Homotopy lifting

We adapt the Stationary Lifting Property [17, Theorem 6.2].

Lemma 5.6 *Let $\delta > 0$. Let $A \subseteq \mathbb{R}^n$ be a G-subset. Let Y be a normal G-space. Suppose $p : E \to \mathbb{R}^n$ is a G-δ-fibration over A. If $H : Y \times I \to A$ is a G-homotopy and $h : Y \to E$ is a G-map such that $ph = H(-, 0)$, then there exists a G-homotopy $\tilde{H} : Y \times I \to E$ such that:*

1. *$h = \tilde{H}(-, 0)$,*
2. *$p\tilde{H}$ is δ-close to H, and*
3. *$\tilde{H}(\{y\} \times I) = \tilde{H}(\{y\} \times \{0\})$ if $H(\{y\} \times I) = H(\{y\} \times \{0\})$.*

Proof Define a G-subset

$$C := \{y \in Y \mid H(\{y\} \times I) = H(\{y\} \times \{0\})\}.$$

Note that C is the inverse image of $\{0\}$ under the G-map $(y \mapsto \operatorname{diam} H(\{y\} \times I))$. Hence C is a *closed G-G_δ-subset of Y;* that is, C is a closed G-subset of Y and C is a countable intersection of open G-subsets of Y. Then, since Y/G is normal, by the strong Urysohn lemma, there exists a G-map $\upsilon : Y \to I$ such that $C = \upsilon^{-1}\{0\}$. Define a G-homotopy

$$H^* : Y \times I \longrightarrow A; \quad (y, s) \longmapsto \begin{cases} H(y, s/\upsilon(y)) & \text{if } 0 \le s < \upsilon(y) \\ H(y, 1) & \text{if } \upsilon(y) \le s \le 1. \end{cases}$$

Note that $\upsilon^{-1}\{0\} \subseteq C$ implies $H^*(-, 0) = H(-, 0) = ph$. Since p is a G-δ-fibration over A, there exists a G-homotopy $\tilde{H}^* : Y \times I \to E$ such that $h = \tilde{H}^*(-, 0)$ and $p\tilde{H}^*$ is δ-close to H^*. Now define G-homotopy

$$\tilde{H} : Y \times I \longrightarrow E; \quad (y, s) \longmapsto \tilde{H}^*(y, s\upsilon(y)).$$

Note $\tilde{H}(-, 0) = \tilde{H}^*(-, 0) = h$ and $\|p\tilde{H}(y, s) - H(y, s)\| = \|p\tilde{H}^*(y, s\upsilon(y)) - H^*(y, s\upsilon(y))\| < \delta$ for all $(y, s) \in Y \times I$. Furthermore, if $y \in C \subseteq \upsilon^{-1}\{0\}$, then note $\tilde{H}(y, s) = \tilde{H}^*(y, 0) = \tilde{H}(y, 0)$. This completes the proof. $\qquad\square$

5.3 Blending bounds

We adapt [17, Lemma 4.7]. This result finds a jointly close solution to the homotopy lifting problem for a prototypical kind of homotopy. We say that a neighborhood N of a subset A of a metric space (X, d) is *metric* if N equals the α-neighborhood $\{x \in X \mid d(x, A) < \alpha\}$ of A in X for some $\alpha > 0$.

Lemma 5.7 *Let $K \subset \operatorname{int} K' \subset V \subset C \subset U$ be G-subsets of $\mathbb{R}^n$ such that:*

1. *U, V are open subsets of $\mathbb{R}^n$,*
2. *K', K are closed subsets of $\mathbb{R}^n$, and*
3. *U contains a metric neighborhood of C.*

Let Z be a finite-dimensional, locally compact, metric, separable G-space. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\mu > 0$ there exists $\nu > 0$ satisfying: if $p : E \to B$ is a G-δ-fibration over U and a G-ν-fibration over V, and if $F : Z \times I \to C$ is a G-homotopy with tracks in $\{\operatorname{int} K', C \backslash K\}$, and if $f : Z \to E$ is a G-map such that $pf = F(-, 0)$, then there exists a G-homotopy $\tilde{F} : Z \times I \to E$ such that $f = \tilde{F}(-, 0)$ and $p\tilde{F}$ is (ε, μ)-close to F with respect to K.

Proof Let $\varepsilon > 0$. Select $0 < \delta < \varepsilon$. Let $\mu > 0$. Select $0 < \nu \le \min\{\varepsilon - \delta, \mu\}$ such that the ν-neighborhood of C is contained in U. The G-homotopy $\tilde{F} : Z \times I \to E$ is constructed in two steps.

The first step is to consider the G-subsets

$$Z_1 := \{z \in Z \mid F(\{z\} \times I) \cap K \ne \varnothing\}$$
$$Z_1' := \{z \in Z \mid F(\{z\} \times I) \subset \operatorname{int} K'\}.$$

Since $K \subset \operatorname{int} K'$, by hypothesis on F, we have $Z_1 \subseteq Z_1'$. Since K and K' are closed in $\mathbb{R}^n$, we have that Z_1 is closed and Z_1' is open in Z. Since p is a ν-fibration over $V \supset K'$, there exists a G-homotopy $\tilde{F}_0 : Z_1' \times I \to E$ such that $f|Z_1' = \tilde{F}_0(-, 0)$ and $p\tilde{F}_0$ is ν-close to $F|Z_1' \times I$. Then, by Corollary 5.5, there exists a G-homotopy $F' : Z \times I \to \mathbb{R}^n$ extending $p\tilde{F}_0$ such that $pf = F'(-, 0)$ and F' is ν-close to F. Hence $\tilde{F}_0$ is a partial lift of F' and the image of F' is contained in U.

The second step is to use Lemma 5.3. Since Z is a normal G-space, there exists a G-homotopy $R : (Z \times I) \times I \to Z \times I$ such that $R(-, 0)$ has image in $Z \times \{0\} \cup Z_1' \times I$ and that $R(y, s) = y$ if $s = 1$ or $y \in Z \times \{0\} \cup Z_1 \times I$. Consider the G-homotopy $F'R : (Z \times I) \times I \to \mathbb{R}^n$ with initial G-lift $(f \cup \tilde{F}_0)R(-, 0) : Z \times I \to E$. Since p is a δ-fibration over U, by Lemma 5.6, there exists a G-homotopy $\tilde{F}_I : (Z \times I) \times I \to E$ such that:

- $(f \cup \tilde{F}_0)R(-, 0) = \tilde{F}_I(-, 0)$,
- $p\tilde{F}_I$ is δ-close to $F'R$, and
- $\tilde{F}_I(y, s) = (f \cup \tilde{F}_0)(y)$ if $y \in Z \times \{0\} \cup Z_1 \times I$.

Now define a G-homotopy

$$\tilde{F} := \tilde{F}_I(-, 1) : Z \times I \longrightarrow E.$$

Note $\tilde{F}(-, 0) = \tilde{F}_I((-, 0), 1) = f$. Also note $\|p\tilde{F}(y) - F(y)\| \leq \|p\tilde{F}_I(y, 1) - F'R(y, 1)\| + \|F'(y) - F(y)\| < \delta + \nu \leq \varepsilon$ for all $y \in Z \times I$. Furthermore, if $F(z, t) \in K$, then $z \in Z_1$, so note $\|p\tilde{F}(y) - F(y)\| = \|p\tilde{F}_0(y) - F(y)\| < \nu \leq \mu$. This completes the proof. $\qquad\square$

Finally, we adapt [17, Theorem 4.8]. This result finds a jointly close solution to the homotopy lifting problem for an arbitrary homotopy.

