ERRATUM TO: ON CONNECTED SUMS OF REAL PROJECTIVE SPACES

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1. The error and the downgrade

In the author's thesis, [Kha06, Theorem 3.1.3] is incorrect as stated and proven.

Theorem. *Let $f : R \longrightarrow R'$ be a morphism of rings with involution such that the induced map $f_* : \widehat{H}^j(R) \longrightarrow \widehat{H}^j(R')$ is an isomorphism (implicitly, of R-modules) for both $j = 0, 1$. Suppose R satisfies (2.4.1.1) and that R' is a flat right R-module. Then for all $n \in \mathbb{Z}$, the induced map $f_* : NQ_n(R) \longrightarrow NQ_n(R')$ is an isomorphism.*

Recall the 2-periodic j-th Tate cohomology of the cyclic group C_2 of order two:

$$\widehat{H}^j(R) \ := \ \widehat{H}^j(\mathsf{C}_2; R) \ = \ \frac{\{a \in R \mid a = \epsilon\bar{a}\}}{\{b + \epsilon\bar{b} \mid b \in R\}}$$

with coefficients in the additive group of the ring R, where $\epsilon := (-1)^j$ and whose $\mathbb{Z}[\mathsf{C}_2]$-module structure is given by the involution [Kha06, p47]. Furthermore, the abelian group $\widehat{H}^j(R)$ has the structure of a *left R-module* by the 'quadratic' action

$$r \cdot [s] \ := \ [rs\bar{r}].$$

The hypothesis (2.4.1.1) is that each $\widehat{H}^j(R)$ has a 1-dimensional resolution by finitely generated projective left R-modules; this is true for any Dedekind domain R.

However, the winnowed and published version [Kha09, Proposition 25] is correct.

Proposition. *Let R be a Dedekind domain with involution. Suppose that the Tate cohomology groups vanish: $\widehat{H}^*(R) = 0$. Its nilpotent L-groups vanish: $NL^h_*(R) = 0$.*

Moreover, it is my opinion that it is essentially the only recovery of that theorem.

Remark. So now delete [Kha06, Lemma 3.2.3(3)] and reduce [Kha06, 3.3.1(2)]. Only the publication [Kha09] uses this sort of material; the error is avoided therein.

2. The minor source of error

For about ten years, I thought that the only mistake was in the first half of the proof, where one constructs a resolution of $\widehat{H}^*(R'[x])$ using data from $\widehat{H}^*(R[x])$. This works by induction to $R'[x]$ (not by restriction to $R[x]$) by instead assuming:

the induced map $f_* : R' \otimes_R \widehat{H}^j(R) \longrightarrow \widehat{H}^j(R')$ is an isomorphism of R'-modules.

Here is an illustration with the Gaussian integers and a cyclotomic number ring.

Example 2.1. Consider the inclusion $f : \mathbb{Z}[i] \longrightarrow \mathbb{Z}[\zeta]$, where $i := \sqrt{-1} \longmapsto \zeta^2$, and $\zeta := e^{\pi i/4}$ is a primitive 8-th root of unity, whose involution is $\overline{\zeta} := \zeta^{-1} = \zeta^7$. Recall $\zeta^4 + 1 = 0$. Note the first induced map is an isomorphism of $\mathbb{Z}[i]$-modules:

$$f_* : \widehat{H}^j(\mathbb{Z}[i]) = \frac{\mathbb{Z}[i]}{(1-i)} \cdot \begin{cases} 1 & \text{if } j=0 \\ i & \text{if } j=1 \end{cases} \longrightarrow \widehat{H}^j(\mathbb{Z}[\zeta]) = \frac{\mathbb{Z}[\zeta]}{(1-\zeta)} \cdot \begin{cases} 1 & \text{if } j=0 \\ \zeta^2 & \text{if } j=1. \end{cases}$$

However, note the second induced map is not an isomorphism of $\mathbb{Z}[\zeta]$-modules:

$$f_* : \mathbb{Z}[\zeta] \otimes \widehat{H}^j(\mathbb{Z}[i]) = \frac{\mathbb{Z}[\zeta]}{(1-\zeta^2)} \cdot \begin{cases} 1 & j=0 \\ \zeta^2 & j=1 \end{cases} \longrightarrow \widehat{H}^j(\mathbb{Z}[\zeta]) = \frac{\mathbb{Z}[\zeta]}{(1-\zeta)} \cdot \begin{cases} 1 & j=0 \\ \zeta^2 & j=1. \end{cases}$$

Notice that $\mathbb{Z}[\zeta]$ is a free right $\mathbb{Z}[i]$-module with basis $\{1, \zeta\}$, a fortiori a flat module.

3. The major source of error

In 2019, I discovered that the second half of the proof of [Kha06, Theorem 3.1.3] has a fatal error, notwithstanding the above adjustment that remedies the first half.

The *fatal error* therein was in imprecisely applying [Kha06, Lemma 2.4.5(2)]: calculations ensuing from a Künneth-type spectral sequence [Kha06, Remark 2.4.4].

Specifically, even assuming the replacement hypothesis in Section 2, say for $H_0(C'^t \otimes_{R'[x]} C')$ it still boils down to having the following canonical map be onto:

$$H_0(C)^t \otimes_{R[x]} H_0(C) \longrightarrow H_0(C)^t \otimes_{R[x]} \boldsymbol{R'[x]} \otimes_{R[x]} H_0(C)$$

and similarly for Tor^1. These only seem to hold if the modules vanish [Kha09, 25]. Here a right A-module M^t is a given left one M made 'right' by the involution on A.

References

[Kha06] Qayum Khan. *On connected sums of real projective spaces*. PhD thesis, Indiana University, 2006.

[Kha09] Qayum Khan. Reduction of UNil for finite groups with normal abelian Sylow 2-subgroup. *J. Pure Appl. Algebra*, 213(3):279–298, 2009.

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"And other seeds fell on rocky ground..."
Matthew 13:5

*"Split a piece of wood, and I am there.
Lift up the stone, and you will find me there."*
Thomas 77

This work is devoted to the Word: the Lamb, the Lion, the Gate.

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Abstract

Part I computes the unitary nilpotent groups UNil for certain classes of virtually cyclic groups. UNil originates as the obstruction to splitting a homotopy equivalence between manifolds along a two-sided hypersurface. However, for the past fifteen years, UNil has become highlighted in connection with the isomorphism conjecture in algebraic L-theory. Part II focuses on the connected sum problem for real projective spaces. It consists of an assortment of geometric phenomena, namely: the behavior under passage to a self-similar cover, smooth splitting in dimension five, and topological destabilization / smoothing in dimension four. Overall, the algebra of Part I and the topology of Part II are related by the infinite dihedral group.

Contents

Introduction

0.1. Background

0.1.1. The splitting problem. Let $g : W \to Y$ be a simple homotopy equivalence between connected closed manifolds of dimension $n > 5$. Let $i : X \to Y$ be the inclusion of a connected, separating, codimension one submanifold such that the induced homomorphism

$$i_* : \pi_1(X) \longrightarrow \pi_1(Y)$$

is injective. Then we obtain decompositions

$$\begin{aligned} Y &= Y_- \cup_X Y_+ \\ \pi_1(Y) &= \pi_1(Y_-) *_{\pi_1(X)} \pi_1(Y_+). \end{aligned}$$

DEFINITION 0.1.1. We call g **split along** X if g is transversal to X, and the following restrictions are homotopy equivalences:

$$\begin{aligned} g &: g^{-1}(X) \longrightarrow X \\ g &: g^{-1}(Y_\pm) \longrightarrow Y_\pm. \end{aligned}$$

We call g **splittable along** X if $g : W \to Y$ is homotopic to a split map $g' : W \to Y$.

The goals of this dissertation are to study the splitting problem, especially for connected sums of real projective spaces, and how it applies to the computation of the surgery L-groups of certain infinite groups with torsion.

0.1.2. Splitting obstruction theory. Let us start with a little bit of background. Sylvain Cappell, in the mid 1970's, developed a functor called UNil_n, the unitary nilpotent group in dimension $n \in \mathbf{Z}$, from the category of triples, consisting of a ring R with involution and two (R, R)-bimodules $\mathscr{B}_-, \mathscr{B}_+$ with involution, to the category of abelian groups.

DEFINITION 0.1.2. An **involution** $^{-} : R \to R$ on a ring R is an additive homomorphism such that, for all $r, s \in R$, we have

$$\begin{aligned} \overline{r+s} &= \overline{r} + \overline{s} \\ \overline{\overline{r}} &= r. \end{aligned}$$

An **involution** $^{\wedge} : \mathscr{B}_{\pm} \to \mathscr{B}_{\pm}$ on an (R,R)-bimodule $\mathscr{B}_{\pm}$ is an additive homomorphism such that, for all $b \in \mathscr{B}_{\pm}$ and $r, s \in R$, we have

$$(r \cdot b \cdot s)^{\wedge} = \overline{s} \cdot b^{\wedge} \cdot \overline{r}.$$

For each $n \in \mathbf{Z}$, Sylvain Cappell [**Cap74b**] algebraically defines an abelian group, which is natural in R and $\mathscr{B}_{\pm}$ and is 4-periodic in n:

$$\mathrm{UNil}_n^h(R; \mathscr{B}_-, \mathscr{B}_+) \cong \mathrm{UNil}_{n+4}^h(R; \mathscr{B}_-, \mathscr{B}_+).$$

In the splitting problem, the involutions $^{-}$ on the group rings

$$\begin{aligned} R &= \mathbf{Z}[\pi_1 X] \\ A_{\pm} &= \mathbf{Z}[\pi_1 Y_{\pm}] \end{aligned}$$

are defined on group elements by

$$\overline{a} = a^{-1}.$$

The involution $^{\wedge}$ on the (R,R)-bimodule

$$\mathscr{B}_{\pm} = \mathbf{Z}[\pi_1 Y_{\pm} \setminus \pi_1 X]$$

is the restriction of the involution $^{-}$ on $A_{\pm}$. Observe that R is a subring of $A_{\pm}$ and there is a decomposition of (R,R)-bimodules:

$$A_{\pm} = R \oplus \mathscr{B}_{\pm}.$$

THEOREM 0.1.3 (Cappell [**Cap74a**]). *There is an element*

$$\mathrm{split}_L^h(g; X) \in \mathrm{UNil}_{n+1}^h := \mathrm{UNil}_{n+1}^h(\mathbf{Z}[\pi_1 X]; \mathbf{Z}[\pi_1 Y_- \setminus \pi_1 X], \mathbf{Z}[\pi_1 Y_+ \setminus \pi_1 X])$$

which vanishes if and only if g is splittable along X.

THEOREM 0.1.4 (Cappell [**Cap74b**]). *Let* $n \in \mathbf{Z}$.

(1) *The abelian group* UNil^h_{n+1} *is a summand of* $L^h_{n+1}(\mathbf{Z}[\pi_1 Y])$.

(2) *There is a Mayer-Vietoris exact sequence*

$$\cdots \longrightarrow L^h_{n+1}(\mathbf{Z}[\pi_1 Y_-]) \oplus L^h_{n+1}(\mathbf{Z}[\pi_1 Y_+]) \longrightarrow \frac{L^h_{n+1}(\mathbf{Z}[\pi_1 Y])}{\mathrm{UNil}^h_{n+1}}$$
$$\xrightarrow{\partial} L^h_n(\mathbf{Z}[\pi_1 X]) \longrightarrow L^h_n(\mathbf{Z}[\pi_1 Y_-]) \oplus L^h_n(\mathbf{Z}[\pi_1 Y_+]) \longrightarrow \cdots .$$

0.1.3. The connected sum of real projective spaces. Consider the splitting problem for the manifolds

$$\begin{aligned} Y &= \mathbf{RP}^n \# \mathbf{RP}^n = (\mathbf{RP}^{n-1} \times -1) \cup_f (S^{n-1} \times D^1) \cup_f (\mathbf{RP}^{n-1} \times 1) \\ X &= S^{n-1} \times 0. \end{aligned}$$

Here $f : S^{n-1} \to \mathbf{RP}^{n-1}$ is the two-fold covering map. Just like the circle S^1, for each $k > 0$, there exists a k-fold (irregular) self-covering map $p_k : Y \to Y$ defined by

$$p_k(x) := \begin{cases} (x_0, \cos(k\pi(x_1+1)/2)) & \text{if } x = (x_0, x_1) \in S^{n-1} \times D^1 \\ (y_0, (-x_1)^k) & \text{if } x = (y_0, x_1) \in \mathbf{RP}^{n-1} \times \pm 1. \end{cases}$$

In Section 0.2, We shall study what happens to then splitting problem of a homotopy equivalence to Y along X when we pass to self-similar covers $p_k : Y \to Y$.

0.1.4. The infinite dihedral group. Define the infinite dihedral group

$$\mathbf{D}_\infty := \mathrm{Isom}(\mathbf{Z}).$$

Here one should think of the integers $\mathbf{Z}$ as the vertices of a very large regular polygon. Let a and b be reflection through 0 and $1/2$, and let $t = ab$ be translation by -1. Then there are semidirect and free product decompositions:

$$\begin{aligned} \mathbf{D}_\infty &= \mathbf{C}_\infty \rtimes_{-1} \mathbf{C}_2 &&= \langle t, a \mid ata^{-1} = t^{-1}, a^2 = 1 \rangle \\ \mathbf{D}_\infty &= \mathbf{C}_2 * \mathbf{C}_2 &&= \langle a, b \mid a^2 = b^2 = 1 \rangle . \end{aligned}$$

In Section 0.2, for any $n > 2$, we shall use

$$\pi_1(\mathbf{RP}^n \# \mathbf{RP}^n) = \mathbf{D}_\infty.$$

In Section 0.4, for any finite group F, we shall use

$$F \times \mathbf{D}_\infty = (F \times \mathbf{C}_2) *_F (F \times \mathbf{C}_2).$$

For any $n \in \mathbf{Z}$, we shall abbreviate

$$\mathrm{UNil}^h_n(\mathbf{Z}[F]) := \mathrm{UNil}^h_n(\mathbf{Z}[F]; \mathbf{Z}[F], \mathbf{Z}[F]),$$

which is a summand of the abelian group $L^h_n(\mathbf{Z}[F \times \mathbf{D}_\infty])$.

0.2. Splitting homotopy equivalences in finite covers

Let $n > 2$, and let M be a compact, connected, smooth m-manifold such that $m + n > 5$. For example, if $n > 5$ then we can take M to be a point.