Proof (*Proof of Theorem 5.1*) To begin, we shall set up additional parameters. Select closed G-subsets K_1, K_2, K_3, C_1 of $\mathbb{R}^n$ such that

$$K \subset \text{int } K_1 \subset \text{int } K_2 \subset K_3 \subset V \subset C \subset \text{int } C_1 \subset C_1 \subset U$$

and U, K_1 (resp. $C \setminus K_1$, int K_3) contains a metric neighborhood of C_1, K (resp. $C_1 \setminus K$, int K_2). Let $\varepsilon > 0$. Select $0 < \delta' \leq \varepsilon/3$ such that $C_1 \setminus K$ (resp. int K_3) contains the $2\delta'$-neighborhood of $C \setminus K_1$ (resp. int K_2). Select $0 < \delta < \delta'$. Let $\mu > 0$. Select $0 < \nu' \leq \mu/3$. Select $0 < \nu \leq \min\{\delta' - \delta, \nu'\}$ such that U contains the ν-neighborhood of C_1.

Next, let Z be a compact, metric G-space. Let $F : Z \times I \to C$ be a G-homotopy. Let $f : Z \to E$ be a G-map such that $pf = F(-, 0)$. Since F is continuous, each $z \in Z$ has a neighborhood W^z in Z and a finite partition $\mathcal{P}^z$ of I:

$$\mathcal{P}^z = \{0 = t_0^z < \cdots < t_i^z < \cdots < t_{n^z}^z = 1\}$$

such that the partial-track $F(\{z\} \times [t_i^z, t_{i+1}^z])$ lies in either $C \setminus K_1$ or int K_2. Since Z is compact, the open cover $\{W^z | z \in Z\}$ admits a finite subcover, and the common refinement $\mathcal{P}$ of the associated partitions is finite:

$$\mathcal{P} = \{0 = t_0 < \cdots < t_i < \cdots < t_n = 1\}.$$

Thus, for each $z \in Z$ and $0 \leq i < n$, the partial-track $F(\{z\} \times [t_i, t_{i+1}])$ lies in either $C \setminus K_1$ or int K_2, depending on z.

Lastly, for each $0 \leq i \leq n$, we shall inductively define maps $\tilde{F}_i : Z \times [0, t_i] \to E$ such that:

- $\tilde{F}_0 = f$ and $\tilde{F}_i$ extends $\tilde{F}_{i-1}$ if $i > 0$,
- $p\tilde{F}_i$ is (ε, μ)-close to $F|Z \times [0, t_i]$ with respect to K, and
- $p\tilde{F}_i|Z \times \{t_i\}$ is (δ', ν')-close to $F|Z \times \{t_i\}$ with respect to K if $i < n$.

Hence $\tilde{F} := \tilde{F}_n : Z \times I \to E$ shall be the desired homotopy.

Since $pf = F(-, 0)$, note $\tilde{F}_0 := f$ satisfies the above properties. Assume, for some $0 \leq i < n$, that there exists $\tilde{F}_i$ satisfying the three properties. Since F is continuous and X is compact, by the tube lemma, there exists $t_i < s_{i+1} < t_{i+1}$ such that diam $F(\{z\} \times [t_i, s]) < \delta'$ (resp. $< \nu'$) for all $z \in Z$ (resp. if $F(z, t_i) \in K$). Select $t_i < s_i < s_{i+1}$. Define a G-homotopy H from $p\tilde{F}_i | Z \times \{t_i\}$ to $F | Z \times \{t_{i+1}\}$ by

$$H : Z \times [t_i, t_{i+1}] \longrightarrow \mathbb{R}^n;$$

$$(z, t) \longmapsto \begin{cases} \dfrac{s_i - t}{s_i - t_i} p\tilde{F}_i(z, t_i) + \dfrac{t - t_i}{s_i - t_i} F(z, t_i) & \text{if } t \in [t_i, s_i] \\[2ex] F\left(z, t_i + \dfrac{s_{i+1} - t_i}{s_{i+1} - s_i}(t - s_i)\right) & \text{if } t \in [s_i, s_{i+1}] \\[2ex] F(z, t) & \text{if } t \in [s_{i+1}, t_{i+1}]. \end{cases}$$

Note, for all $(z, t) \in Z \times [t_i, t_{i+1}]$, that

$$\|H(z, t) - F(z, t)\|$$

$$\leq \begin{cases} \dfrac{s_i - t}{s_i - t_i} \|p\tilde{F}_i(z, t_i) - F(z, t_i)\| + \|F(z, t_i) - F(z, t)\| & \text{if } t \in [t_i, s_i] \\[2ex] \text{diam } F(\{z\} \times [t_i, s_{i+1}]) & \text{if } t \in [s_i, s_{i+1}] \\[2ex] 0 & \text{if } t \in [s_{i+1}, t_{i+1}]. \end{cases}$$

Then observe that:

- H is $(2\delta', 2\nu')$-close to $F | Z \times [t_i, t_{i+1}]$ with respect to K,
- H has image in C_1, and
- H has tracks in $\{\text{int } K_3, C_1 \backslash K\}$.

Since p is a δ-fibration over U and a ν-fibration over V, by Lemma 5.7 and the epsilonics in its proof, there exists a G-homotopy $\tilde{H} : Z \times [t_i, t_{i+1}] \to E$ such that $\tilde{F}_i(-, t_i) = \tilde{H}(-, t_i)$ and $p\tilde{H}$ is (δ', ν')-close to H with respect to K. Now define

$$\tilde{F}_{i+1} := \tilde{F}_i \cup \tilde{H} : Z \times [0, t_{i+1}] \longrightarrow E.$$

Note, for all $(z, t) \in Z \times [t_i, t_{i+1}]$, that

$$\|p\tilde{F}_{i+1}(z, t) - F(z, t)\| \leq \|p\tilde{H}(z, t) - H(z, t)\| + \|H(z, t) - F(z, t)\|.$$

Hence $p\tilde{F}_{i+1}$ is (ε, μ)-close to F on $Z \times [t_i, t_{i+1}]$ with respect to K. Furthermore $p\tilde{F}_{i+1}|$ is (δ', ν')-close to $H| = F|$ on $Z \times \{t_{i+1}\}$ with respect to K. This concludes the inductive construction of the desired homotopy $\tilde{F}$. $\qquad\square$

6 Equivariant sucking over Euclidean space

Throughout Sect. 6, we assume that G is a finite subgroup of $O(n)$, that G acts freely on S^{n-1} and on M, and that M is a manifold of dimension $m > 4$.

The following theorem is the main result herein; the proof is located at the end of the section. It is an equivariant version of the first part of Corollary 3.35 and appears in the Introduction as Theorem 1.4.

Theorem 6.1 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if $p: M \to \mathbb{R}^n$ is a proper G-δ-fibration, then p is G-ε-homotopic to a G-MAF.*

The proof of Theorem 6.1 shows that δ is independent of M but is dependent on $\dim(M)$.

The following corollary is an equivariant version of the second part of Corollary 3.35. The scaling trick in the proof is due to Chapman [4] in the non-equivariant case.

Corollary 6.2 *If $p: M \to \mathbb{R}^n$ is a proper G-bounded fibration, then p is G-boundedly homotopic to a G-MAF.*

Proof Obtain $\delta > 0$ from Theorem 6.1 with $\varepsilon = 1$. Let $p: M \to \mathbb{R}^n$ be a proper G-bounded fibration. That is, there exists $\lambda > 0$ such that p is a proper G-λ-fibration. There exists $L > 0$ such that $\lambda/L < \delta$. Then, since $G \subseteq O(n)$, note that $\frac{1}{L} p$ is a proper G-δ-fibration. So, by Theorem 6.1, there exists a G-1-homotopy $H : M \times I \to \mathbb{R}^n$ such that $H(-, 0) = \frac{1}{L} p$ and $p_1 := H(-, 1)$ is a G-MAF. Therefore the scaled map $L \cdot H : M \times I \longrightarrow \mathbb{R}^n$ is a bounded G-homotopy from p to a G-MAF Lp_1. □

The rest of this section is devoted to the proof of Theorem 6.1.

Lemma 6.3 *Let $0 < r_1 < r_2$ be given. For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying: if $p: M \to \mathbb{R}^n$ is a proper G-δ-fibration and a G-MAF over $\mathbb{R}^n \backslash \mathrm{B}^n_{r_1}$, then, for every $\nu > 0$, the map p is a G-(ε, ν)-fibration over $(\mathbb{R}^n, \mathbb{R}^n \backslash \mathrm{B}^n_{r_2})$ for the class of compact, metric G-spaces.*

Proof This is an immediate consequence of Corollary 5.2. □

Lemma 6.4 *Let $r > 0$ and in the following commutative diagram*

$$
\begin{array}{ccc}
M & \xrightarrow{q_M} & N \\
p \downarrow & & \downarrow p/G \\
\mathbb{R}^n & \xrightarrow{q_{\mathbb{R}^n}} & \overset{\circ}{\mathrm{c}}(X)
\end{array}
$$

suppose that p is a proper G-map and p/G is a MAF over $\overset{\circ}{\mathrm{c}}(X) \backslash \mathrm{c}_r(X)$. Then p is a G-MAF over $\mathbb{R}^n \backslash \mathrm{B}^n_r$.

Proof Let Z be a G-space, and denote the quotient map $q_Z : Z \to Z/G$. Let $\varepsilon > 0$. Let $f : Z \to M$ and $F : Z \times I \to \mathbb{R}^n$ be the data for an ε-lifting problem: $F(z, t) = pf(z) \in \mathbb{R}^n \backslash \mathrm{B}^n_r$. Consider the induced ε-lifting problem, consisting of the continuous maps $\bar{f} : Z/G \to N$ and $\bar{F} : Z/G \times I \to \overset{\circ}{\mathrm{c}}(X)$ of quotient spaces such that $\bar{F}(w, t) = q_M \bar{f}(w) \in \overset{\circ}{\mathrm{c}}(X) \backslash \mathrm{c}_r(X)$. Since p/G is an ε-fibration over $\overset{\circ}{\mathrm{c}}(X) \backslash \mathrm{c}_r(X)$, there exists an ε-solution: $\bar{H} : Z/G \times I \to N$ such that $\bar{H}(w, 0) = \bar{f}(w)$ and $d(p/G \circ \bar{H}, \bar{F}) < \varepsilon$. Define a G-invariant map

$$ H := \bar{H}(q_Z \times \mathrm{id}_I) : Z \times I \longrightarrow N. $$

Note that $H(z, 0) = \bar{f} q_Z(z) = q_M f$ and that $p/G \circ H$ is ε-close to $p/G \circ \bar{F} = q_{\mathbb{R}^n} F$.

Now, since G acts freely on M implies that q_M is a covering map, by the Homotopy Lifting Property, there exists a unique map $\tilde{F} : Z \times I \to M$ such that $\tilde{F}(z, 0) = f(z)$ and $q_M \tilde{F} = H$. Note, for any $z \in Z$ and $\gamma \in G$, since f is G-equivariant and H is G-invariant, that both the paths $\gamma \tilde{F}(z, -)$ and $\tilde{F}(\gamma z, -)$ have common initial point $\gamma f(z) = f(\gamma z)$ and have common q_M-composition $H(z, -) = H(\gamma z, -)$. Thus, by the uniqueness property of path lifts in a covering space, we must have $\gamma \tilde{F}(z, t) = \tilde{F}(\gamma z, t)$ for all $\gamma \in G$, $z \in Z$, $t \in I$. Therefore $\tilde{F} : Z \times I \to M$ is a G-homotopy. Since $q_{\mathbb{R}^n}$ is distance non-increasing, note

$$ d(p\tilde{F}, F) \le d(q_{\mathbb{R}^n} p \tilde{F}, q_{\mathbb{R}^n} F) = d(p/G \circ q_M \tilde{F}, q_{\mathbb{R}^n} F) = d(p/G \circ H, q_{\mathbb{R}^n} F) < \varepsilon. $$

Thus $\tilde{F}$ is an ε-solution to the lifting problem given by f and F. Therefore p is an ε-fibration over $\mathbb{R}^n \backslash \overline{B}^n_r$, for all $\varepsilon > 0$. □

The proof of the following lemma follows immediately from the definition.

Lemma 6.5 *Let $r > 0$ and in the following commutative diagram*

$$
\begin{array}{ccccc}
M & \xrightarrow{\ q_M\ } & N & \xleftarrow{\ \text{incl}\ } & N' = g^{-1}(X \times (0, \infty)) \\
\ \downarrow p & & \ \downarrow p/G & & \ \downarrow p/G| \\
\mathbb{R}^n & \xrightarrow{\ q_{\mathbb{R}^n}\ } & \overset{\circ}{c}(X) & \xleftarrow{\ \text{incl}\ } & X \times (0, \infty)
\end{array}
$$

suppose that p is a proper G-map. If $p/G|$ is a MAF over $X \times (r, \infty)$, then p/G is a MAF over $\overset{\circ}{c}(X) \backslash c_r(X)$. □

Lemma 6.6 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that: if $p : M \to \mathbb{R}^n$ is a proper G-δ-fibration, then p is G-ε-homotopic to a map p' that is a G-MAF over $\mathbb{R}^n \backslash \bar{B}^n_\varepsilon$.*

Proof Let $\varepsilon > 0$ and $m > 4$ be given. By Lemma 4.5, there exists $0 < \varepsilon_0 \leq \varepsilon/4$ such that: if Z is a topological space and $F : Z \times I \to \mathbb{R}^n \backslash B^n_{\varepsilon/2}$ is a homotopy and $q_{\mathbb{R}^n} \circ F$ is an ε_0-homotopy, then F is an ε_0-homotopy. By Proposition 3.13, the open cone $\overset{\circ}{c}(X) = \mathbb{R}^n/G$ with the induced metric has finite isometry type. Then, since $X = S^{n-1}/G$ is a closed smooth manifold, by Proposition 3.16 and Corollary 3.34, there exists $\delta > 0$ such that: if P is an m-manifold and $f : P \to X \times (0, \infty)$ is a proper δ-fibration over $(\varepsilon/2, \infty)$, then f is ε_0-homotopic rel $f^{-1}(X \times (0, \varepsilon - \varepsilon_0])$ to a map f' that is a MAF over $X \times (\varepsilon, \infty)$.

Let $p : M \to \mathbb{R}^n$ be a proper G-δ-fibration. By Proposition 4.3, the induced map $g := p/G : N \to \overset{\circ}{c}(X)$ is a proper stratified δ-fibration. Consider the m-manifold $P := N \backslash g^{-1}\{v\}$ and $f := g|P : P \to X \times (0, \infty)$. Since f is a proper δ-fibration over $(\varepsilon/2, \infty)$, there exists an ε_0-homotopy $H : P \times I \to X \times (0, \infty)$ rel $f^{-1}(X \times (0, \varepsilon - \varepsilon_0])$ from f to a MAF f' over $X \times (\varepsilon, \infty)$. This extends uniquely to an ε_0-homotopy $\bar{H} : N \times I \to \overset{\circ}{c}(X)$ rel $g^{-1}(c_{\varepsilon - \varepsilon_0}(X))$ from g to some $g' := \bar{H}(-, 1)$. Note, by Lemma 6.5, that g' is a MAF over the frustum $\overset{\circ}{c}(X) \backslash c_\varepsilon(X)$.