THEOREM 0.2.1 (cf. Thm. 6.1.1, Cor. 6.2.4). *Suppose W is a compact smooth $(m + n)$-manifold and*

$$g : W \longrightarrow M \times \mathbf{RP}^n \# \mathbf{RP}^n$$

is a simple homotopy equivalence that restricts to a diffeomorphism on the boundary. Then g is splittable along the two-sided hypersurface

$$M \times S^{n-1}$$

when lifted to some odd self-similar cover.

Let $d := m + n - (-1)^n$, and consider the ring $R = \mathbf{Z}[\pi_1 M]$ with involution. The involution on the Laurent extension $R[t, t^{-1}]$ is given by $\bar{t} = t$. Then the fundamental theorem of algebraic L-theory [**Ran74**] states that there is a natural decomposition

$$L^h_d(R[t, t^{-1}]) = L^h_d(R) \oplus L^p_d(R) \oplus N^- L^h_d(R) \oplus N^+ L^h_d(R),$$

where

$$N^\pm L^h_d(R) := \mathrm{Ker}\left(\mathrm{aug}_1 : L^h_d(R[t^{\pm 1}]) \longrightarrow L^h_d(R)\right).$$

THEOREM 0.2.2 (cf. Cor. 6.2.2). *Suppose W is a compact smooth $(m+n)$-manifold and*

$$g : W \longrightarrow M \times \mathbf{RP}^n \# \mathbf{RP}^n$$

is a simple homotopy equivalence that restricts to a diffeomorphism on the boundary. Then g is splittable along the one-sided hypersurface

$$M \times \mathbf{RP}^{n-1} \# \mathbf{RP}^{n-1}$$

when lifted to some self-similar cover, if and only if its Browder-Livesay invariant

$$BL(g) \in L^h_d(R[t, t^{-1}])$$

has zero components in the summands

$$L^p_d(R) \quad \text{and} \quad L^h_d(R).$$

Instead of looking at one splitting problem at a time, we instead could study those splitting problems that become solved in a fixed self-similar cover. The following theorem negatively resolves a question of Shmuel Weinberger.

THEOREM 0.2.3 (cf. Thm. 6.3.2). *Let $k > 1$. Then the kernel of the k-fold self-similar transfer homomorphism*

$$p^!_k : L_2(\mathbf{Z}[\mathbf{D}_\infty]) \longrightarrow L_2(\mathbf{Z}[\mathbf{D}_\infty])$$

is not finitely generated.

0.3. Homotopy structures in lower dimensions

Let X, Y be compact connected topological 4-manifolds.

DEFINITION 0.3.1. We say Y **is stably homeomorphic to** X if $Y \# r(S^2 \times S^2)$ is homeomorphic to $X \# r(S^2 \times S^2)$ for some $r \geq 0$.

For example, we can take $X = \mathbf{RP}^4 \# \mathbf{RP}^4$ and aim at a homeomorphism classification of manifolds in its stable homeomorphism type.

THEOREM 0.3.2 (cf. Cor. 8.1.3). *Suppose $\pi_1(X) = \mathbf{D}_\infty$. If Y is stably homeomorphic to X, then $Y \# 3(S^2 \times S^2)$ is homeomorphic to $X \# 3(S^2 \times S^2)$.*

The next theorem incorporates stable homeomorphism into splitting problems. Let M and M' be closed non-orientable TOP 4-manifolds with fundamental group

$$\pi_1(M) = \pi_1(M') = \mathbf{C}_2.$$

Consider their connected sum X along the 3-sphere S:

$$X = M \# M'.$$

Select an element

$$\vartheta \in \mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-).$$

There exists a unique homeomorphism class $(X_\vartheta, h_\vartheta)$, consisting of a closed TOP 4-manifold X_ϑ and a tangential homotopy equivalence

$$h_\vartheta : X_\vartheta \longrightarrow X,$$

such that it has L-theoretic splitting obstruction

$$\mathrm{split}_L^h(h_\vartheta; S) = \vartheta.$$

THEOREM 0.3.3 (cf. Thm. 8.2.4). *Consider the pair $(X_\vartheta, h_\vartheta)$.*

(1) *The 3-stabilization $X_\vartheta \# 3(S^2 \times S^2)$ is homeomorphic to $X \# 3(S^2 \times S^2)$ and admits a smooth structure if and only if X does.*

(2) *There exists a 5-dimensional* TOP *normal bordism F between h_ϑ and $\mathbf{1}_X$ with surgery obstruction*

$$\sigma(F) = \vartheta \in L_5^h(\mathbf{Z}[\mathbf{C}_2^- * \mathbf{C}_2^-])$$

consisting of exactly six 2-handles and six 3-handles.

In order to perform certain cobordism constructions smoothly in dimension five, we must first examine the extent to which piecewise-linear surgery is effective in dimension four.

Let X be a closed, connected, oriented PL 4-manifold with fundamental group π. The 2-dimensional component of the assembly map [**TW79**] is a homomorphism

$$\kappa_2 : H_2(\pi; \mathbf{Z}_2) \longrightarrow L_4^h(\mathbf{Z}[\pi]).$$

THEOREM 0.3.4 (cf. Cor. 7.1.4). *The surgery exact sequence*

$$\mathcal{S}^s_{\mathrm{PL}}(X) \longrightarrow \mathcal{N}_{\mathrm{PL}}(X) \longrightarrow L_4^s(\mathbf{Z}[\pi])$$

of based sets is exact if κ_2 is injective.

Now we focus our attention to splitting obstruction theory for homotopy equivalences between smooth 5-manifolds. It is well-known that any PL manifold of dimension four or five admits a unique compatible smooth structure.

Let Y be a closed, connected, oriented, smooth 5-manifold. Suppose that X is a connected, oriented, smooth 4-submanifold of Y such that:

(1) X is separating and π_1-injective in Y, and
(2) the assembly map for $\pi_1(X)$ is injective (cf. Thm. 0.3.4), and
(3) Wall realization exists as follows:

$$L_5^h(\mathbf{Z}[\pi_1 X]) \times \mathcal{S}^h_{\mathrm{PL}}(X) \longrightarrow \mathcal{S}^h_{\mathrm{PL}}(X).$$

THEOREM 0.3.5 (cf. Thm. 7.2.1). *Let W be a closed smooth 5-manifold, and let*

$$g : W \longrightarrow Y$$

be a simple homotopy equivalence. Then the map g is splittable along X if and only if the following L-theory element vanishes:

$$\mathrm{split}^h_L(g; X) \in \mathrm{UNil}^h_6 \subseteq L^h_6(\mathbf{Z}[\pi_1(Y)]).$$

0.4. Computations of UNil for certain virtually cyclic groups

The Farrell-Jones isomorphism conjecture reduces the computation of L-groups of arbitrary groups Γ to the computation of UNil-groups of finite groups F (see Chapter 2 for details).

Let's first review some recent calculations of Frank Connolly, Jim Davis, and Andrew Ranicki. We start with the field $\mathbf{F}_2$.

THEOREM 0.4.1 ([**CK95, CD04**]). *Let $n \in \mathbf{Z}$.*

(1) *If $n \equiv 0 \pmod 2$, then the Arf invariant over the function field $\mathbf{F}_2(x)$ induces an isomorphism*

$$\mathrm{Arf} : \mathrm{UNil}_n(\mathbf{F}_2) \longrightarrow \frac{x\mathbf{F}_2[x]}{\langle f^2 - f\rangle}.$$

(2) *If $n \equiv 1 \pmod 2$, then*

$$\mathrm{UNil}_n(\mathbf{F}_2) = 0.$$

Next, we pass to the integers $\mathbf{Z}$. Recall that $\mathrm{UNil}_n(\mathbf{Z})$ is a summand of $L_n(\mathbf{Z}[\mathbf{D}_\infty])$.

THEOREM 0.4.2 ([**CR05, CD04, BR06**]). *Let $n \in \mathbf{Z}$.*

(1) *If $n \equiv 0, 1 \pmod 4$, then*

$$\mathrm{UNil}_n(\mathbf{Z}) = 0.$$

(2) *If $n \equiv 2 \pmod 4$, then there is an induced isomorphism*

$$\mathrm{UNil}_n(\mathbf{Z}) \longrightarrow \mathrm{UNil}_n(\mathbf{F}_2).$$

(3) *If $n \equiv 3 \pmod 4$, then there is a non-split short exact sequence*

$$0 \longrightarrow \frac{x\mathbf{F}_2[x]}{\langle f^2 - f\rangle} \longrightarrow \mathrm{UNil}_n(\mathbf{Z}) \longrightarrow x\mathbf{F}_2[x] \times x\mathbf{F}_2[x] \longrightarrow 0.$$

Part I of the dissertation is an extension to $\mathbf{Z}[F]$ for certain finite groups F.

THEOREM 0.4.3 (cf. Cor. 3.2.2). *Suppose F is a finite group of odd order. Then, for all $n \in \mathbf{Z}$, there is an induced isomorphism*

$$\mathrm{incl}_* : \mathrm{UNil}_n(\mathbf{Z}[1]) \longrightarrow \mathrm{UNil}_n(\mathbf{Z}[F]).$$

THEOREM 0.4.4 (cf. Thm. 4.1.2). *Suppose F is a finite group that contains a normal Sylow 2-subgroup of exponent two. Then*

$$\mathrm{UNil}_0(\mathbf{Z}[F]) = \mathrm{UNil}_1(\mathbf{Z}[F]) = 0$$

and there is an induced isomorphism

$$\mathrm{UNil}_2(\mathbf{Z}[F]) \longrightarrow \mathrm{UNil}_2(\mathbf{F}_2).$$

THEOREM 0.4.5 (cf. Thm. 4.1.3). *Consider the cyclic group C_2 of order two. There exists a decomposition*

$$\mathrm{UNil}_3(\mathbf{Z}[\mathbf{C}_2]) \cong \mathrm{UNil}_0(\mathbf{F}_2) \oplus \mathrm{UNil}_3(\mathbf{Z}) \oplus \mathrm{UNil}_3(\mathbf{Z}).$$

THEOREM 0.4.6 (cf. Thm. 3.2.1). *Suppose F is a finite group that contains a normal abelian Sylow 2-subgroup S. Then, for all $n \in \mathbf{Z}$, there is an induced isomorphism*

$$\mathrm{incl}_* : \mathrm{UNil}_n(\mathbf{Z}[S])_{F/S} \longrightarrow \mathrm{UNil}_n(\mathbf{Z}[F]).$$

PART I. ALGEBRA: Obstructions to splitting homotopy equivalences

CHAPTER 1

Preliminaries in algebraic K- and L-theory

1.1. Definition of UNil with decorations in Nil

The purpose of this section is to generalize J. Brookman's algebraic surgery definition of Cappell's unitary nilpotent groups. This slight modification allows intermediate decorations in Waldhausen's nilpotent groups.

1.1.1. Definition of Nil and the assembly viewpoint. First, we quickly define Friedhelm Waldhausen's nilpotent category and its algebraic K-theory.

DEFINITION 1.1.1 ([**Wal78**, p. 148], compare [**CK95**] [**Gru05**]). Let R be a ring, and let $\mathscr{B}_-, \mathscr{B}_+$ be (R,R)-bimodules that are flat as left R-modules. Define an exact category[1]

$$\mathrm{NIL} := \mathrm{NIL}^{\mathrm{proj}}(R; \mathscr{B}_-, \mathscr{B}_+)$$

as follows. Each **object** is a quadruple $x = (P_-, P_+; p_-, p_+)$, where $P_\pm$ are finitely generated projective left R-modules and $p_\pm : P_\pm \to \mathscr{B}_\pm \otimes_R P_\mp$ are morphisms of left R-modules, satisfying the following nilpotence condition for some $n \in \mathbf{Z}_{\geq 0}$:

$$(p_- \circ p_+)^n = 0 \in \mathrm{End}_{T(\mathscr{B})}(T(\mathscr{B}) \otimes_R P_+).$$

Here the morphisms

$$p_\pm : T(\mathscr{B}) \otimes_R P_\pm \longrightarrow T(\mathscr{B}) \otimes_R \mathscr{B}_\pm \otimes_R P_\mp$$

are extensions to the tensor algebra $T(\mathscr{B})$ on the (R,R)-bimodule $\mathscr{B} := \mathscr{B}_- \otimes_R \mathscr{B}_+$. Each **morphism** $f : x \to x'$ is a pair (f_-, f_+), where $f_\pm : P_\pm \to P'_\pm$ is a left

[1] This notion of Quillen is equivalent to being a full additive subcategory of an abelian category such that it is closed under extensions of objects. In our case, the abelian category is $\mathrm{END}^{\mathrm{all}}(R; \mathscr{B}_-, \mathscr{B}_+)$, which is defined similarly to NIL except we do not require that P_-, P_+ are projective and $p_- \circ p_+$ is nilpotent.

R-module morphism, such that the following diagram commutes:

$$\begin{tikzcd} P_\pm \arrow[r, "p_\pm"] \arrow[d, "f_\pm"'] & \mathscr{B}_\pm \otimes_R P_\mp \arrow[d, "\mathbf{1}\otimes f_\mp"'] \\ P'_\pm \arrow[r, "p'_\pm"] & \mathscr{B}_\pm \otimes_R P'_\mp . \end{tikzcd}$$

□

DEFINITION 1.1.2 ([**Wal78**]). For each $n \in \mathbf{Z}_{\geq 0}$, define the abelian group

$$\mathrm{Nil}_n(R; \mathscr{B}_-, \mathscr{B}_+) := K_n(\mathrm{NIL}) = \pi_{n+1}(BQ\mathrm{NIL}).$$

It is the $(n+1)$-st homotopy group of the classifying space of Quillen's Q-construction, which defines the algebraic K-theory of the exact category NIL. The reduced Nil-group is defined to fit into the natural decomposition

$$\mathrm{Nil}_n(R; \mathscr{B}_-, \mathscr{B}_+) = K_n(R) \oplus K_n(R) \oplus \widetilde{\mathrm{Nil}}_n(R; \mathscr{B}_-, \mathscr{B}_+).$$

Namely, it is the split kernel of the homomorphism on K_n-groups induced by the forgetful functor $\mathrm{NIL} \to \mathrm{PROJ}(R) \times \mathrm{PROJ}(R)$ defined by

$$(P_-, P_+; p_-, p_+) \longmapsto (P_-, P_+).$$

In regard to decorations for L-theory, our main interest shall be $\widetilde{\mathrm{Nil}}_0(R; \mathscr{B}_-, \mathscr{B}_+)$. This abelian group can be more directly defined [**Wal69**] as the reduced Grothendieck group of Waldhausen's exact category $\mathrm{NIL} = \mathrm{NIL}^{\mathrm{proj}}(R; \mathscr{B}_-, \mathscr{B}_+)$. □

DEFINITION 1.1.3 ([**Wal78**, §I.1]). A pushout (colimit) in the category of rings:

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & R \end{tikzcd}$$

is **pure** if α and β are injective and there exist (C, C)-bimodule decompositions

$$A = \alpha(C) \oplus A' \quad \text{and} \quad B = \beta(C) \oplus B'.$$

It is written $R = A *_C B$.