By Lemma 4.4, there is a unique G-homotopy $\tilde{H} : M \times I \to \mathbb{R}^n$ such that $\tilde{H}(-, 0) = p$ and $q_{\mathbb{R}^n} \circ \tilde{H} = \bar{H} \circ (q_M \times \text{id}_I)$. Observe that $\tilde{H}$ restricts to a constant homotopy on $M_0 := p^{-1}(B^n_{\varepsilon - \varepsilon_0})$. Furthermore, there exists a restriction

$$\bar{H}| : (N \backslash q_M(M_0)) \times I \longrightarrow \overset{\circ}{c}(X) \backslash c_{\varepsilon - 2\varepsilon_0}(X) \subseteq \overset{\circ}{c}(X) \backslash c_{\varepsilon/2}(X).$$

Then the restriction $\tilde{H}| : (M \backslash M_0) \times I \to \mathbb{R}^n \backslash B^n_{\varepsilon/2}$ exists and is an ε_0-homotopy. Hence $\tilde{H} : M \times I \to \mathbb{R}^n$ is an G-ε-homotopy. Finally, consider the map $p' := \tilde{H}(-, 1) : M \to \mathbb{R}^n$. Since p' is a proper G-map covering g', by Lemma 6.4, we conclude that p' is a G-MAF over $\mathbb{R}^n \backslash B^n_\varepsilon$. □

Proposition 6.7 *For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying: if $p : M \to \mathbb{R}^n$ is a proper G-δ-fibration, then p is G-ε-homotopic to a map p' that is, for every $v > 0$, a proper G-(ε, v)-fibration over $(\mathbb{R}^n, \mathbb{R}^n \backslash B^n_\varepsilon)$ for the class of compact, metric G-spaces.*

Proof Let $\varepsilon > 0$ be given. Let $r_1 = \varepsilon/2$ and $r_2 = \varepsilon$. Let $\delta_1 > 0$ be given by Lemma 6.3 for ε, r_1, and r_2. Let $\delta_2 > 0$ be given by Lemma 6.6 for $\min(\varepsilon/2, \delta_1/2)$.

Let $\delta = \min(\delta_2, \delta_1/2)$. If $p : M \to \mathbb{R}^n$ is a proper G-δ-fibration, then Lemma 6.6 implies that p is G-$\delta_1/2$-homotopic to a map p', where p' is a G-MAF over $\mathbb{R}^n \backslash B^n_{\varepsilon/2}$. Now p' is a G-δ_1-fibration. It follows from Lemma 6.3 that, for every $v > 0$, the map p' is a G-(ε, v)-fibration over $(\mathbb{R}^n, \mathbb{R}^n \backslash B^n_\varepsilon)$ for the class of compact, metric G-spaces. □

Definition 6.8 Let $r > 0$. The **radial crush with parameter** r is the G-map

$$\rho_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \quad x \longmapsto \begin{cases} (1 - \frac{r}{\|x\|})x & \text{if } r < \|x\| \\ 0 & \text{if } \|x\| \leq r. \end{cases}$$

Lemma 6.9 *Let $r > 0$.*

1. *There is a G-r-homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to ρ_r.*
2. *The map ρ_r is distance non-increasing.*
3. *The map ρ_r is a G-MAF.*

Proof Part (1) is given by the straight-line homotopy

$$H : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n; \quad (x, t) \longmapsto \begin{cases} (1 - \frac{tr}{\|x\|})x & \text{if } r < \|x\| \\ (1 - t)x & \text{if } \|x\| \leq r. \end{cases}$$

This is easily checked to be a G-r-homotopy.

Part (2) follows from the calculation

$$\|\rho_r(x) - \rho_r(y)\|^2 = \begin{cases} 0 & \text{if } \|x\|, \|y\| \leq r \\ (\|y\| - r)^2 & \text{if } \|x\| \leq r \leq \|y\| \\ \|x - y\|^2 - 2r(1 - \cos\theta)(\|x\| + \|y\| - r) & \text{if } r \leq \|x\|, \|y\| \end{cases}$$
$$\leq \|x - y\|^2.$$

The inequality in the second case follows from the Triangle Inequality. The equality in the third case follows from the Law of Cosines, where $\cos\theta = \frac{\langle x, y \rangle}{\|x\|\|y\|}$.

For Part (3), consider the above straight-line homotopy $H = \{H_t : \mathbb{R}^n \to \mathbb{R}^n\}_{t \in I}$. Observe, for each $t \in [0, 1)$, that H_t is a G-homeomorphism. Moreover, for each $t \in [0, 1)$, note, for all $y \in \mathbb{R}^n$, that

$$\|y - \rho_r H_t^{-1}(y)\| = \begin{cases} (1 - t)r & \text{if } (1 - t)r < \|y\| \\ \|y\| & \text{if } \|y\| \leq (1 - t)r \end{cases}$$

and hence, for all $x \in \mathbb{R}^n$, that

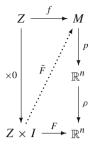$$\|\rho_r(x) - \rho_r H_t^{-1}\rho_r(x)\| = \begin{cases} (1 - t)r & \text{if } r < \|x\| \\ 0 & \text{if } \|x\| \leq r. \end{cases}$$

Let $\varepsilon > 0$. Select $T \in [0, 1)$ such that $(1 - T)r < \varepsilon$. Then, the straight-line homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to $\rho_r H_T^{-1}$ is a G-ε-homotopy, and the straight-line homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to $H_T^{-1}\rho_r$ is a G-ε-homotopy, when measured in the target $\mathbb{R}^n$ using ρ_r. Hence $\rho_r : \mathbb{R}^n \to \mathbb{R}^n$ is a G-ε-homotopy equivalence. Therefore, by (b) $\implies$ (c) in [26, Theorem 3.4], we conclude that ρ_r is a G-approximate fibration. □

Lemma 6.10 *Let $a, \varepsilon > 0$. Suppose, for all $\nu > 0$, that $p : M \to \mathbb{R}^n$ is a proper G-(ε, ν)-fibration over $(\mathbb{R}^n, \mathbb{R}^n \setminus B_a^n)$ for the class of compact, metric G-spaces. If $\rho = \rho_{a+\varepsilon} : \mathbb{R}^n \to \mathbb{R}^n$ is the radial crush with parameter $a + \varepsilon$, then the composite $\rho \circ p : M \to \mathbb{R}^n$ is a G-MAF.*

Proof Let Z be a compact, metric G-space. Let $\mu > 0$. Suppose the following diagram is a G-homotopy lifting problem:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & M \\
\times 0 \downarrow & \tilde{F} \nearrow \ \ \mathbb{R}^n & \downarrow p \\
& & \downarrow \rho \\
Z \times I & \xrightarrow{\ F\ } & \mathbb{R}^n
\end{array}
$$

Then the following is also a G-homotopy lifting problem:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ pf\ } & \mathbb{R}^n \\
\times 0 \downarrow & \hat{F} \nearrow & \downarrow \rho \\
Z \times I & \xrightarrow{\ F\ } & \mathbb{R}^n
\end{array}
$$

By Lemma 6.9(3) this latter problem has a G-$\mu/2$-solution $\hat{F}\colon Z \times I \to \mathbb{R}^n$. It follows that

$$
\begin{array}{ccc}
Z & \xrightarrow{\ f\ } & M \\
\times 0 \downarrow & \tilde{F} \nearrow & \downarrow p \\
Z \times I & \xrightarrow{\ \hat{F}\ } & \mathbb{R}^n
\end{array}
$$

is also a G-homotopy lifting problem, which by hypothesis has a G-$(\varepsilon, \mu/2)$-solution $\tilde{F}\colon Z \times I \to M$ over $(\mathbb{R}^n, \mathbb{R}^n \backslash B_a^n)$. We show that $\rho p \tilde{F}$ is μ-close to F, as follows.