The main ingredient of Waldhausen's K-theory Mayer-Vietoris sequence [**Wal78**, Thms. 1–3] is the identification of $\widetilde{\mathrm{Nil}}$ as the defect in excisive approximation by a certain assembly map. This leads to modified Mayer-Vietoris sequence in K-theory; see §1.3.1. We strongly refer the reader to §1.3.3 for a detailed discussion of the analogous viewpoint in L-theory.

1.1.2. The algebraic surgery definition of UNil**.** Now suppose that the ring R and the (R,R)-bimodules $\mathscr{B}_\pm$ have compatible involutions $^-$ and $^\wedge$:

$$(r \cdot b \cdot s)^\wedge = \overline{s} \cdot b^\wedge \cdot \overline{r} \quad \text{for all} \quad r, s \in R \text{ and } b \in \mathscr{B}_\pm.$$

Definition 1.1.4 ([**Bro04**, Defn. 4.6], compare [**Ran92a**] [**CK95**]). On the above exact category, define the **involution functor**

$$* : \mathrm{NIL} \longrightarrow \mathrm{NIL}^{\mathrm{op}};$$

$$x^* := (P_+^*, P_-^*; -p_-^*, -p_+^*) \quad \text{and} \quad f^* := (f_+^*, f_-^*).$$

Here, for any (finitely generated projective) left R-module P, the **dual module**

$$P^* := \mathrm{Hom}_R(P, R)$$

has *left* R-module structure given by $r \cdot \varphi := (m \mapsto \varphi(m)\overline{r})$. □

Now, we define Sylvain Cappell's even-dimensional unitary nilpotent groups of algebraic L-theory.

Definition 1.1.5 (decorated version of [**Cap74b**], compare [**Bro04**, Defn. 7.14]). Let ν be a $*$-invariant subgroup of $\mathrm{Nil}_0(R; \mathscr{B}_-, \mathscr{B}_+)$, and let $\epsilon = \pm 1$. A **ν-decorated, nonsingular ϵ-quadratic unilform over** $(R; \mathscr{B}_-, \mathscr{B}_+)$ is a pair $(P_-, \theta_-; P_+, \theta_+)$ of bimodule-valued ϵ-quadratic forms

$$\theta_\pm : P_\pm \longrightarrow \mathscr{B}_\pm \otimes_R P_\mp$$

such that $P_\mp = P_\pm^*$ as finitely generated projective left R-modules[2] and

$$x = (P_-, P_+; \theta_- + \epsilon\theta_-^*, \theta_+ + \epsilon\theta_+^*) \in \mathrm{NIL} \quad \text{with projective class} \quad [x] \in \nu.$$

[2]The nil class $[x]$ lies in $\widetilde{\mathrm{Nil}}_0$ if and only if $P_\pm$ are stably finitely generated free modules (1.1.2).

A **lagrangian** is a pair (V_-, V_+) of summands of (P_-, P_+) such that the following three conditions hold. First, $V_\pm$ is a summand of $P_\pm$ such that for every $m, m' \in V_\pm$:

$$\theta_\pm(m)(m') = b \mp b^\wedge \quad \text{for some} \quad b \in \mathscr{B}_\pm.$$

Second, with respect to the evaluation pairing[3] $P_- \times P_+ \to R$, the submodules V_- and V_+ annihilate each other:

$$V_\mp = V_\pm^\circ := \left\{ f \in V_\pm^* \mid f(V_\pm) = 0 \right\}.$$

Third, their well-defined, nonsingular quotient must have a restricted nil class

$$[(P_-/V_-, P_+/V_+; \theta_- + \epsilon\theta_-^*, \theta_+ + \epsilon\theta_+^*)] \in \nu.$$

Let $k \in \mathbf{Z}$ and write $\epsilon := (-1)^k$. Define the abelian group

$$\mathrm{UNil}_{2k}^\nu(R; \mathscr{B}_-, \mathscr{B}_+)$$

as the Witt group of ν-decorated, nonsingular ϵ-quadratic unilforms modulo lagrangians. □

REMARK 1.1.6. Similar to L-groups, there is a notion of sublagrangian of unilforms [**CK95, CD04**], where we instead require $V_\mp \subseteq V_\pm^\circ$. On occasion, we use the Witt-equivalent **sublagrangian construction**

$$(P_-/V_+^\circ, \theta_-; P_+/V_-^\circ, \theta_+).$$

Next we re-work Jeremy Brookman's definition of the odd-dimensional unitary nilpotent groups.

DEFINITION 1.1.7 (decorated version of [**Bro04**, §13.1, §13.2]). Let ν be a $*$-invariant subgroup of $\mathrm{Nil}_0(R; \mathscr{B}_-, \mathscr{B}_+)$, and let $\epsilon = \pm 1$. A **ν-decorated, short odd, ϵ-quadratic Poincaré nilcomplex** is a pair (C, ψ) satisfying the following three conditions. First, it consists of a 1-dimensional chain complex in the NIL category:

$$C = \left\{ C_1 \xrightarrow{d} C_0 \right\} \quad \text{with projective Euler characteristic} \quad [C] = [C_0] - [C_1] \in \nu.$$

[3]Here we use nonsingularity of the unilform: $P_- = P_+^*$.

Second, it consists of an ϵ-quadratic structure

$$\psi = \left\{ C_0^* \xrightarrow{\psi_0} C_1, C_0^* \xrightarrow{\psi_1} C_0 \right\}$$

satisfying the cycle condition

$$d \circ \psi_0 + (\psi_1 - \epsilon\psi_1^*) = 0.$$

Third, the mapping cone of its Poincaré duality must be a contractible complex in the NIL category:

$$0 \longrightarrow C_0^* \xrightarrow{\left(\begin{smallmatrix} -\psi_0 \\ -d^* \end{smallmatrix}\right)} C_1 \oplus C_1^* \xrightarrow{\left(\begin{smallmatrix} d & \epsilon\psi_0^* \end{smallmatrix}\right)} C_0 \longrightarrow 0.$$

A **null-cobordism** of (C, ψ) is a pair $(f : C \to D, (\delta\psi, \psi))$ satisfying the following four conditions. First, it consists of a highly connected 2-dimensional chain complex in the NIL category:

$$D = \{ 0 \longrightarrow D_1 \longrightarrow 0 \} \quad \text{with projective Euler characteristic} \quad [D] = -[D_1] \in \nu.$$

Second, it consists of a morphism of chain complexes

$$f = \left\{ C_1 \xrightarrow{f_1} D_1 \right\}.$$

Third, it consists of a ϵ-quadratic structure[4]

$$\delta\psi = \left\{ D_1^* \xrightarrow{\delta\psi_0} D_1 \right\}.$$

Fourth, the mapping cone of its Poincaré duality must be a contractible complex in the NIL category:

$$0 \longrightarrow C_1 \oplus D^1 \xrightarrow{-\left(\begin{smallmatrix} f_1 & \delta\psi_0 - \epsilon\delta\psi_0^* \\ -d & \epsilon\psi_0^* \circ f_1^* \end{smallmatrix}\right)} D_1 \oplus C_0 \longrightarrow 0.$$

Let $k \in \mathbf{Z}$ and write $\epsilon := (-1)^k$. Define the abelian group

$$\mathrm{UNil}^{\nu}_{2k+1}(R; \mathscr{B}_-, \mathscr{B}_+)$$

as the cobordism group of ν-decorated, short odd, ϵ-quadratic Poincaré nilcomplexes. □

[4]In the highly-connected case, the ϵ-quadratic pair $(\delta\psi, \psi)$ is automatically a relative cycle.

DEFINITION 1.1.8. Let $n \in \mathbf{Z}$. The unitary nilpotent groups with simple, free, and projective decorations are defined as

$$\begin{aligned}
\mathrm{UNil}_n^s(R;\mathscr{B}_-,\mathscr{B}_+) &:= \mathrm{UNil}_n^0(R;\mathscr{B}_-,\mathscr{B}_+)\\
\mathrm{UNil}_n^h(R;\mathscr{B}_-,\mathscr{B}_+) &:= \mathrm{UNil}_n^{\widetilde{\mathrm{Nil}}_0}(R;\mathscr{B}_-,\mathscr{B}_+)\\
\mathrm{UNil}_n^p(R;\mathscr{B}_-,\mathscr{B}_+) &:= \mathrm{UNil}_n^{\mathrm{Nil}_0}(R;\mathscr{B}_-,\mathscr{B}_+).
\end{aligned}$$

One can show the following result, which allows us to focus our computations on the Witt groups $\mathrm{UNil}^\nu_{\mathrm{even}}$.

PROPOSITION 1.1.9. *Consider the following group ring and bimodules with involution as Laurent extensions:*

$$\begin{aligned}
A &:= R[\mathbf{C}_\infty]\\
\mathscr{B}_\pm[\mathbf{C}_\infty] &:= A \otimes_R \mathscr{B}_\pm \otimes_R A.
\end{aligned}$$

Then, for all $n \in \mathbf{Z}$, there are Shaneson-type exact sequences

$$0 \longrightarrow \mathrm{UNil}_n^s(R;\mathscr{B}_-,\mathscr{B}_+) \longrightarrow \mathrm{UNil}_n^s((R;\mathscr{B}_-,\mathscr{B}_+)[\mathbf{C}_\infty]) \longrightarrow \mathrm{UNil}_{n-1}^h(R;\mathscr{B}_-,\mathscr{B}_+) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{UNil}_n^h(R;\mathscr{B}_-,\mathscr{B}_+) \longrightarrow \mathrm{UNil}_n^h((R;\mathscr{B}_-,\mathscr{B}_+)[\mathbf{C}_\infty]) \longrightarrow \mathrm{UNil}_{n-1}^p(R;\mathscr{B}_-,\mathscr{B}_+) \longrightarrow 0.$$

□

1.1.3. Definition of lower L-groups. We start with the fundamental theorem of algebraic K-theory, due to Bass-Heller-Swan for $n = 1$ and to Quillen for $n > 1$.

THEOREM 1.1.10 (cf. [**Bas68**, Thm. XII.7.4]). *Let R be a ring and $n \in \mathbf{Z}_{>0}$. There is a split exact sequence of abelian groups, natural in R:*

$$0 \longrightarrow K_n(R) \longrightarrow K_n(R[x]) \oplus K_n(R[x^{-1}]) \longrightarrow K_n(R[x,x^{-1}]) \longrightarrow K_{n-1}(R) \longrightarrow 0.$$

Furthermore, there is a natural decomposition

$$K_n(R[x,x^{-1}]) = K_n(R) \oplus K_{n-1}(R) \oplus \mathrm{Nil}_n(R) \oplus \mathrm{Nil}_n(R).$$

This theorem motivated Hyman Bass to recursively define the lower K-groups.

DEFINITION 1.1.11 ([**Bas68**, p. 664]). Let $n \in \mathbf{Z}_{\leq 0}$. Suppose that the functor K_n is defined from the category of rings to the category of abelian groups. Then define the **lower K-functor** K_{n-1} by the formula

$$K_{n-1}(R) := \operatorname{Cok}\left(K_n(R[x]) \oplus K_n(R[x^{-1}]) \longrightarrow K_n(R[x, x^{-1}])\right).$$

The following vanishing result is due to D.W. Carter.

THEOREM 1.1.12 ([**Car80**]). *Let $\mathcal{O}$ be the ring of integers in a number field, and let F be finite group. Then, for all $n < -1$, the following abelian group vanishes:*

$$K_n(\mathcal{O}[F]) = 0.$$

Next, we turn to algebraic L-theory. The involution on the Laurent extension $R[\mathbf{C}_\infty]$ is the nontrivial[5] involution of the group-ring:

$$\overline{rg} = \overline{r} g^{-1} \quad \text{for all} \quad r \in R \text{ and } g \in \mathbf{C}_\infty.$$

The following theorem consists of the Ranicki-Shaneson sequences, which can be used in part to prove Proposition 1.1.9.

THEOREM 1.1.13 ([**Ran73a**, Thm. 1.1]). *Let R be a ring with involution and $m \in \mathbf{Z}$. There are functorial split exact sequences of abelian groups:*

$$0 \longrightarrow L^s_m(R) \longrightarrow L^s_m(R[\mathbf{C}_\infty]) \longrightarrow L^h_{m-1}(R) \longrightarrow 0$$
$$0 \longrightarrow L^h_m(R) \longrightarrow L^h_m(R[\mathbf{C}_\infty]) \longrightarrow L^p_{m-1}(R) \longrightarrow 0.$$

This theorem motivated Andrew Ranicki to define recursively the lower L-groups, where the classical ones are indexed as follows:

$$L^s_m = L^{\langle 2\rangle}_m \quad \text{and} \quad L^h_m = L^{\langle 1\rangle}_m \quad \text{and} \quad L^p_m = L^{\langle 0\rangle}_m.$$

DEFINITION 1.1.14 ([**Ran92b**, Defn. 17.1]). Let $n \in \mathbf{Z}_{\leq 0}$. Suppose that the functor $L^{\langle n\rangle}_m$ is defined from the category of rings with involution to the category of abelian groups. Then define the **lower L-functor** $L^{\langle n-1\rangle}_{m-1}$ by the formula

$$L^{\langle n-1\rangle}_{m-1}(R) := \operatorname{Cok}\left(L^{\langle n\rangle}_m(R) \longrightarrow L^{\langle n\rangle}_m(R[\mathbf{C}_\infty])\right).$$

[5]There is an analogous fundamental theorem of algebraic L-theory for $R[x, x^{-1}]$ where $\overline{x} = x$.