Let $(z, t) \in Z \times I$. There are two cases to consider. First suppose $\hat{F}(z, t) \in \mathbb{R}^n \backslash B_a^n$. Then $p\tilde{F}(z, t)$ is $\mu/2$-close to $\hat{F}(z, t)$ because $\tilde{F}$ is a $(\varepsilon, \mu/2)$-solution. Therefore, $\rho p \tilde{F}(z, t)$ is $\mu/2$-close to $\rho \hat{F}(z, t)$ because ρ is distance non-increasing by Lemma 6.9(2). Also $\rho \hat{F}(z, t)$ is $\mu/2$-close to $F(z, t)$ because $\hat{F}$ is a $\mu/2$-solution. Thus, $\rho p \tilde{F}(z, t)$ is μ-close to $F(z, t)$. Second suppose $\hat{F}(z, t) \in B_a^n$. In particular, $\rho \hat{F}(z, t) = 0$. Then $p\tilde{F}(z, t)$ is ε-close to $\hat{F}(z, t)$ because $\tilde{F}$ is an ε-solution. Therefore, $p\tilde{F}(z, t) \in B_{a+\varepsilon}^n$ and so $\rho p \tilde{F}(z, t) = 0$. Also $F(z, t)$ is $\mu/2$-close to $\rho \hat{F}(z, t) = 0$ because $\hat{F}$ is a $\mu/2$-solution. Thus, $F(z, t)$ is $\mu/2$-close to $\rho p \tilde{F}(z, t)$. Hence $\rho p \tilde{F}$ is μ-close to F. In any case, we have shown $\rho p \tilde{F}$ is μ-close to F.

Thus $\rho p \colon M \to \mathbb{R}^n$ is an approximate G-fibration for the class of compact, metric G-spaces. Therefore, by a result of S. Prassidis [26, Proposition 2.18], we conclude that ρp is an approximate G-fibration for the class of all G-spaces. □

Proof (*Proof of Theorem 6.1*) Given $\varepsilon > 0$, let δ be given by Proposition 6.7 for $\varepsilon/3$. It follows that p is G-$\varepsilon/3$-homotopic to a map $p'\colon M \to \mathbb{R}^n$ that is, for all $\nu > 0$, a proper G-$(\varepsilon/3, \nu)$-fibration over $(\mathbb{R}^n, \mathbb{R}^n \backslash B_{\varepsilon/3}^n)$ for the class of compact, metric G-spaces. Let $\rho = \rho_{2\varepsilon/3}\colon \mathbb{R}^n \to \mathbb{R}^n$, the radial crush with parameter $2\varepsilon/3$. Lemma 6.10 implies that $\rho p'\colon M \to \mathbb{R}^n$ is a G-MAF. Lemma 6.9(1) implies that there is a G-$2\varepsilon/3$-homotopy from $\mathrm{id}_{\mathbb{R}^n}$ to ρ. Therefore, p is G-ε-homotopic to $\rho p'$, a G-MAF. □

7 Bounded fibrations from discontinuous actions

First, we show certain Γ-spaces W admit an equivariant map to the real line.

Proposition 7.1 *Let W be a connected topological manifold equipped with a free, discontinuous C_∞-action (resp. with a free, discontinuous D_∞-action). Then there exists a C_∞-map (resp. D_∞-map) $\overline{f} : W \to \mathbb{R}$. Hence $\overline{f}$ is the infinite cyclic cover of a map (resp. C_2-map) $f : M \to S^1$ of topological manifolds.*

Proof Consider the infinite group $\Gamma := C_\infty$ (resp. $\Gamma := D_\infty$). Since the Γ-action on W is free and discontinuous, the quotient map $q : W \to N := W/\Gamma$ is a regular covering map. Since N is a connected topological manifold, by [24, Essay III, Theorem 4.1.3], there exists a (locally finite) simplicial complex N' and a homotopy equivalence $h : N' \to N$. Then the pullback $W' := h^*(W)$ has a canonical map $q' : W' \to N'$. Since q is a covering map, we have that q' is a covering map. Then W' has an induced simplicial structure and free, simplicial Γ-action. By replacing the simplicial structure on W with the first barycentric subdivision, we obtain that W' is a free Γ-CW-complex. Furthermore, by covering space theory, the canonical map $g : W' \to W$ (satisfying $q \circ g = h \circ q'$) is a Γ-homotopy equivalence. Then there exists a Γ-map $g' : W \to W'$ that is a Γ-homotopy inverse to g. Since W' is a Γ-CW-complex with isotropy in the family fin of finite subgroups of Γ, by [29, Theorem I.6.6], there exists a Γ-map $c : W' \to E_{fin}\Gamma = \mathbb{R}$. Now define the desired Γ-map by $\overline{f} := c \circ g' : W \longrightarrow \mathbb{R}$. $\qquad\square$

The main theorem herein shows that C_2-bands unwrap to C_2-bounded fibrations.

Definition 7.2 A **manifold band** (M, f) consists of a closed, connected topological manifold M and a continuous map $f : M \to S^1$ such that the induced infinite cyclic cover $\overline{M} := f^*(\mathbb{R})$ is connected and finitely dominated (cf. [16, Defn. 15.3]). Let the circle S^1 have the C_2-action generated by complex conjugation $R : z \mapsto \overline{z}$. A C_2-**manifold band** (M, f, R) consists of a manifold band (M, f) and a C_2-action on M, generated by a homeomorphism $R : M \to M$, such that the continuous map $f : M \to S^1$ is C_2-equivariant.

We do not assume that the C_2-action on M is free.

Definition 7.3 Let (M, f, R) be a C_2-manifold band. The **infinite cyclic cover** $(\overline{M}, \overline{f}, R)$ is defined by the pullback diagram

$$
\begin{array}{ccc}
\overline{M} & \xrightarrow{\ \overline{f}\ } & \mathbb{R} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \exp} \\
M & \xrightarrow{\ f\ } & S^1
\end{array}
$$

and the diagonal C_2-action

$$R : \overline{M} \longrightarrow \overline{M}; \quad (x, t) \longmapsto (Rx, -t).$$

The covering translation is defined by

$$T : \overline{M} \longrightarrow \overline{M}; \quad (x, t) \longmapsto (x, t + 1)$$

and satisfies the dihedral relation $RT = T^{-1}R$. Note that all the maps in the diagram are proper and C_2-equivariant, and moreover that $\overline{f}$ is D_∞-equivariant. Observe that (M, f) is a connected manifold band if and only if $\overline{M}$ is a connected, finitely dominated manifold.

Example 7.4 Let $n > 0$. An example of a free C_2-manifold band is

$$(S^n \times S^1, \text{proj}_{S^1}, (x, z) \longmapsto (-x, \overline{z})) \, .$$

The following infinite cyclic cover possesses a free D_∞-action:

$$(S^n \times \mathbb{R}, \text{proj}_{\mathbb{R}}, (x, t) \longmapsto (-x, -t)) \, .$$

The following statements are equivariant versions of [16, Props. 17.13, 17.14].

Lemma 7.5 *Let* (M, f, R) *be a* C_2-*manifold band. Then there exists a* C_2-*homotopy*

$$\{K_s : \overline{M} \times \mathbb{R} \to \overline{M} \times \mathbb{R}\}_{s \in I},$$

called a C_2-**sliding domination***, and a constant* $N > 1$ *such that:*

1. $K_0 = \text{id}_{\overline{M} \times \mathbb{R}}$,
2. $\text{proj}_{\mathbb{R}} \circ K_s = \text{proj}_{\mathbb{R}}$ *for all* $s \in I$,
3. $K_s | \text{Graph}(\overline{f} : \overline{M} \to \mathbb{R})$ *is the inclusion for all* $s \in I$, *and*
4. $K_1(\overline{M} \times \mathbb{R}) \subseteq \{(x, t) \mid |\overline{f}(x) - t| \leq N\}$.

We perform a folding trick to construct K indirectly.

Proof By [16, Proposition 17.13], there exists a homotopy

$$\{K'_s : \overline{M} \times \mathbb{R} \to \overline{M} \times \mathbb{R}\}_{s \in I},$$

called a **sliding domination**, and a constant $N > 1$ satisfying Properties (1)–(4). Then, by Property (3) for K', there is a well-defined, continuous C_2-map

$$K : I \times \overline{M} \times \mathbb{R} \longrightarrow \overline{M} \times \mathbb{R}; \quad (s, x, t) \longmapsto \begin{cases} K'_s(x, t) & \text{if } \overline{f}(x) \leq t \\ R K'_s(x, t) & \text{if } \overline{f}(x) \geq t. \end{cases}$$

One checks easily that K and N satisfy Properties (1)–(4). $\qquad \square$

Theorem 7.6 *Let* (M, f, R) *be a* C_2-*manifold band. Then the proper map* $\overline{f} : \overline{M} \to \mathbb{R}$ *is a* C_2-*bounded fibration.*

Proof We show that $\overline{f}$ is an $(C_2, N + \varepsilon)$-fibration for any $\varepsilon > 0$. Consider the C_2-sliding domination K with constant $N > 1$ of Lemma 7.5. Let $g : Z \to \overline{M}$ be a C_2-map and $G : Z \times I \to \mathbb{R}$ be a C_2-homotopy such that $G(z, 0) = \overline{f}g(z)$. First, define a C_2-homotopy

$$\hat{G} : Z \times I \longrightarrow \overline{M} \times \mathbb{R}; \quad (z, t) \longmapsto (g(z), G(z, t)).$$

Next, define a C_2-homotopy

$$\tilde{G} : Z \times I \longrightarrow \overline{M}; \quad \tilde{G} := \text{proj}_{\overline{M}} \circ K_1 \circ \hat{G}.$$

Note Lemma 7.5(2) implies $K_1 \hat{G} = (\tilde{G}, G)$. Then Lemma 7.5(4) implies $d(\overline{f}\tilde{G}, G) \leq N$. Therefore $\overline{f} : \overline{M} \to \mathbb{R}$ is a C_2-bounded fibration. $\qquad \square$

8 Dihedral wrapping up over the real line

The main theorem herein constructs a dihedral covering translation. The non-C_2-version is given in [16, Theorem 17.1], and we modify some of its techniques.

Theorem 8.1 *Let W is a topological manifold of dimension > 4 and equipped with a free C_2-action generated by $R: W \to W$. Suppose $p: W \to \mathbb{R}$ is a C_2-MAF.*

1. *There exists a cocompact, free, discontinuous D_∞-action on W extending the C_2-action.*
2. *The C_2-MAF $p: W \to \mathbb{R}$ is properly C_2-homotopic to a D_∞-MAF $\tilde{p}: W \to \mathbb{R}$.*

We shall construct a covering translation $J_1: W \to W$ satisfying the dihedral relation $R J_1 = J_1^{-1} R$. The basic technique is to cut, fill in, and paste embeddings.

Proof (Proof of Theorem 8.1(1))] First, we decompose W and obtain a certain isotopy G. Consider a decomposition $W = C_- \cup_{\partial_- B} B \cup_{\partial_+ B} C_+$ into closed subsets:

$$B := p^{-1}[-\mu, \mu], \quad \partial_- B := p^{-1}\{-\mu\}, \quad \partial_+ B := p^{-1}\{\mu\},$$
$$C_- := p^{-1}(-\infty, -\mu], \quad C_+ := p^{-1}[\mu, \infty).$$

Also consider the auxiliary compact subsets

$$K := p^{-1}[-1, -\mu], \quad \partial^- K := p^{-1}\{-1\}.$$

Define an isotopy $g: I \times \mathbb{R} \to \mathbb{R}$ from the identity:

$$\{g_s: \mathbb{R} \longrightarrow \mathbb{R}; \quad t \longmapsto s + t\}_{s \in I}.$$

Select $0 < \mu < \frac{3}{7}$. Then, by Corollary 2.8, there exists an isotopy $G: I \times W \to W$ of homeomorphisms such that $G_0 = \mathrm{id}_W$ and pG_s is $\mu/3$-close to $g_s p$ for all $s \in I$.

Second, we construct the homeomorphism $J_1: W \to W$ in four steps, as follows. In fact, we shall construct an isotopy $J: I \times W \to W$ of homeomorphisms from $J_0 = \mathrm{id}_W$ to the desired J_1 such that pJ_s is μ-close to $g_s p$.

1. Define an isotopy H of embeddings from the inclusion:

$$\left\{ H_s := RG_s^{-1}R|K \cup G_s|C_+ : K \cup C_+ \longrightarrow W \right\}_{s \in I}.$$

   Indeed each H_s is injective, since $RG_s^{-1}R(K) \cap G_s(C_+) = \varnothing$.
2. Since $K \cup \partial_+ B$ is compact and $H|K \cup \partial_+ B$ is the restriction of a proper isotopy of a neighborhood of $K \cup \partial_+ B$ in W, by the Isotopy Extension Theorem of Edwards–Kirby [9, Cor. 1.2], we obtain that $H|K \cup \partial_+ B$ extends to an isotopy $\bar{H}$ from the identity:

$$\left\{ \bar{H}_s: W \longrightarrow W \right\}_{s \in I}.$$

   Then $H|K \cup C_+$ extends to an isotopy of embeddings from the inclusion:

$$\left\{ J_s^+ := H_s|K \cup \bar{H}_s|B \cup H_s|C_+ : K \cup B \cup C_+ \longrightarrow W \right\}_{s \in I}.$$

   Since each $\bar{H}_s(B)$ is the unique compact subset of W with frontier $H_s(\partial_- B \cup \partial_+ B)$, we obtain $\bar{H}_s(\mathrm{int}\, B) \cap H_s(K \cup C_+) = \varnothing$. So J_s^+ is indeed injective.
3. Since $H_s(\partial_- K) \subset B$, we obtain $C_+ \subset J_s^+(K \cup B \cup C_+)$. Then we can define an isotopy J^- of embeddings from the inclusion:

$$\left\{ J_s^- := R(J_s^+)^{-1}R|C_- : C_- \longrightarrow W \right\}_{s \in I}.$$

4. Note $J_s^- | K = RG_s^{-1}R | K = J_s^+ | K$. Therefore, we can define an isotopy J of homeomorphisms from id_W as a union:

$$\{J_s := J_s^- | C_- \cup J_s^+ | (B \cup C_+) : W \longrightarrow W\}_{s \in I} .$$

Third, we verify that pJ_s is μ-close to $g_s p$ for all $s \in I$. Note, for all $x \in C_-$, writing $y := (J_s^+)^{-1}(Rx)$, that $|pJ_s^-(x) - g_s p(x)| = |pR(J_s^+)^{-1}R(x) - Rg_s^{-1}Rp(x)| = |g_s p(y) - pJ_s^+(y)|$. Hence it suffices to verify that pJ_s^+ is μ-close to $g_s p$. If $x \in C_+$ then $|pJ_s^+(x) - g_s p(x)| = |pG_s(x) - g_s p(x)| < \mu/3$. If $x \in K$, writing $y := G_s^{-1}(Rx)$, then note $|pJ_s^+(x) - g_s p(x)| = |pRG_s^{-1}R(x) - Rg_s^{-1}Rp(x)| = |g_s p(y) - pG_s(y)| < \mu/3$. Otherwise, suppose $x \in B$. Since $\bar{H}$ is an isotopy, we have $s - 2\mu/3 < \inf pRG_s^{-1}R(\partial_- B) \leq p\bar{H}_s(x) \leq \sup pG_s(\partial_+ B) < s + 2\mu/3$. Then $|pJ_s^+(x) - g_s p(x)| = |p\bar{H}_s(x) - g_s p(x)| \leq |g_s^{-1}p\bar{H}_s(x)| + |p(x)| < 2\mu/3 + \mu/3 = \mu$. Therefore pJ_s is μ-close to $g_s p$ for all $s \in I$.