Then there are Rothenberg-type exact sequences involving the forgetful map.

$$\cdots \longrightarrow L_m^{\langle n\rangle}(R) \xrightarrow{\text{forget}} L_m^{\langle n-1\rangle}(R) \xrightarrow{\chi} \widehat{H}^m(\mathbf{C}_2; K_{n-1}(R)) \xrightarrow{\partial} L_{m-1}^{\langle n\rangle}(R) \longrightarrow \cdots .$$

Definition 1.1.15 ([**Ran92b**, 17.7]). Define the **ultimate lower L-functor** $L_m^{-\infty}$ as their direct limit under the forgetful maps:

$$L_m^{\langle -\infty\rangle}(R) := \operatorname*{colim}_{n\leq 2} L_m^{\langle n\rangle}(R).$$

Remark 1.1.16. Let $\mathcal{O}$ be the ring of integers in a number field, and let F be a finite group. Then, as a corollary of Carter's vanishing theorem (1.1.12) and the Ranicki-Rothenberg exact sequence, there is an induced isomorphism for all $m \in \mathbf{Z}$:

$$L_m^{\langle -1\rangle}(\mathcal{O}[F]) \xrightarrow{\cong} L_m^{\langle -\infty\rangle}(\mathcal{O}[F]).$$

1.2. Splitting h-cobordisms and homotopy equivalences

Throughout this dissertation, we fix the following notation for compact CAT *manifolds of dimension $n > 5$, where the manifold category is either* DIFF *(smooth),* PL *(piecewise linear), or* TOP *(topological).*

1.2.1. Topological notation. Consider a properly embedded, connected incompressible[6], two-sided $(n-1)$-submanifold $(X, \partial X)$ of a compact connected CAT n-manifold $(Y, \partial Y)$. Write $i : (X, \partial X) \to (Y, \partial Y)$ as the inclusion. Denote G as the fundamental group of Y and $\omega : G \to \mathbf{C}_2 = \{\pm 1\}$ its orientation character. Also denote H as the fundamental group of X. Since X is two-sided in Y, its orientation character is the restriction $\omega \circ i : H \to \mathbf{C}_2$.

Denote $J = J_- \cup J_+$ as the fundamental groupoid of $Y \setminus X$. Write the induced homomorphisms $i_\pm : H \to J_\pm$ and $j_\pm : J_\pm \to G$ of vertex groups. Since X is incompressible in Y, the maps $i_\pm$ and $j_\pm$ are also injective, and the orientation character of $Y \setminus X$ is also the restriction $\omega \circ j : J \to \mathbf{C}_2$. Observe that if X is separating in Y (see [**Wal99**, §12A]), then $J = J_- \sqcup J_+$ and we have a pushout of

[6]We call X **incompressible in** Y if the induced map $i_* : \pi_1(X) \to \pi_1(Y)$ is injective.

connected spaces and groups:

$$\begin{array}{ccc} X & \xrightarrow{i_+} & Y_+ \\ {\scriptstyle i_-}\downarrow & & \downarrow{\scriptstyle j_+} \\ Y_- & \xrightarrow{j_-} & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H & \xrightarrow{i_+} & G_+ \\ {\scriptstyle i_-}\downarrow & & \downarrow{\scriptstyle j_+} \\ G_- & \xrightarrow{j_-} & G. \end{array}$$

Hence $G = G_- *_H G_+$ is an *injective amalgam of groups*, by the Seifert-VanKampen theorem.

Otherwise, observe that if X is non-separating in Y (see [**Wal99**, §12B]), then $J = J_- = J_+$ and we have another pushout (colimit) of connected spaces and groups:

$$X \underset{i_-}{\overset{i_+}{\rightrightarrows}} Y \setminus X \xrightarrow{j} Y \qquad \text{and} \qquad H \underset{i_-}{\overset{i_+}{\rightrightarrows}} J \xrightarrow{j} G.$$

Hence $G = *_J H$ is a *Higman-Neumann-Neumann (HNN) extension of groups.*

For simplicity, we shall associate a triad

$$\Phi := (\mathbf{Z}[H]; \mathbf{Z}[J_- \setminus H], \mathbf{Z}[J_+ \setminus H])\,,$$

consisting of a ring and two free bimodules. Then denote

$$\Phi^\omega := (\mathbf{Z}[H^\omega]; \mathbf{Z}[(J_- \setminus H)^\omega], \mathbf{Z}[(J_+ \setminus H)^\omega])\,,$$

consisting of a ring with involution and two free bimodules with involution.

1.2.2. Splitting obstructions. The reader is reminded from §1.2.1 that the compact CAT manifold $(Y, \partial Y)$, of dimension $n > 5$, contains a two-sided hypersurface $(X, \partial X)$.

Consider the following subgroups of the projective class and Whitehead groups:

$$\begin{aligned} I &:= \operatorname{Ker}\left(\widetilde{K}_0(\mathbf{Z}[H]) \xrightarrow{i} \widetilde{K}_0(\mathbf{Z}[J])\right) \\ B &:= \operatorname{Im}\left(\operatorname{Wh}(J) \xrightarrow{j} \operatorname{Wh}(G)\right). \end{aligned}$$

The following theorem on splitting h-cobordisms is due essentially to F. Waldhausen. The technique of proof is to handle-exchange in the non-compact cover $\widehat{W}$ corresponding to the subgroup H of $G = \pi_1(W)$.

Theorem 1.2.1 ([**Wal69**, Thms. §5,6]). *Let $(W; Y, Y')$ be an h-cobordism* rel ∂Y *of compact manifolds. Then it can be decomposed as a union of h-cobordisms with* *compact bases* $X \times D^1$ *and* $Y \setminus X \times \mathring{D}^1$ *if and only if its Whitehead torsion* $\tau \in \mathrm{Wh}(G)$ *satisfies the following vanishing condition:*

$$\partial_K(\tau) \oplus \mathrm{split}_K(\tau) = 0 \in I \oplus \widetilde{\mathrm{Nil}}_0(\Phi) \cong \mathrm{Wh}(G)/B.$$

PROOF. This is immediate from exactness at $\mathrm{Wh}(G)$ in Waldhausen's Mayer-Vietoris sequence (1.3.4). □

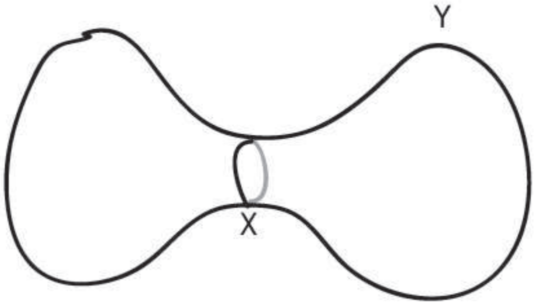

FIGURE 1.2.1. The two-sided submanifold X has a neighborhood $X \times D^1$ in the ambient manifold Y.

The existence of the latter isomorphism is part of our combined statement. The analogous theorem, on splitting homotopy equivalences between manifolds of the same dimension, is due to S. Cappell.

THEOREM 1.2.2 ([**Cap74a**, Thms. 1, 5]). *Let* $g : (W, \partial W) \to (Y, \partial Y)$ *be a homotopy equivalence between compact manifolds that restricts to an isomorphism* $g : \partial W \to \partial Y$*. Then it is h-bordant to a union of homotopy equivalences with compact targets* $X \times D^1$ *and* $Y \setminus X \times \mathring{D}^1$*, if and only if the following vanishing conditions are satisfied:*

$$[\partial_K(\tau(g))] = 0 \in \widehat{H}^n(\mathbf{C}_2; I^\omega) \quad \textit{and} \quad \mathrm{split}^h_L(g) = 0 \in \mathrm{UNil}^h_{n+1}(\Phi^\omega).$$

Furthermore, it is homotopic to this union of homotopy equivalences if and only if

$$\partial_K(\tau(g)) \oplus \mathrm{split}_K(\tau(g)) = 0 \in \mathrm{Wh}(G)/B \quad \textit{and} \quad \mathrm{split}^s_L(g) = 0 \in \mathrm{UNil}^s_{n+1}(\Phi^\omega).$$

□

In both parts of the theorem, the secondary obstruction in L-theory is only defined if the primary obstruction in K-theory vanishes.

The **splitting obstructions** $\partial_K, \mathrm{split}_K, \mathrm{split}_L$ are topologically defined in terms of CW-complexes and handle-exchanges (see [**Wal69**, §5] and [**Cap76b**, Ch. I]). They constitute the key ingredients in the proofs of the existence of Mayer-Vietoris sequences in algebraic K- and L-theory (stated in §1.3), at least for injective amalgams of finitely presented groups.

1.2.3. Decomposition of homotopy structures. The geometric significance of splitting homotopy equivalences (1.2.2) is the existence of a decomposition of the **homotopy structure set** $\mathcal{S}^{\kappa}_{\mathrm{CAT}}(Y, \partial Y)$, where κ is a $*$-invariant subgroup[7] of torsions $\mathrm{Wh}(G)$. Recall that it consists of κ-torsion CAT h-bordism classes of κ-torsion homotopy equivalences

$$g : (W, \partial W) \longrightarrow (Y, \partial Y)$$

whose restrictions $g : \partial W \to \partial Y$ are CAT isomorphisms.

Our statement of Cappell's decomposition theorem is an interpolation with respect to Whitehead torsions. Therein, the **subset of split structures** $\mathcal{S}^{\sigma}_{\mathrm{split}}(Y, \partial Y; X)$ is defined similarly to the structure set except that the elements are split σ-torsion h-bordism classes of split σ-torsion homotopy equivalences.

THEOREM 1.2.3 ([**Cap74a**, Thms. 3, 7]). *Let ν and σ be $*$-invariant subgroups of $\widetilde{\mathrm{Nil}}_0(\Phi)$ and B. Then, for any manifold category, the action of the surgery L-group restricts to a bijection*

$$\mathrm{act} : \mathrm{UNil}^{\nu}_{n+1}(\Phi^{\omega}) \times \mathcal{S}^{\sigma}_{\mathrm{split}}(Y, \partial Y; X) \longrightarrow \mathcal{S}^{\nu \oplus \sigma}(Y, \partial Y).$$

□

[7] The action of the group $\mathbf{C}_2$ is given by the conjugate-transpose $*$ with respect to the involution on the ring $\mathbf{Z}[G^{\omega}]$.

REMARK 1.2.4. The proof of this theorem is entirely geometric. The partial inverse of Wall's realization

$$\mathrm{act} : L^{\kappa}_{n+1}(\mathbf{Z}[G^{\omega}]) \times \mathcal{S}^{\kappa}(Y, \partial Y) \longrightarrow \mathcal{S}^{\kappa}(Y, \partial Y)$$

is **Cappell's nilpotent normal cobordism construction**

$$\mathrm{nncc} : \mathcal{S}^{\nu\oplus\sigma}(Y, \partial Y) \longrightarrow \mathrm{UNil}^{\nu}_{n+1}(\Phi^{\omega}) \times \mathcal{S}^{\sigma}_{\mathrm{split}}(Y, \partial Y; X).$$

For their handlebody-type definitions, we refer the reader to [**Wal99**, Theorem 10.5] and [**Cap76b**, §II.1]. It turns out that the surgery obstruction of Cappell's nncc coincides with the splitting obstruction:

$$\mathrm{split}_L = \mathrm{proj} \circ \mathrm{nncc} : \mathcal{S}^{\nu\oplus\sigma}(Y, \partial Y) \longrightarrow \mathrm{UNil}^{\nu}_{n+1}(\Phi^{\omega}).$$

1.3. Mayer-Vietoris sequences with decorations

1.3.1. The failure of excision in K-theory.

DEFINITION 1.3.1 ([**Wal78**, Intro.]). A pushout (colimit) of rings

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & R \end{array}$$

is **pure** if α and β are monomorphisms and there exist left projective summands A' and B' such that

$$A = \alpha(C) \oplus A' \quad \text{and} \quad B = \beta(C) \oplus B'$$

as (C, C)-bimodules. (Similarly for HNN-extensions of rings.) Its associated triad is denoted

$$\Phi := (C; A', B').$$

The following theorem of F. Waldhausen is stated in terms of Quillen's algebraic K-theory of rings, all of whose constructions are categorical in nature. A ring C is **regular coherent** if every finitely presented module admits a finite resolution by finitely generated projective modules.

THEOREM 1.3.2 ([**Wal78**, Thms. 1,2,4]). *Consider any pure pushout of rings $R = A *_C B$. Then there exists a homotopy fibration of based spaces*

$$\Omega K(R) \longrightarrow \widetilde{K}\mathrm{Nil}(\Phi) \oplus K(C) \xrightarrow{\left(\begin{smallmatrix} 0 & \alpha \\ 0 & -\beta \end{smallmatrix}\right)} K(A) \times K(B).$$

Furthermore, the exotic $\widetilde{K}\mathrm{Nil}(\Phi)$-term is contractible if C is a regular coherent ring. $\square$

The induced sequence on homotopy groups is the Mayer-Vietoris sequence in algebraic K-theory.

COROLLARY 1.3.3 (Waldhausen). *Consider any pure pushout of rings $R = A *_C B$. Then for all $n \in \mathbf{Z}$, there exists an exact sequence of abelian groups:*

$$K_{n+1}(R) \xrightarrow{\left(\begin{smallmatrix} \mathrm{split}_K \\ \partial_K \end{smallmatrix}\right)} \widetilde{\mathrm{Nil}}_n(\Phi) \oplus K_n(C) \xrightarrow{\left(\begin{smallmatrix} 0 & \alpha \\ 0 & -\beta \end{smallmatrix}\right)} K_n(A) \oplus K_n(B) \longrightarrow K_n(R).$$

Moreover, $\widetilde{\mathrm{Nil}}_n(\Phi)$ is a summand of $K_{n+1}(R)$. $\square$

COROLLARY 1.3.4 (see Theorem 1.2.1). *Consider any injective amalgam or HNN-extension of groups (denoted in §1.2.1). Then there exists a (non-split) short exact sequence of abelian groups:*

$$0 \longrightarrow B \longrightarrow \mathrm{Wh}(G) \xrightarrow{\left(\begin{smallmatrix} \mathrm{split}_K \\ \partial_K \end{smallmatrix}\right)} \widetilde{\mathrm{Nil}}_0(\Phi) \oplus I \longrightarrow 0.$$

Moreover, $\widetilde{\mathrm{Nil}}_0(\Phi)$ is a summand of $\mathrm{Wh}(G)$. In particular, $\mathrm{split}_K(B) = 0$ and $\partial_K(B) = I$. $\square$

1.3.2. The failure of excision in L-theory. Using handlebody techniques, S. Cappell proves a Mayer-Vietoris sequence in algebraic L-theory. Our combined re-statement interpolates the K-theory decorations using Corollary 1.3.4. For simplicity, we write the group G instead of the ring $R[G^\omega]$ with involution, where R is a subring of $\mathbf{Q}$.

THEOREM 1.3.5 ([**Cap74b**, Thms. 1–5]). *Consider any injective amalgam or HNN-extension of finitely presented groups (denoted in §1.2.1). Suppose ν and σ*

are $*$-invariant subgroups of $\widetilde{\mathrm{Nil}}_0(\Phi)$ *and* $\mathrm{Wh}(G)$ *such that* $B \subseteq \sigma$. *Then for all* $n \in \mathbf{Z}$, *there exists an exact sequence of abelian groups:*

$$L_{n+1}^{\nu\oplus\sigma}(G) \xrightarrow{\begin{pmatrix}\mathrm{split}_L\\ \partial_L\end{pmatrix}} \mathrm{UNil}_{n+1}^{\nu}(\Phi^{\omega}) \oplus L_n^{\partial_K(\sigma)}(H) \xrightarrow{i_- - i_+} L_n^h(J) \xrightarrow{j} L_n^{\nu\oplus\sigma}(G).$$

Write the associated group triad $\widehat{\Phi} := (G; J; H)$. *Then the term*

$$\mathrm{UNil}_n^{\nu}(\Phi^{\omega}) = LS_{n-1}(\widehat{\Phi}^{\omega}) = L_n(\widehat{\Phi}^{\omega})$$

is a summand of $L_n^{\nu\oplus\sigma}(G)$. *Furthermore, this exotic* UNil*-term has exponent 4 (see* [**Far79**]*). This abelian group vanishes if 2 is a unit in* R *or if* H *is square-root closed*[8] *in* G. □

REMARK 1.3.6. Using the above vanishing condition, Cappell applied this theorem and a five-lemma argument to show that the Novikov conjecture (with **Q**-coefficients) is true under taking closure by a certain kind of injective Bass-Serre graphs of groups [**Cap76a**, Thm. 1].

REMARK 1.3.7. We outline the geometric definition of the connecting homomorphism, using Remark 1.2.4:

$$\begin{pmatrix}\mathrm{split}_L\\ \partial_L\end{pmatrix} : L_{n+1}^{\nu\oplus\sigma}(G) \longrightarrow \mathrm{UNil}_{n+1}^{\nu}(\Phi^{\omega}) \oplus L_n^{\partial_K(\sigma)}(H).$$

Choose appropriate manifolds $(Y, \partial Y)$ and $(X, \partial X)$ with these fundamental groups and orientation character (see §1.2.1 for notation). Let $x \in L_{n+1}^{\nu\oplus\sigma}(G)$ and consider its Wall realization

$$\partial(x) := \mathrm{act}(x, \mathbf{1}_Y) \in \mathcal{S}^{\nu\oplus\sigma}(Y, \partial Y).$$

Define the composite

$$\mathrm{split}_L(x) := (\mathrm{proj}_1 \circ \mathrm{nncc} \circ \partial)(x) \in \mathrm{UNil}_{n+1}^{\nu}(\Phi^{\omega}).$$

Consider the other projection

$$[g] = (\mathrm{proj}_2 \circ \mathrm{nncc} \circ \partial)(x) \in \mathcal{S}_{\mathrm{split}}^{\sigma}(Y, \partial Y; X).$$

[8]We call a subgroup H **square-root closed in** G if $g \in G$ and $g^2 \in H$ imply $g \in H$.

There is a normal bordism G between the split homotopy equivalence g and the identity $\mathbf{1}_Y$, with surgery obstruction $x - \mathrm{split}_L(x)$. It is obtained by uniting Wall's normal bordism between the homotopy equivalence $\partial(x)$ and the identity $\mathbf{1}_Y$, with surgery obstruction x, to Cappell's normal bordism between the homotopy equivalence $\partial(x)$ and the split homotopy equivalence g, with surgery obstruction $\mathrm{split}_L(x)$. Observe that the transversal restriction F of G to X is an n-dimensional normal bordism, whose lower boundary is the identity $\mathbf{1}_X$, and whose upper boundary is a homotopy equivalence $g|_{g^{-1}(X)}$. Therefore it has a well-defined surgery obstruction

$$\partial_L(x) := \sigma(F) \in L_n^{\partial_K(\sigma)}(H).$$

1.3.3. The controlled/assembly viewpoint of UNil**.** The failure of L-theory being excisive is embodied in the homotopy cofiber of an assembly map, and the success of its homological approximation is measured by controlled topology.

Let G be an injective amalgam $G = J_- *_H J_+$ or an HNN-extension $G = *_H J$ (as denoted in §1.2.1). The associated **Bass-Serre tree** T is a contractible simplicial 1-complex, on which G has an effective transitive simplicial action without inversions, such that its vertex isotropy groups are conjugate to the vertex groups of the groupoid $J = J_- \cup J_+$, and that its edge isotropy groups are conjugate to H.

Proposition 1.3.8 (Decorated, controlled version of [**Wei94**, Exercise §6.2A]). *Consider the control map*

$$p : EG \times_G T \longrightarrow T \longrightarrow T/G = \begin{cases} D^1 & \text{if } G \text{ is an injective amalgam} \\ S^1 & \text{if } G \text{ is an injective HNN-extension.} \end{cases}$$

Suppose ν and σ and π are $*$-invariant subgroups of $\widetilde{\mathrm{Nil}}_0(\Phi)$ and $\mathrm{Wh}(J)$ and I such that $\partial_K(j\sigma) = \pi$. Then for all $n \in \mathbf{Z}$, the following sequence of abelian groups is split exact:

$$0 \longrightarrow L_n^{\pi \to \sigma}(T/G; p) \xrightarrow{\text{forgetcontrol}} L_n^{\nu \oplus j\sigma}(G) \xrightarrow{\mathrm{split}_L} \mathrm{UNil}_n^{\nu}(\Phi^\omega) \longrightarrow 0.$$

Remark 1.3.9. We describe the context of the left-hand term, without decorations. For any block fibration $p : E \to X$ between finite polyhedra, M. Yamasaki showed [**Yam87**, Thm. 3.9] that there exists a certain functorial homotopy equivalence A. Consequently, the following diagram of spectra homotopy commutes:

$$\begin{tikzcd}
\mathbb{H}(X; \mathbf{L}.(p)) \arrow[r, "\text{blasmb}"] \arrow[d, "A"', "\simeq"] & \mathbf{L}.(E) \arrow[d, "A"', "\simeq"] \\
\mathbb{L}(X; p) \arrow[r, "\text{forgetcontrol}"] & \mathbb{L}(\text{pt}; E \to \text{pt}).
\end{tikzcd}$$

The upper-right term $\mathbf{L}.(E)$ is the classical $L_*^{\langle -\infty \rangle}$-spectrum of the space E, and blasmb denotes the assembly map blocked over X. Following the notation of F. Quinn (see [**Qui82**, Defn. 8.1] and [**Ran92a**] for the definitions), we write homotopy groups in each dimension $n \in \mathbf{Z}$ as follows. The controlled L-group of p is denoted $L_n(X; p)$, and the blocked $\mathbf{L}$.-homology group of p is denoted $H_n(X; \mathbf{L}.(p))$. Both of these functors, from the category of block fibrations to the category of Ω-spectra, are homotopy invariant and excisive. Originally, Quinn developed [**Qui79**] the above diagram and homotopy equivalence A for the pseudo-isotopy functor $\mathscr{P}$.

Proof of Proposition 1.3.8. We may assume $n > 6$ by periodicity isomorphisms, such as

$$\otimes \sigma^*(\mathbf{CP}^2) : L_n(E) \longrightarrow L_{n+4}(E).$$

Consider Yamasaki's identification

$$A_* : H_n^{\pi \to \sigma}(T/G; \mathbf{L}.(p)) \xrightarrow{\cong} L_n^{\pi \to \sigma}(T/G; p)$$

in Remark 1.3.9, but with intermediate decorations. It is equivalent to show the split exactness of the following sequence of abelian groups:

$$0 \longrightarrow H_n^{\pi \to \sigma}(T/G; \mathbf{L}.(p)) \xrightarrow{\text{blasmb}} L_n^{\nu \oplus j\sigma}(G) \xrightarrow{\text{split}_L} \text{UNil}_n^{\nu}(\Phi^{\omega}) \longrightarrow 0,$$

which we abbreviate to

$$0 \longrightarrow H \xrightarrow{\text{blasmb}} L \xrightarrow{\text{split}} U \longrightarrow 0.$$

Recall that, in the setting of rings and bimodules with involution, Cappell algebraically defines a map $\psi : U \to L$. In the case of group rings [**Cap74b**, Thm. 2], ψ is a monomorphism split by the geometrically-defined map split : $L \to U$. We shall define an isomorphism between L and $H \oplus U$.

Construct a connected closed DIFF $(n-1)$-manifold Y and two-sided submanifold X, with desired fundamental groups G and H and orientation character ω, as follows. First, start with the $(n-1)$-disc and attach 1- and 2-handles to obtain a connected compact DIFF $(n-1)$-manifold Y_0 with boundary X, having finitely presented fundamental group H^ω. Next, perform interior 0- and 1-surgeries on Y_0 to obtain a compact DIFF $(n-1)$-manifold Z with boundary X and finitely presented fundamental groupoid J^ω. Finally, using the monomorphism $i : H \to J$, glue to obtain a connected closed manifold

$$Y = \bigcup_{X \times D^1} Z.$$

Now define a set map $\varphi : L \to H$ as follows. First, by Wall realization [**Wal99**, Thm. 10.4], represent any element $x \in L$ as the surgery obstruction of a normal map of manifold triads

$$F : (V^n; Y, W) \longrightarrow (Y \times [0,1]; Y \times 0, Y \times 1)$$

such that $F_0 : Y \to Y$ is the identity and $F_1 : W \to Y$ is a $(\nu \oplus j\sigma)$-torsion homotopy equivalence. Next, since the pushout G is injective, by Cappell's nilpotent normal cobordism construction [**Cap76b**, §II.1 §III.2] applied to F_1 with reversed orientation, there exists a normal bordism of manifold triads

$$F' : (V'; W, W') \longrightarrow (Y \times [1,2]; Y \times 1, Y \times 2)$$

from F_1 to a $(j\sigma)$-torsion homotopy equivalence $F_2' : W' \to Y$ split along X. Finally, glue to obtain a normal map of manifold triads

$$F'' := F \cup F' : (V''; Y, W') \longrightarrow (Y \times [0,2]; Y \times 0, Y \times 2)$$

with surgery obstruction

$$\sigma(F'') = x - (\psi \circ \mathrm{split})(x) \in \mathrm{Ker}(\mathrm{split} : L \to U)$$

such that both homotopy equivalences $F_0'' = F_0$ and $F_2'' = F_1'$ are split along X. As outlined in Remark 1.3.7, we may cut F'' open along the transverse inverse image of $X \times [0,2]$ to obtain a normal map

$$\overline{F''} : (\overline{V}; Z, \overline{W'}, N; X, M') \longrightarrow (Z \times [0,2]; Z \times 0, Z \times 2, X \times [0,2]; X \times 0, X \times 2),$$

whose restrictions to the outer boundary pieces Z and $\overline{W'}$ are homotopy equivalences. Its surgery kernel

$$\sigma_*(\overline{F''}) = (y; z)$$

is an n-dimensional algebraic quadratic Poincaré pair (see [**Ran81**, Prop. 2.2.1, Dfn.]) over the morphism $\mathbf{Z}[H^\omega] \to \mathbf{Z}[G^\omega_\pm]$ of rings with involution. Therefore, we can define an n-dimensional $\mathbf{L}.(p)$-cycle $(y; z)$ with homology class (see [**Ran92a**, Defn. 12.17])

$$\varphi(x) := [(y; z)] \in H.$$

Repeat the above construction for any element $y \in L$, turn its normal bordism G'' upside down, which induces the reversed orientation on the identity map $Y \to Y$, and glue to obtain

$$E := F'' \cup -G''.$$

Note that both boundary pieces are homotopy equivalences split along X. Also note

$$\sigma(E) = (x - y) - (\psi \circ \mathrm{split})(x - y),$$

which induces the element $\varphi(x - y) \in H$, by definition of φ. However

$$\overline{E} = \overline{F''} \cup -\overline{G''},$$

which induces the element $\varphi(x) - \varphi(y) \in H$, by definition of H. Therefore these elements are equal. Thus φ is subtractive, and hence φ is a group morphism.

It now suffices to show the following group morphisms are inverses:

$$\begin{pmatrix} \varphi \\ \mathrm{split} \end{pmatrix} : L \to H \oplus U \quad \text{and} \quad \begin{pmatrix} \mathrm{blasmb} & \psi \end{pmatrix} : H \oplus U \to L.$$

Note from the definition of H that $\mathrm{split} \circ \mathrm{blasmb} = 0$, and from the definition of φ that $\varphi \circ \mathrm{blasmb} = \mathbf{1}_H$. Also $\psi - \psi \circ \mathrm{split} \circ \psi = 0$ implies $\varphi \circ \psi = 0$. Therefore

$$\begin{pmatrix} \varphi \\ \mathrm{split} \end{pmatrix} \circ \begin{pmatrix} \mathrm{blasmb} & \psi \end{pmatrix} = \begin{pmatrix} \mathbf{1}_H & 0 \\ 0 & \mathbf{1}_U \end{pmatrix}.$$

Also note from the definition of φ that $\mathrm{blasmb} \circ \varphi = \mathbf{1} - \psi \circ \mathrm{split}$. Therefore

$$\begin{pmatrix} \mathrm{blasmb} & \psi \end{pmatrix} \circ \begin{pmatrix} \varphi \\ \mathrm{split} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_L \end{pmatrix}.$$

$\square$

REMARK 1.3.10. The above explicit proof is equivalent to a five-lemma argument applied to the commutative ladder between the Mayer-Vietoris sequences [**Qui82**, Prop. 8.4] and [**Cap74b**, Thm. 5.2]:

$$\begin{tikzcd}
H_n^{\pi}((-1,1);p) \arrow[r, "i_- - i_+"] \arrow[d, "\cong"] & H_n^{\sigma}([-1,1);p) \oplus H_n^{\sigma}((-1,1];p) \arrow[r, "j"] \arrow[d, "\cong"] & H_n^{\pi\to\sigma}([-1,1];p) \arrow[r, "\partial"] \arrow[d] & H_{n-1}^{\pi}((-1,1);p) \arrow[d, "\cong"] \\
L_n^{\pi}(H) \arrow[r, "i_- - i_+"] & L_n^{\sigma}(J_-) \oplus L_n^{\sigma}(J_+) \arrow[r, "j"] & \dfrac{L_n^{\nu\oplus j\sigma}(G)}{\mathrm{UNil}_n^{\nu}(\Phi^{\omega})} \arrow[r, "\partial"] & L_{n-1}^{\pi}(H).
\end{tikzcd}$$

(Similarly for S^1.) The indicated vertical isomorphisms are Yamasaki's normalization lemma [**Yam87**], which applies to $\mathbf{L}$.-homology with cosheaf coefficients where there is a single contractible stratum.