Fourth, we verify that J_1 satisfies the dihedral relation. On the one hand, recall $C_+ \subset J_s^+(K \cup B \cup C_+)$ and $J_s^- | K = J_s^+ | K$. On the other hand, observe $\mu/3 < \frac{1}{7}$ implies $J_1^+(\partial_- B) \subset \mathrm{int}\, C_+$ hence $C_- \cup B \subset J_1(C_-)$. Then

$$J_1^{-1}R | C_- = (J_1^+)^{-1}R | C_- = RJ_1^- | C_- = RJ_1 | C_-$$
$$J_1^{-1}R | (B \cup C_+) = (J_1^-)^{-1}R | (B \cup C_+) = RJ_1^+ | (B \cup C_+) = RJ_1 | (B \cup C_+).$$

Therefore the dihedral relation $J_1^{-1}R = RJ_1$ holds. (In fact, the relation $J_s^{-1}R = RJ_s$ holds for all $s \in [7\mu/3, 1]$.) Hence $\{R, J_1\}$ generates a D_∞-action on W.

Finally, we verify that the D_∞-action is cocompact. Let $x \in W$. Define a neighborhood $U := p^{-1}(p(x) - \mu, p(x) + \mu)$ of x in W. Since pJ_1 is μ-close to $g_1 p$ and $\mu < \frac{1}{2}$, we have $pJ_1(U) \cap p(U) = \varnothing$. Thus the C_∞-action generated by J_1 is free and discontinuous. That is, the quotient map $W \to W/J_1$ is a covering map. Furthermore, define a $J_1 R$-invariant subset

$$V := p^{-1}[0, \infty) \cap J_1 p^{-1}(-\infty, 0]$$

with frontier $\partial V := p^{-1}\{0\} \sqcup J_1 p^{-1}\{0\}$ in W. Since pJ_1 is μ-close to $g_1 p$, it follows that V is compact. Note $W = \bigcup_{n \in \mathbb{Z}}(J_1)^n(V)$ and $(J_1)^n(V) \cap (J_1)^m(V) \subset \partial V$ for all $m, n \in \mathbb{Z}$. Therefore V is a fundamental domain for J_1 and W/J_1 is compact. □

We shall construct a D_∞-MAF $\tilde{p} : W \to \mathbb{R}$ according to the following outline: (1) use Urysohn functions to interpolate crudely between p and $g_1 pJ_1^{-1}$, (2) use relative sucking to sharpen the bounded fibration to an approximate fibration, (3) use a crushing map to force C_2-equivariance about $\frac{1}{2}$, and (4) use the fundamental domain V to copy-and-paste across W to force C_∞-equivariance.

Proof (*Proof of Theorem 8.1(2)*) It suffices to construct the desired D_∞-MAF $\tilde{p} : W \to \mathbb{R}$, since the straight-line homotopy is a proper C_2-homotopy from p to $\tilde{p}$:

$$H : I \times W \longrightarrow \mathbb{R}; \quad (s, x) \longmapsto (1 - s)p(x) + s\tilde{p}(x).$$

First, by the relative sucking principle (Proposition 2.6), we may re-select $0 < \mu < \frac{1}{48}$ so that: if $f : W \to \mathbb{R}$ is a proper 2μ-fibration which is an approximate fibration over $(-\infty, \frac{7}{24}) \cup (\frac{17}{24}, \infty)$, then f is homotopic rel $f^{-1}((-\infty, \frac{1}{4}) \cup (\frac{3}{4}, \infty))$ to an approximate fibration $f' : W \to \mathbb{R}$. Then, since $\mu < \frac{1}{3}$, by the Urysohn lemma, there exists a map $u : W \to I$ such that $p^{-1}(-\infty, \frac{1}{3}] \subseteq u^{-1}\{0\}$ and $J_1 p^{-1}[-\frac{1}{3}, \infty) \subseteq u^{-1}\{1\}$. Define a proper map

$$p' : W \longrightarrow \mathbb{R}; \quad x \longmapsto u(x) + pJ_{u(x)}^{-1}(x).$$

Second, note $|p'(x) - p(x)| = |g_s p(y) - p J_s(y)| < \mu$ for all $x \in W$, where we abbreviate $s := u(x)$ and $y := J_s^{-1}(x)$. Since p is a μ-fibration, we conclude that p' is a 2μ-fibration. Furthermore, since $\mu < \frac{1}{24}$ implies $(p')^{-1}(-\infty, \frac{7}{24}) \subset p^{-1}(-\infty, \frac{1}{3}]$, we obtain that p' restricts to the approximate fibration p over $(-\infty, \frac{7}{24})$. Moreover, since $\mu < \frac{1}{48}$ implies $(p')^{-1}(\frac{17}{24}, \infty) \subset J_1 p^{-1}[-\frac{1}{3}, \infty)$, we obtain that p' restricts to the approximate fibration $g_1 p J_1^{-1}$ over $(\frac{17}{24}, \infty)$. Therefore p' is homotopic rel $(p')^{-1}((-\infty, \frac{1}{4}) \cup (\frac{3}{4}, \infty))$ to an approximate fibration $p'' : W \to \mathbb{R}$.

Third, consider a proper cell-like map

$$\kappa : \mathbb{R} \longrightarrow \mathbb{R}; \quad t \longmapsto \begin{cases} 2t & \text{if } t \in [0, \frac{1}{4}] \\ \frac{1}{2} & \text{if } t \in [\frac{1}{4}, \frac{3}{4}] \\ 2t + 1 & \text{if } t \in [\frac{3}{4}, 1] \\ t & \text{if } t \in (-\infty, 0] \cup [1, \infty). \end{cases}$$

By the composition principle [7, p. 38], we obtain that $p''' := \kappa \circ p'' : W \to \mathbb{R}$ is also an approximate fibration. Furthermore, since $pR = Rp$ and $J_1 R = R J_1^{-1}$, note $p''' \circ J_1 R = g_1 R \circ p'''$; roughly speaking, p''' is C_2-equivariant about $\frac{1}{2}$.

Fourth, recall that $V = p^{-1}[0, \infty) \cap J_1 p^{-1}(-\infty, 0]$. It follows that $p'''|p^{-1}\{0\} = p|$ and $p'''|J_1 p^{-1}\{0\} = g_1 p J_1^{-1}|$. Then we can define a proper map

$$\tilde{p} : W \longrightarrow \mathbb{R}; \quad x \longmapsto g_1^m p''' J_1^{-m}(x) \text{ where } x \in J_1^m(V).$$

Since $\tilde{p} \circ J_1 = g_1 \circ \tilde{p}$ and $\tilde{p} \circ J_1 R = g_1 R \circ \tilde{p}$, we obtain that $\tilde{p}$ is a proper D_∞-map. Furthermore, by the uniformization principle [7, p. 43], we obtain that $\tilde{p}$ is an approximate fibration. Therefore, by Lemma 8.2 (see below), we conclude that $\tilde{p}$ is in fact a D_∞-MAF. □

Lemma 8.2 *Let W be a topological manifold equipped with a free, discontinuous D_∞-action. Suppose $\tilde{p} : W \to \mathbb{R}$ is a proper D_∞-map and an approximate fibration. Then $\tilde{p}$ is a D_∞-MAF.*

Proof First, we show that the induced C_2-map $\tilde{p}/J_1 : W/J_1 \to \mathbb{R}/\mathbb{Z} = S^1$ is indeed a C_2-MAF. By Coram–Duvall's uniformization principle [7, p. 43], we conclude that $\tilde{p}/J_1 : W/J_1 \to S^1$ is an approximate fibration. Since C_2 acts freely on W, that the fixed-set restriction $(\tilde{p}/J_1)^R : (W/J_1)^R = \varnothing \to (S^1)^R = \{1, -1\}$ is trivially an approximate fibration. By Jaworowski's recognition principle [23, Theorem 2.1], both the finite-dimensional spaces W/J_1 and S^1 are C_2-ENRs, hence they are C_2-ANRs for the class of separable metric spaces. Therefore, by Prassidis's recognition principle [26, Theorem 3.1], we conclude that $\tilde{p}/J_1$ is a C_2-approximate fibration.