CHAPTER 2

L-theory of type III virtually cyclic groups: Equivalences

Unless otherwise specified in the rest of Part I, all orientation characters are trivial, and all K-theory decorations are excluded: $L_* = L_*^{\langle -\infty \rangle}$ (defined in §1.1.3).

The Farrell-Jones isomorphism conjecture in L-theory [**FJ93**] states for any group Γ that $L_*(\mathbf{Z}[\Gamma])$ is determined by $L_*(\mathbf{Z}[V])$ of all virtually cyclic subgroups V of Γ together with certain homological information. More specifically, there are spectral sequences (Atiyah-Hirzebruch [**Qui82**] and p-chain Davis-Lück [**DL03**]) which converge to the fibered $\mathbf{L}$.-homology of the classifying space $B_{\mathcal{VC}}(\Gamma)$ for Γ-actions with virtually cyclic isotropy. A group V is **virtually cyclic** if it contains a cyclic subgroup of finite index. Equivalently:

(I) V is a finite group, or

(II) V is a group extension $1 \to F \to V \to \mathbf{C}_\infty \to 1$ for some finite group F, or

(III) V is a group extension $1 \to F \to V \to \mathbf{D}_\infty \to 1$ for some finite group F (see §5.1 for more on $\mathbf{D}_\infty$).

The L-theory of type I, with various decorations, is determined classically by Wall and others [**Wal76**]. The L-theory of type II is determined by a combination the L-theory of type I and the monodromy map $(\mathbf{1} - \alpha_*)$ in the Cappell-Ranicki-Shaneson-Wall sequence [**Wal99**, §12B] [**Ran73b**]. The groups V of type III admit a decomposition

$$V = V_- *_F V_+$$

as an injective amalgam with

$$[V_\pm : F] = 2.$$

Thus the Mayer-Vietoris sequence [**Cap74a**] determines the groups $L_*(\mathbf{Z}[V])$ as a combination of the L-groups of the type I groups F, V_-, V_+ and of Cappell's splitting obstruction groups

$$\mathrm{UNil}_*(\mathbf{Z}[F];\mathbf{Z}[V_- \setminus F],\mathbf{Z}[V_+ \setminus F]).$$

Recent computations [**CK95, CR05, CD04, BR06**] of these UNil-groups for $F=1$ provide a starting point for our determination of the L-groups $L_*(\mathbf{Z}[V])$ for certain classes of type III virtually cyclic groups V.

The various equivalences established in this chapter are the fabric through which the tapestry of computations in Chapters 3 and 4 are woven.

2.1. Foundational results on UNil

2.1.1. Twisted generalization of the Connolly-Ranicki theorem. As desired for our study of type III virtually cyclic groups, we generalize the Connolly-Ranicki theorem [**CR05**, Thm. A], which is recovered below from $K=1$. Our generalization includes that which was observed by Connolly-Davis [**CD04**] for their computation of $\mathrm{UNil}^h_*(\mathbf{Z};\mathbf{Z}^-,\mathbf{Z})$, corresponding to $u=-1$.

Definition 2.1.1 (generalizes [**Ran81**, §5.1]). Let A be a ring with involution, α an automorphism of A, and $u \in A^\times$ a unit. Suppose for all $a \in A$ that the following identities are satisfied:

$$u\alpha(\overline{u}) = 1 \quad \text{and} \quad (\overline{\alpha})^2(a) = (\overline{u})a(\overline{u})^{-1}, \quad \text{where} \quad \overline{\alpha}(a) := \overline{\alpha(a)}.$$

The (α,u)**-twisted polynomial extension of** A, denoted $(A^u_\alpha[x],{}^-)$, is the following ring with involution. Define the (A,A)-bimodule

$$A^u_\alpha[x] := \bigoplus_{n\in\mathbf{Z}_{\geq 0}} x^n A \quad \text{with commutation rule} \quad ax^n = x^n\alpha^n(a).$$

For all $a \in A, n \in \mathbf{Z}_{\geq 0}$, its involution is given by

$$\overline{x^n a} = \overline{a}(xu^{-1})^n.$$

Let F be a covariant functor from rings with antistructure to abelian groups, and let $\varepsilon \in A$. Define an abelian group

$$N^u_{\varepsilon,\alpha}F(A) := \mathrm{Ker}\left(\mathrm{aug}_\varepsilon : F(A^u_\alpha[x]) \to F(A)\right).$$

For each $\varepsilon \in A$, there is a decomposition

$$F(A^u_\alpha[x]) = F(A) \oplus N^u_{\varepsilon,\alpha}F(A).$$

Abbreviate

$$N^u_\alpha F(A) := N^u_{0,\alpha}F(A) \quad \text{and} \quad NF(A) := N^1_{\mathrm{id}}F(A);$$

the latter coincides with the notation of Bass [**Bas68**, Ch. XII].

THEOREM 2.1.2. *Let G be a group extension*

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbf{D}_\infty \longrightarrow 1,$$

and let R a ring with involution. Write the injective amalgam

$$G = G_- *_K G_+ \quad \textit{with} \quad [G_\pm : K] = 2.$$

Suppose the extension is ***semitrivial:***

$$G_+ = K \times \mathbf{C}_2.$$

Select a right coset representative $t \in G_- \setminus K$. Define an R-algebra automorphism

$$\alpha : R[K] \longrightarrow R[K]; \qquad \alpha(g) := t^{-1}gt,$$

and define a unit

$$u := t^2 \in K \subset R[K]^\times.$$

Then for all $n \in \mathbf{Z}$ and decorations $\kappa = h, p$, there exists an isomorphism

$$r : \mathrm{UNil}^\kappa_n(R[K]; R[G_- \setminus K], R[G_+ \setminus K]) \longrightarrow N^u_\alpha L^\kappa_n(R[K]).$$

The map r is natural in rings R with involution and in semitrivial extensions G. If n is even, then r is given in terms of the indeterminate x by

$$r[P_-, \theta_-; P_+, \theta_+] = \left[(P_- \oplus P_+)[x], \begin{pmatrix} x\theta_- & 1 \\ 0 & \theta_+ \end{pmatrix} \right].$$

COROLLARY 2.1.3 ([**CR05**, Theorem A]). *For any ring R with involution, there is a natural isomorphism*

$$r : \mathrm{UNil}^h_n(R; R, R) \longrightarrow NL^h_n(R).$$

PROOF. Take $K = 1$. Then $\alpha = \mathrm{id}$ and $u = 1$. So $R[G_{\pm} \setminus K] \cong R$ as (R, R)-bimodules with involution. $\square$

REMARK 2.1.4. In the determination of these particular UNil-groups, the advantage of the Connolly-Ranicki isomorphism r is that the NL-groups fit into several exact sequences. The NL-groups in some cases can be computed using homological algebra [**CR05, BR06**] and linking forms [**CD04, BDK**], whereas the UNil-groups were originally designed to fit into Cappell's L-theory Mayer-Vietoris sequence [**Cap74b**]. A mirage of this isomorphism is proven earlier by Connolly-Koźniewski [**CK95**, Thm. 3.9], in terms of a certain additive category $(\mathbb{A}, \alpha)$ with involution. However our result is not apparently its corollary.

LEMMA 2.1.5. *Let A be a ring with involution, let $\mathscr{B}$ be an (A, A)-bimodule with involution $^\wedge$, and let P be a finitely generated projective left A-module. Suppose $\theta : P \times P \to \mathscr{B}$ is a sequilinear form over A with values in $\mathscr{B}$. For any $n \in \mathbf{Z}_{\geq 0}$, denote the tensor power*

$$\mathscr{B}^n := \underbrace{\mathscr{B} \otimes_A \mathscr{B} \otimes_A \cdots \otimes_A \mathscr{B}}_{n \text{ copies}}.$$

In particular, $\mathscr{B}^0 := A$.

(1) *The* ***tensor algebra***

$$T(\mathscr{B}) := \bigoplus_{n \in \mathbf{Z}_{\geq 0}} \mathscr{B}^n$$

is a ring with its involution $^\wedge$ defined by

$$(\beta_1 \otimes \cdots \otimes \beta_n)^\wedge := (\beta_n)^\wedge \otimes \cdots \otimes (\beta_1)^\wedge.$$

It admits a split monomorphism $\mathrm{power}_0 : A \to T(\mathscr{B})$ *of rings with involution and a split monomorphism* $\mathrm{power}_1 : \mathscr{B} \to T(\mathscr{B})$ *of (A, A)-bimodules with involution.*

(2) *The identification*

$$\Phi_{\mathscr{B}} : \mathscr{B} \otimes_A P^* \longrightarrow \mathrm{Hom}_A(P, \mathscr{B}); \qquad \beta \otimes f \longmapsto (p \mapsto f(p)\, \beta^\wedge)$$

is an isomorphism of left A-modules. Its extension

$$\Phi_{T(\mathscr{B})} : T(\mathscr{B}) \otimes_A P^* \longrightarrow \mathrm{Hom}_A(P, T(\mathscr{B})); \qquad \gamma \otimes f \longmapsto (p \mapsto f(p)\,\gamma^\wedge)$$

is an isomorphism of left $T(\mathscr{B})$-modules.

(3) *The left $\mathscr{B}$-adjoint*

$$\theta_{\mathscr{B}} : P \to \mathscr{B} \otimes_A P^*; \qquad p \longmapsto (\Phi_{\mathscr{B}})^{-1}(q \mapsto \theta(p,q))$$

is a morphism of left A-modules. Moreover, its extension

$$\theta_{T(\mathscr{B})} : T(\mathscr{B}) \otimes_A P \longrightarrow T(\mathscr{B}) \otimes_A P^*; \qquad \gamma \otimes p \longmapsto (\Phi_{T(\mathscr{B})})^{-1}(q \mapsto \theta(p,q) \otimes \gamma^\wedge)$$

is a morphism of left $T(\mathscr{B})$-modules. □

Remark 2.1.6. Our main theorem generalizes to semitrivial extensions where $\mathbf{D}_\infty = \mathbf{C}_2 * \mathbf{C}_2$ is replaced with $Q * \mathbf{C}_2$ for any group Q. The resulting NL-group corresponds to the (α, u)-twisted noncommutative polynomial ring $R[K]\{x_i\}_{i\in|Q|-1}$, isomorphic to the tensor algebra $T(R[G_- \setminus K])$. Here α and u are multi-indices corresponding to choices of nontrivial right coset representatives $\{t_i\}_{i\in|Q|-1}$. However we do not pursue this generalization because our focus is on type III virtually cyclic groups.

Definition 2.1.7. In the setting of Theorem 2.1.2, consider the ring $A := R[K]$ with involution. Define A^u_α as the (A, A)-bimodule with involution, whose: right A-module structure is the standard structure on A, left A-module structure is defined by $a \cdot b := \alpha(a)b$, and involution is defined by $(rg)^\wedge := \overline{r}\alpha(g^{-1})u^{-1}$. Indeed, the left and right A-actions do associate, and the involution does distribute over products. Furthermore, observe that

$$\mathrm{Hom}_A(P, Q) \longrightarrow \mathrm{Hom}_A(P, A^u_\alpha \otimes_A Q); \qquad f \longmapsto (p \mapsto 1 \otimes f(p))$$

is an isomorphism of abelian groups for all left A-modules P, Q.

Lemma 2.1.8. *Consider the (A, A)-bimodule A^u_α with involution.*

(1) *The identification*

$$R[G_\pm \setminus K] \longrightarrow A^u_\alpha; \qquad rt_\pm g \longmapsto rg$$

is an isomorphism of (A,A)-bimodules with involution.

(2) *The identification*

$$\Psi : T(A^u_\alpha) \longrightarrow A^u_\alpha[x]; \qquad \beta_1 \otimes \cdots \otimes \beta_n \longmapsto x\beta_1 \cdots x\beta_n$$

is an isomorphism of A-algebras with involution. It restricts to a morphism $x : A^u_\alpha \to A^u_\alpha[x]$ of (A,A)-bimodules. □

The following lemma is key in the proof of the theorem. It validates our definition of the ring $B := A^u_\alpha[x]$ with involution, as introduced in Definition 2.1.1 and Theorem 2.1.2.

Lemma 2.1.9. *Let $\epsilon = \pm 1$. Consider a nonsingular ϵ-quadratic unilform $u = (P_-, \theta_-; P_+, \theta_+)$ over $(A; A^u_\alpha, A)$. Using Lemmas 2.1.5 and 2.1.8, write*

$$P_\pm[x] := B \otimes_A P_\pm \quad \textit{and} \quad \theta_\pm : P_\pm[x] \to P_\mp[x].$$

Then the following ϵ-quadratic form over B is nonsingular:

$$r(u) := \left((P_- \oplus P_+)[x], \begin{pmatrix} x\theta_- & 1 \\ 0 & \theta_+ \end{pmatrix} \right).$$

Proof. Consider the sesquilinear forms

$$\theta_- : P_- \times P_- \to \mathscr{B} := A^u_\alpha \quad \text{and} \quad \theta_+ : P_+ \times P_+ \to A.$$

The adjoint of their ϵ-symmetrizations λ_- and λ_+ are defined by

$$\lambda_- : P_- \times P_- \to \mathscr{B} \quad \text{and} \quad \lambda_+ : P_+ \times P_+ \to A; \quad p \longmapsto \Phi^{-1}(q \mapsto \theta_\pm(p,q) + \epsilon\theta_\pm(q,p)^\wedge).$$

The unilform $u = (P_-, \theta_-; P_+, \theta_+)$ satisfies the nilpotence condition: there exists $N > 0$ such that

$$((\lambda_-)_{\mathscr{B}} \circ (\lambda_+)_A)^N = 0 : P_+ \longrightarrow (\mathscr{B} \otimes_A A)^N \otimes_A P_+.$$

Observe, by extension to the tensor algebra, that

$$((\lambda_-)_{T(\mathscr{B})} \circ (\lambda_+)_{T(\mathscr{B})})^N = 0 \in \mathrm{End}_{T(\mathscr{B})}(T(\mathscr{B}) \otimes_A P_+).$$

Hence by Lemma 2.1.8 we obtain

$$(x\lambda_- \circ \lambda_+)^N = 0 \in \mathrm{End}_B(P_+[x]).$$

Define a left B-module morphism Λ by

$$\Lambda := \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \sum_{n=0}^{2N} (-1)^n \begin{pmatrix} 0 & \epsilon x\lambda_- \\ \lambda_+ & 0 \end{pmatrix}^n : (P_- \oplus P_+)[x] \longrightarrow (P_+ \oplus P_-)[x].$$

Note that our nilpotence condition implies on $(P_- \oplus P_+)[x]$ that the "geometric series" Λ satisfies

$$\Lambda \begin{pmatrix} x\lambda_- & 1 \\ \epsilon 1 & \lambda_+ \end{pmatrix} = \mathrm{id} = \begin{pmatrix} x\lambda_- & 1 \\ \epsilon 1 & \lambda_+ \end{pmatrix} \Lambda.$$

Therefore $r(u)$ is a nonsingular ϵ-quadratic form over B. $\square$

Higman linearization is the mechanism for defining the inverse of r, but we must first check that it works in the generality of (α, u)-twisted cyclic extensions. Below we again write $P[x] := A^u_\alpha[x] \otimes_A P$ and identify $P[x]^* = P^*[x]$ as (left) $A^u_\alpha[x]$-modules using Lemma 2.1.5.

Lemma 2.1.10. *Let A be a ring with involution, and let (α, u) with any x satisfy Definition 2.1.1. Consider $\vartheta \in L^p_{2k}(A^u_\alpha[x])$. Then there exist a finitely generated projective A-module P and morphisms $f_0, f_1 : P \to P^*$ such that*

$$\vartheta = [P[x], f_0 + x f_1].$$

Remark 2.1.11. The case of (α, u) trivial and x indeterminate is [**Ran74**, Lemma 4.2]. Its generalization to the polynomial extension of an additive category is provided in [**CK95**, Lemma 3.6a, Proposition 3.6b].

Proof. There exist a f.g. projective A-module P, a degree $N > 0$, and morphisms $f_j : P \to P^*$ for all $0 \leq j \leq N$ such that

$$\vartheta = \left[P[x], f := \sum_{j=0}^{N} x^j f_j \right].$$

If $N = 1$ then we are done, so assume $N > 1$. We proceed by backwards induction on N. Define a f.g. projective A-module

$$P' := P \oplus P \oplus P^*$$

and a morphism f' of degree $N-1$ by

$$f' := \begin{pmatrix} f - x^N f_N & 0 & x \\ -x^{N-1} f_N & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} : P'[x] \longrightarrow P'^*[x].$$

Define an A-module automorphism

$$\varphi := \begin{pmatrix} 1 & 0 & 0 \\ -\overline{\alpha}(u)^{-1} x & 1 & 0 \\ x^{N-1} f_N & 0 & 1 \end{pmatrix} : P'[x] \to P'[x].$$

Note that the pullback of the quadratic form f' along φ is

$$\varphi^* \circ f' \circ \varphi = \begin{pmatrix} 1 & -x & (x^{N-1} f_N)^* \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have an isomorphism of $(-1)^k$-quadratic forms over $A^u_\alpha[x]$:

$$(\varphi, 0) : (P[x], f) \oplus \mathscr{H}(P[x]) \longrightarrow (P'[x], f').$$

Therefore the Witt class

$$\vartheta = \left[P'[x], f' = \sum_{j=0}^{N-1} x^j f'_j \right]$$

is represented by a form of one lesser degree (but thrice the rank). □

Now we induce homomorphisms between the desired L-groups.

LEMMA 2.1.12. *Let $k \in \mathbf{Z}$, and let $\kappa = s, h, p$ be a decoration. The function r defined in Lemma 2.1.9 induces a natural homomorphism*

$$r : \mathrm{UNil}^\kappa_{2k}(A; A^u_\alpha, A) \longrightarrow N^u_\alpha L^\kappa_{2k}(A).$$

PROOF. Write $\epsilon := (-1)^k$. Let $u = (P_-, \theta_-; P_+, \theta_+)$ be a nonsingular ϵ-quadratic unilform over $(A; A^u_\alpha, A)$. The image of $r(u)$ under $\mathrm{aug}_0 : B \to A$ is

$$\mathrm{aug}_0(r(u)) \cong \mathscr{H}(P_-[x]).$$

Then, since $r(u)$ is nonsingular over $B = A^u_\alpha[x]$ by Lemma 2.1.9, we obtain an equivalence class

$$[r(u)] \in N^u_\alpha L^\kappa_{2k}(A).$$

Let $u' = (P'_-, \theta'_-; P'_+, \theta'_+)$ be a nonsingular ϵ-quadratic unilform over $(A; A^u_\alpha, A)$. Denote the switch automorphism

$$\mathrm{Sw} := \mathbf{1}_{P_-[x]} \oplus \mathrm{sw} \oplus \mathbf{1}_{P'_+[x]}, \quad \text{where} \quad \mathrm{sw} : P_+[x] \oplus P'_-[x] \longrightarrow P'_-[x] \oplus P_+[x].$$

Observe

$$\mathrm{Sw}^*(r(u \oplus u')) = r(u) \oplus r(u').$$

Hence

$$[r(u \oplus u')] = [r(u)] + [r(u')].$$

Thus $[r]$ is additive.

Suppose (V_-, V_+) is a lagrangian of u; see Definition 1.1.5. Observe that

$$(V_- \oplus V_+)[x]$$

is a lagrangian of $r(u)$. Thus r induces a homomorphism, well-defined on Witt classes. □

LEMMA 2.1.13. *Let P be an A-module, and write $P[x] := B \otimes_A P$. A morphism*

$$\nu : P \to A^u_\alpha \otimes_A P$$

is nilpotent if and only if

$$\mathbf{1} - x\nu : P[x] \to P[x]$$

is an isomorphism.

PROOF. The inverse must be given by a finite "geometric series"

$$\sum_{n \in \mathbf{Z}_{\geq 0}} (x\nu)^n$$

as in Proof 2.1.9. □

LEMMA 2.1.14. *Let $k \in \mathbf{Z}$; write $\epsilon := (-1)^k$. Consider the function s defined by*

$$s(P[x], f_0 + x f_1) := \left(P, f_1; P^*, -(f_0 + \epsilon f_0^*)^{-1} f_0^* (f_0 + \epsilon f_0^*)^{-1}\right),$$

in terms of A-module morphisms

$$f_0 : P \longrightarrow A \otimes_A P^* \quad \textit{and} \quad f_1 : P \longrightarrow A^u_\alpha \otimes_A P^*.$$

Then for all decorations $\kappa = s, h, p$, the function s induces a homomorphism

$$s : N^u_\alpha L^\kappa_{2k}(A) \longrightarrow \mathrm{UNil}^\kappa_{2k}(A; A^u_\alpha, A).$$

Proof. Observe the isomorphism

$$f_0 + \epsilon f_0^* : P \longrightarrow P^*.$$

Since the ϵ-quadratic form $v = (P[x], f_0 + x f_1)$ over $B = A^u_\alpha[x]$ is nonsingular, the following map is an isomorphism:

$$(f_0 + \epsilon f_0^*) + x(f_1 + \epsilon f_1^\wedge) : P[x] \longrightarrow P^*[x].$$

Then, by Lemma 2.1.13, we conclude that the following map is a nilpotent morphism:

$$(f_0 + \epsilon f_0^*)^{-1}(f_1 + \epsilon f_1^\wedge) : P \longrightarrow (A^u_\alpha \otimes_A A) \otimes_A P.$$

Then $s(v)$ is a nonsingular ϵ-quadratic unilform over $(A; A^u_\alpha, A)$, so we obtain an equivalence class

$$[s(v)] \in \mathrm{UNil}^\kappa_{2k}(A; A^u_\alpha, A).$$

Let $v' = (P'[x], f_0' + x f_1')$ be another such linearized ϵ-quadratic form. Then

$$s(v \oplus v') = s(v) \oplus s(v')$$

on the nose, hence

$$[s(v \oplus v')] = [s(v)] + [s(v')].$$

Thus $[s]$ is additive.

Suppose S is a lagrangian of v. Necessarily, it is of the form $S = P_-[x]$, where

$$P_- := S \cap P \subset P[x].$$

Then P_- is a lagrangian of both (P, f_0) and (P, f_1) with annihilator

$$(f_0 + \epsilon f_0^*)P_- = P_-^\circ,$$

since S is a $(f_0 + xf_1)$-lagrangian. Observe $(V_-, V_+) = (P_-, P_-^\circ)$ is a lagrangian of $s(v)$, see Proof 2.1.12. Thus by Lemma 2.1.10 the domain of the function $[s]$ extends to all nonsingular ϵ-quadratic forms ϑ over B with null-augmentation over A:

$$\vartheta = (P[x], f = \sum_{j=0}^{N} x^j f_j).$$

Therefore we obtain an induced homomorphism

$$s : N_\alpha^u L_{2k}^\kappa(A) \to \mathrm{UNil}_{2k}^\kappa(A; A_\alpha^u, A),$$

well-defined on Witt classes. □

We are ready to prove the main theorem of this section.

Proof of Theorem 2.1.2. Observe that Lemma 2.1.8 performs the identification

$$\mathrm{UNil}_n^\kappa(R[K]; R[G_- \setminus K], R[G_+ \setminus K]) \cong \mathrm{UNil}_n^\kappa(A; A_\alpha^u, A).$$

Here $A := R[K]$ is the group ring with involution $\overline{rg} := \overline{r}g^{-1}$, and the (A, A)-bimodule A_α^u is defined above the lemma. By our definition (1.1.8) of UNil_n^κ for n odd, the Ranicki-Shaneson sequence in L^κ-groups, and naturality, it suffices to prove the theorem for $n = 2k$ even. That is, it remains to check that the homomorphisms r and s defined in Lemmas 2.1.12 and 2.1.14 are inverses.

Write $\epsilon := (-1)^k$. Let $u = (P_-, \theta_-; P_+, \theta_+)$ be a nonsingular ϵ-quadratic unilform over $(A; A_\alpha^u, A)$. Note

$$(s \circ r)(u) = \left(P_- \oplus P_+, \begin{pmatrix} \theta_- & 0 \\ 0 & 0 \end{pmatrix}; P_+ \oplus P_-, \begin{pmatrix} \theta_+ & -1 \\ 0 & 0 \end{pmatrix} \right).$$

Observe that it has a sublagrangian

$$(V_-, V_+) = (0 \oplus P_+, 0),$$

whose annihilator is

$$(V_+^\circ, V_-^\circ) = (P_- \oplus P_+, P_+ \oplus 0).$$

So we obtain the Witt equivalent unilform $(P_-, \theta_-; P_+, \theta_+)$, supported on the sublagrangian construction $(V_+^\circ/V_-; V_-^\circ/V_+)$. Therefore

$$(s \circ r)[u] = [u].$$

Using Lemma 2.1.13, let $v = (P[x], f_0 + xf_1)$ be a nonsingular ϵ-quadratic form over B with null-augmentation. Note

$$(r \circ s)(v) = \left((P \oplus P^*)[x], \begin{pmatrix} xf_1 & 1 \\ 0 & -(f_0 + \epsilon f_0^*)^{-1} f_0^* (f_0 + \epsilon f_0^*)^{-1} \end{pmatrix} \right).$$

Observe that it has a sublagrangian

$$S = 0 \oplus (f_0 + \epsilon f_0^*) S_0[x],$$

where S_0 is a lagrangian of the nonsingular ϵ-quadratic form (P, f_0), and whose annihilator is

$$S^\perp = S_0[x] \oplus 0 + \mathrm{Graph}(f_0 + \epsilon f_0^* : P \to P^*)[x].$$

So we obtain the Witt equivalent quadratic form $(P[x], f_0 + xf_1)$, supported on the sublagrangian construction $S^\perp / S$. Therefore

$$(r \circ s)[v] = [v].$$

Thus the homomorphisms r and s are inverses. □

2.1.2. Vanishing theorems. The sequel documents and directly proves known results for UNil-groups in general and NL-groups in particular.

THEOREM 2.1.15 (Cappell). *Let $A = A_- *_R A_+$ be a pure pushout of rings with involution. Write $A_\pm = R \oplus \mathscr{B}_\pm$ as (R, R)-bimodules with involution. Suppose 2 is a unit in R. Then, for all $n \in \mathbf{Z}$,:*

$$\mathrm{UNil}_n^h(R; \mathscr{B}_-, \mathscr{B}_+) = 0.$$

REMARK 2.1.16. The statement appears without proof as [**Cap74b**, Theorem 1(ii)]. However, Cappell does indicate the essence of a proof in [**Cap76b**, Page 137, Remark 2]. There it is stated in the case of injective amalgams $G = G_- *_H G_+$ of finitely presented groups: $R = R_0[H]$ and $\mathscr{B}_\pm = R_0[G_\pm \setminus H]$, where $\mathbf{Z}[\frac{1}{2}] \subseteq R_0 \subseteq \mathbf{Q}$.

We begin with a numerical fact, which allows us to use the binomial theorem.

LEMMA 2.1.17. *For any $r \in \mathbf{Z}_{\geq 0}$, the rational number $\binom{-1/2}{r}$ lies in the ring $\mathbf{Z}[\frac{1}{2}]$.*

PROOF. Immediate from the observation
$$\binom{-1/2}{r} = \binom{2r}{r}\left(-\frac{1}{4}\right)^r,$$
proven inductively, and the fact that $\binom{2r}{r} \in \mathbf{Z}$. $\square$

Under the assumption that H is square-root closed in G, instead of the hypothesis $2 \in S^\times$ below, Cappell [**Cap76b**, Lem. II.7,8,9] recursively proved the existence of a square-root of his operator $1-\rho$. Our documentation is geared toward a slightly different operator, compatible with [**Far79, CR05**].

LEMMA 2.1.18. *Suppose ρ is a nilpotent element of a ring S that contains 2 as a unit. Then there exists a unit V of S commuting with ρ such that $V^2 = (1+\rho)^{-1}$.*

PROOF. Recall, by Lemma 2.1.13, that $\rho \in S$ nilpotent implies $1+\rho \in S^\times$. By Lemma 2.1.17 and the nilpotence of ρ, we can define an element $V \in S$ which commutes with ρ:
$$V := \sum_{r=0}^{\infty} \binom{-1/2}{r} \rho^r.$$
Then the binomial theorem implies
$$V^2 = (1+\rho)^{-1}.$$
In particular, we obtain $V \in S^\times$. $\square$

The next theorem states a split injection from UNil to a certain L-group. It has been stated by Andrew Ranicki, but a full proof has not yet appeared in print. The case of finitely presented integral group-rings (2.1.16) was originally obtained in [**Cap74b**] using manifold transversality. Algebraic transversality is now available in [**Ran04**] and should be useful in writing a proof in the general case.