Now, we show, by an elementary argument, that $\tilde{p} : W \to \mathbb{R}$ is a D_∞-MAF. Let Z be a D_∞-space, and let $0 < \varepsilon < \frac{1}{2}$. Let $F : Z \times I \to \mathbb{R}$ be a D_∞-homotopy. Let $f : Z \to W$ be a D_∞-map such that $\tilde{p}f = F(-, 0)$. Since $\tilde{p}/J_1 : W/J_1 \to S^1$ is a C_2-ε-fibration, there exists a C_2-homotopy $\widetilde{F/J_1} : Z/J_1 \times I \to W/J_1$ such that $f/J_1 = (\widetilde{F/J_1})(-, 0)$ and $(\tilde{p}/J_1)(\widetilde{F/J_1})$ is ε-close to F/J_1. Furthermore, consider the normalization map

$$\Theta : \mathbb{R}^2 \backslash \{0\} \longrightarrow S^1; \quad y \longmapsto y/\|y\|.$$

Since $\varepsilon < \frac{1}{2}$, there is the unique great-circle C_2-ε-homotopy

$$H : I \times (Z/J_1 \times I) \longrightarrow S^1 \subset \mathbb{R}^2;$$
$$(s, x) \longmapsto \Theta \left((1-s)(\tilde{p}/J_1)(\widetilde{F/J_1})(x) + s(F/J_1)(x) \right).$$

Since $W \to W/D_\infty$ is a covering map, there exists a unique D_∞-homotopy $\tilde{F} : Z \times I \to W$ such that $f = \tilde{F}(-, 0)$ and $\tilde{F}/J_1 = \widehat{F/J_1}$. Then, since $\mathbb{R} \to S^1$ is a covering map, there exists a unique C_∞-homotopy $\tilde{H} : I \times (Z \times I) \to \mathbb{R}$ such that $\tilde{p}\tilde{F} = \tilde{H}(0, -)$ and $\tilde{p}f = \tilde{H}(-, (-, 0))$ and $F = \tilde{H}(1, -)$ and $\tilde{H}/J_1 = H$. Furthermore, since H is an ε-homotopy and the quotient map $\mathbb{R} \to S^1$ is a local isometry, we conclude that $\tilde{H}$ is an ε-homotopy. Hence $\tilde{p}\tilde{F}$ is ε-close to F. Thus $\tilde{p}$ is a D_∞-ε-fibration for all $0 < \varepsilon < \frac{1}{2}$. Therefore $\tilde{p}$ is a D_∞-MAF. $\square$

Example 8.3 Finally, we illustrate why the proper C_2-homotopy in Part (2) cannot be improved to a bounded C_2-homotopy from p to $\tilde{p}$. Let $m \neq 0, 1$ be a real number. Define a homeomorphism

$$p : \mathbb{R} \longrightarrow \mathbb{R}; \quad x \longmapsto mx.$$

Clearly p is a C_2-MAF. Assume that p is boundedly C_2-homotopic to a D_∞-MAF $\tilde{p} : \mathbb{R} \to \mathbb{R}$. In particular, p is ε-close to a C_∞-map $\tilde{p}$ for some $\varepsilon > 0$. Then an elementary argument shows that $\tilde{p}$ is μ-close to the C_∞-map

$$q : \mathbb{R} \longrightarrow \mathbb{R}; \quad x \longmapsto \tilde{p}(0) + x$$

for any $\mu > \sup_{x \in [0,1]} |\tilde{p}(x) - q(x)|$. Hence p is $(\varepsilon + \mu)$-close to q, a contradiction.

References

1. Beshears, A.: *G*-isovariant structure sets and stratified structure sets. Dissertation. Vanderbilt University, (1997)
2. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (1999)
3. Chapman, T.A.: Approximation results in Hilbert cube manifolds. Trans. Am. Math. Soc. **262**(2), 303–334 (1980)
4. Chapman, T.A.: Approximation results in topological manifolds. Mem. Am. Math. Soc. **34**(251), iii+64 (1981)
5. Chapman, T.A., Ferry, S.: Approximating homotopy equivalences by homeomorphisms. Am. J. Math. **101**(3), 583–607 (1979)
6. Coram, D., Duvall, P.: Approximate fibrations and a movability condition for maps. Pac. J. Math. **72**(1), 41–56 (1977)
7. Coram, D.S.: Approximate fibrations—a geometric perspective. In: Shape theory and geometric topology (Dubrovnik, 1981), Volume 870 of Lecture Notes in Math., pp. 37–47. Springer, Berlin, (1981)
8. Coram, D.S., Duvall, P.F. Jr.: Approximate fibrations. Rocky Mt. J. Math. **7**(2), 275–288 (1977)
9. Edwards, R.D., Kirby, R.C.: Deformations of spaces of imbeddings. Ann. Math. (2) **93**, 63–88 (1971)
10. Farkas, D.R.: Crystallographic groups and their mathematics. Rocky Mt. J. Math. **11**(4), 511–551 (1981)
11. Farrell F.T.: The obstruction to fibering a manifold over a circle. In: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 69–72. Gauthier-Villars, Paris (1971)
12. Farrell, F.T.: The obstruction to fibering a manifold over a circle. Indiana Univ. Math. J. **21**, 315–346, (1971/1972)
13. Ferry, S.: Approximate fibrations over S^1. Hand-written manuscript
14. Hughes, B.: Products and adjunctions of manifold stratified spaces. Topol. Appl. **124**(1), 47–67 (2002)
15. Hughes, B., Prassidis, S.: Control and relaxation over the circle. Mem. Am. Math. Soc. **145**(691), x+96 (2000)

16. Hughes, B., Ranicki, A.: Ends of complexes, volume 123 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1996)
17. Hughes, C.B.: Local homotopy properties in spaces of approximate fibrations. Dissertation. University of Kentucky, (1981)
18. Hughes, C.B.: Approximate fibrations on topological manifolds. Mich. Math. J. **32**(2), 167–183 (1985)
19. Hughes, C.B.: Spaces of approximate fibrations on Hilbert cube manifolds. Compositio Math. **56**(2), 131–151 (1985)
20. Hughes, C.B., Taylor, L.R., Williams, E.B.: Bundle theories for topological manifolds. Trans. Am. Math. Soc. **319**(1), 1–65 (1990)
21. Hughes, C.B., Taylor, L.R., Williams, E.B.: Bounded homeomorphisms over Hadamard manifolds. Math. Scand. **73**(2), 161–176 (1993)
22. Illman, S.: Smooth equivariant triangulations of G-manifolds for G a finite group. Math. Ann. **233**(3), 199–220 (1978)
23. Jaworowski, J.W.: Extensions of G-maps and Euclidean G-retracts. Math. Z. **146**(2), 143–148 (1976)
24. Kirby, R.C., Siebenmann, L.C.: Foundational essays on topological manifolds, smoothings, and triangulations. Princeton University Press, Princeton, NJ, (1977). With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88
25. Mardešić, S., Segal, J.: Shape theory, volume 26 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, (1982). The inverse system approach
26. Prassidis, S.: Equivariant approximate fibrations. Forum Math. **7**(6), 755–779 (1995)
27. Ratcliffe, J.G.: Foundations of hyperbolic manifolds, volume 149 of Graduate Texts in Mathematics. 2nd edn. Springer, New York (2006)
28. Siebenmann, L.C.: A total Whitehead torsion obstruction to fibering over the circle. Comment. Math. Helv. **45**, 1–48 (1970)
29. tom Dieck, T.: Transformation groups, volume 8 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co, Berlin (1987)