THEOREM 2.1.19 ([**FRR95**, Ranicki, Remark 8.7]). *Let $A = A_- *_R A_+$ be a pure pushout of rings with involution. Write $A_\pm = R \oplus \mathscr{B}_\pm$ as (R,R)-bimodules with involution. Then there exists a split monomorphism*
$$\mathrm{UNil}_n^h(R; \mathscr{B}_-, \mathscr{B}_+) \longrightarrow L_n^h(A).$$
$\square$

PROOF OF THEOREM 2.1.15. By Proposition 1.1.9, we may assume that $n = 2k$ is even.

Write $\epsilon := (-1)^k$. Let $[u] \in \mathrm{UNil}_{2k}(R; \mathscr{B}_-, \mathscr{B}_+)$ be a nonsingular ϵ-quadratic unilform $u = (P_-, \theta_-; P_+, \theta_+)$. Recall (cf. [**Far79, CR05**]) that the image of $[u]$ in $L_{2k}(A)$ is $[v] = [(N, \Theta)]$, where v is the nonsingular ϵ-quadratic form over A defined by the f.g. projective A-module N and A-module endomorphism Θ of N:

$$N := A \otimes_R (P_- \oplus P_+) \quad \text{and} \quad \Theta := \begin{pmatrix} \theta_- & \mathbf{1} \\ 0 & \theta_+ \end{pmatrix}.$$

Since 2 is a unit in R hence in A, the ϵ-symmetrization map is an isomorphism:

$$L_n(A) \xrightarrow{1+T_\epsilon} L^n(A).$$

So by Theorem 2.1.19, it is equivalent to show the vanishing of the Witt class

$$(1 + T_\epsilon)[v] = [N, L \circ (1 + \rho)].$$

Here, Cappell [**Cap76b**] denotes L as the standard ϵ-symplectic form on $N = P \oplus P^*$ with $P := A \otimes_R P_-$. Moreover, ρ is necessarily nilpotent:

$$L := \begin{pmatrix} 0 & \mathbf{1} \\ \epsilon\mathbf{1} & 0 \end{pmatrix} : N \to N^* \qquad \text{and} \qquad \rho := \begin{pmatrix} 0 & \epsilon\rho_+ \\ \rho_- & 0 \end{pmatrix} : N \to N.$$

The morphisms $\rho_\pm$ denote the ϵ-symmetrizations of the quadratic forms $\theta_\pm$:

$$\rho_\pm := \theta_\pm + \epsilon\theta_\pm^* : P_\pm \longrightarrow \mathscr{B}_\pm \otimes_R P_\mp.$$

By Lemma 2.1.18, there exists $V \in \mathrm{End}_A(N)^\times$ commuting with ρ such that $V^2 = (1 + \rho)^{-1}$. Observe that $\rho \in \mathrm{End}_A(N)$ is self-adjoint with respect to the nonsingular form (N, L):

$$\rho^* \circ L = \begin{pmatrix} 0 & \epsilon\rho_- \\ \rho_+ & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \mathbf{1} \\ \epsilon\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} \rho_- & 0 \\ 0 & \rho_+ \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \epsilon\mathbf{1} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \epsilon\rho_+ \\ \rho_- & 0 \end{pmatrix} = L \circ \rho.$$

Hence the automorphism V defined in Proof 2.1.18 is also self-adjoint:

$$V^* \circ L = L \circ V.$$

Then note

$$V^* \circ (L \circ (1 + \rho)) \circ V = L \circ V \circ V \circ (1 + \rho) = L.$$

Hence $(N, L\circ(1+\rho))$ is homotopy equivalent to (N, L): its pullback $V^*(N, L\circ(1+\rho))$ along the automorphism V is equal to the symplectic form (N, L). In other words, $V(P)$ is a lagrangian for the ϵ-symmetric form

$$(1 + T_\epsilon)(v) = (N, L \circ (1 + \rho)).$$

Therefore

$$[u] = 0 \in \mathrm{UNil}_{2k}(R; \mathscr{B}_-, \mathscr{B}_+).$$

$\square$

Although the following theorem is a corollary of the previous result (2.1.15) and Theorem 2.1.2, we provide a simple, direct proof. It will be an ingredient in our theorem (3.2.1) on certain finite groups.

THEOREM 2.1.20. *Suppose 2 is a unit in a ring A with involution. Then*

$$N^u_\alpha L^h_n(A) = 0$$

for all $n \in \mathbf{Z}$ and twistings (α, u).

REMARK 2.1.21. A special case in the literature is Karoubi's theorem [**Oja84**, Thm. 7] on the level of symmetric Witt groups: if 2 is a unit in A, then

$$NL_0(A, \epsilon) \xrightarrow{\cong} NL^0(A, \epsilon) = NW(A, \epsilon) = 0 \quad \text{for all } \epsilon = \pm 1.$$

PROOF. By the Ranicki-Shaneson sequence [**Ran73b**] for augmentation kernels:

$$0 \longrightarrow N^u_\alpha L^s_{2k}(A) \longrightarrow N^u_\alpha L^s_{2k}(A[\mathbf{C}_\infty]) \longrightarrow N^u_\alpha L^h_{2k-1}(A) \longrightarrow 0,$$

we may assume that $n = 2k$ is even. Our argument works for both projective and free decorations.

By Higman linearization (2.1.10), let

$$\vartheta = [P[x], f_0 + x f_1] \in N^u_\alpha L_{2k}(A)$$

be a nonsingular ϵ-quadratic form over the twisted polynomial extension $B := A^u_\alpha[x]$ with null-augmentation to A, where $\epsilon := (-1)^k$. Recall, since 2 is a unit in B, that the ϵ-symmetrization map is an isomorphism:

$$L_n(B) \xrightarrow{1+T_\epsilon} L^n(B).$$

In fact, the quadratic refinement is recovered, uniquely up to skew $(-\epsilon)$-even morphisms, as one-half of the symmetric part. So it is equivalent to show the vanishing of the Witt class

$$(1 + T_\epsilon)(\vartheta) = [P[x], \lambda_0 + x\lambda_1].$$

Here, for each $i = 0, 1$, the ϵ-symmetrizations are defined as

$$\lambda_i := f_i + \epsilon f_i^* : P[x] \longrightarrow P[x].$$

There exists a lagrangian P_0 of the ϵ-symmetric form (P, λ_0) over A, since

$$\mathrm{eval}_0(\vartheta) = 0 \in L_{2k}(A).$$

By Lemma 2.1.13, we obtain a nilpotent element of the ring $\mathrm{End}_B(P[x])$:

$$x\nu := \lambda_0^{-1} \circ x\lambda_1.$$

Then by Lemma 2.1.18, there exists

$$V \in \mathrm{End}_B(P[x])^\times$$

commuting with $x\nu$ such that $V^2 = (1 + x\nu)^{-1}$. Observe that $x\nu \in \mathrm{End}_B(P[x])$ is self-adjoint with respect to the nonsingular form $(P[x], \lambda_0)$:

$$(x\nu)^* \circ \lambda_0 = (\lambda_0^{-1} \circ x\lambda_1)^* \circ \lambda_0 = x\lambda_1 \circ \lambda_0^{-1} \circ \lambda_0 = x\lambda_1 = \lambda_0 \circ (x\nu).$$

Hence the automorphism V defined in Proof 2.1.18 is also self-adjoint:

$$V^* \circ \lambda_0 = \lambda_0 \circ V.$$

Then note

$$V^* \circ (\lambda_0 + x\lambda_1) \circ V = V^* \circ \lambda_0 \circ (1 + x\nu) \circ V = \lambda_0 \circ V \circ V \circ (1 + x\nu) = \lambda_0.$$

Hence $(P[x], \lambda_0 + x\lambda_1)$ is homotopy equivalent to $(P[x], \lambda_0)$: its pullback

$$V^*(P[x], \lambda_0 + x\lambda_1)$$

along the automorphism V is equal to the symplectic form $(P[x], \lambda_0)$. In other words, $V(P_0)$ is a lagrangian for the ϵ-symmetric form $(P[x], \lambda_0 + x\lambda_1)$. Therefore

$$\vartheta = [P[x], \lambda_0 + x\lambda_1] = 0 \in N^u_\alpha L_{2k}(A).$$

$\square$

There is an analogous result for Tate $\mathbf{C}_2$-cohomology

$$\widehat{H}^j(A) := \widehat{H}^j(\mathbf{C}_2; A) = \frac{\{\, a \in A \mid a = \epsilon\overline{a} \,\}}{\{\, b + \epsilon\overline{b} \mid b \in A \,\}},$$

where the group $\mathbf{C}_2$ acts via the involution on A and $\epsilon := (-1)^j$.

PROPOSITION 2.1.22. *Suppose 2 is a unit in a ring A with involution. Then $\widehat{H}^j(A) = 0$ for all $j \in \mathbf{Z}$.*

PROOF. Write $\epsilon := (-1)^j$. Suppose $a \in A$ is ϵ-symmetric. Define

$$b := \frac{1}{2}a \in A.$$

Then note $a = \epsilon\overline{a}$ implies

$$a = \frac{1}{2}(a + a) = \frac{1}{2}(a + \epsilon\overline{a}) = b + \epsilon\overline{b}.$$

Therefore every ϵ-symmetric element of A is also ϵ-even. □

The following theorem states that our NL-groups (2.1.1) have exponent four.

THEOREM 2.1.23. *Any ring R with involution satisfies*

$$4 \cdot N^u_\alpha L^h_n(R) = 0$$

for all twistings (α, u) arising from pure pushouts.

REMARK 2.1.24. Originally, Cappell [**Cap74b**] had shown in the group-ring case $G = G_- *_H G_+$ (2.1.16) that UNil is 2-primary:

$$\mathbf{Z}[\tfrac{1}{2}] \otimes_{\mathbf{Z}} \mathrm{UNil} = 0.$$

Then Farrell [**Far79**], cleverly using a factorization of the split monomorphism (2.1.19) through UNil of the amalgam

$$G \times \mathbf{D}_\infty = (G \times \mathbf{C}_2) *_G (G \times \mathbf{C}_2)$$

and techniques from hyperelementary induction [**Dre75**] (see §2.3), sharpened the vanishing result to exponent-four:

$$4 \cdot \mathrm{UNil} = 0.$$

Our result is a direct corollary of his theorem.

PROOF. Consider any pure pushout $A = A_- *_R A_+$ of rings with involution, and write $A_\pm = R \oplus \mathscr{B}_\pm$ as (R, R)-bimodules with involution. Farrell's exponent-four theorem [**Far79**, Thm. 1.3] states that

$$4 \cdot \mathrm{UNil}_n^h(R; \mathscr{B}_-, \mathscr{B}_+) = 0$$

for all $n \in \mathbf{Z}$ and finitely presented group-rings R and $A_\pm$. Its proof remains valid (cf. [**Far79**, Intro.]) for all pure pushouts A of rings with involution, by Ranicki's UNil injectivity (2.1.19).

By hypothesis, there exists a pure pushout A with $\mathscr{B}_\pm = R^u_\alpha$. Therefore we are done by our twisted generalization (2.1.2) of the Connolly-Ranicki isomorphism

$$r : \mathrm{UNil}_n^h(R; R^u_\alpha, R) \longrightarrow N^u_\alpha L^h_n(R).$$

$\square$

2.2. Localization, completion, and excision for NL

The following basic theorem is useful in taking advantage of ring decompositions of group rings.

THEOREM 2.2.1. *Let A be a ring with involution, and let $j, n \in \mathbf{Z}$.*

(1) *For all N odd, the following induced maps are isomorphisms:*

$$\widehat{H}^j(A) \longrightarrow \widehat{H}^j(A[\tfrac{1}{N}]) \quad \text{and} \quad NL_n(A) \longrightarrow NL_n(A[\tfrac{1}{N}]).$$

(2) *The following induced maps are isomorphisms:*

$$\widehat{H}^j(A) \longrightarrow \widehat{H}^j(\widehat{A}_{(2)}) \quad \text{and} \quad NL_n(A) \longrightarrow NL_n(\widehat{A}_{(2)}).$$

PROOF. Let $r > 0$. Consider the localization-completion cartesian square Φ of rings with involution:

$$\Phi := \left(\begin{array}{ccc} A & \longrightarrow & \widehat{A}_{(r)} \\ \downarrow & & \downarrow \\ A[\tfrac{1}{r}] & \longrightarrow & \widehat{A}_{(r)}[\tfrac{1}{r}]. \end{array} \right)$$

Then there exists (see Proof 2.2.3 for details) a Mayer-Vietoris exact sequence

$$\widehat{H}^{j+1}(\widehat{A}_{(r)}[\tfrac{1}{r}]) \xrightarrow{\partial} \widehat{H}^j(A) \longrightarrow \widehat{H}^j(A[\tfrac{1}{r}]) \oplus \widehat{H}^j(\widehat{A}_{(r)}) \longrightarrow \widehat{H}^j(\widehat{A}_{(r)}[\tfrac{1}{r}]).$$

By [**Ran81**, Prop. 6.3.1], and by naturality of the augmentation map aug_0, we obtain $NL_*(\Phi) = 0$. That is, there is a Mayer-Vietoris exact sequence

$$NL_{n+1}(\widehat{A}_{(r)}[\tfrac{1}{r}]) \xrightarrow{\partial} NL_n(A) \longrightarrow NL_n(A[\tfrac{1}{r}]) \oplus NL_n(\widehat{A}_{(r)}) \longrightarrow NL_n(\widehat{A}_{(r)}[\tfrac{1}{r}]).$$

Since N is odd implies that 2 is a unit in $\widehat{A}_{(N)}$ and $\widehat{A}_{(N)}[\frac{1}{N}]$, by the vanishing results (2.1.20) and (2.1.22), we obtain the desired isomorphisms of Part (1). Similarly since 2 is a unit in $A[\frac{1}{2}]$ and $\widehat{A}_{(2)}[\frac{1}{2}]$, by the vanishing results (2.1.20) and (2.1.22), we obtain the desired isomorphisms of Part (2). $\square$

REMARK 2.2.2. It is false that the inclusion $\widehat{A}_{(2)}[x] \hookrightarrow \widehat{A[x]}_{(2)}$ is an isomorphism of rings. The reason is that

$$A[[y]][x] \subsetneq A[x][[y]] :$$

every element of the former has bounded degree in x whereas the latter has elements which do not. Therefore

$$\widehat{A}_{(2)}[x] = A[[y]][x]/(y-2) \subsetneq A[x][[y]]/(y-2) = \widehat{A[x]}_{(2)}.$$

So in general, we should expect that

$$NL_*(A) \cong NL_*(\widehat{A}_{(2)}) \ncong \mathrm{Ker}\left(\mathrm{aug}_0 : L_*(\widehat{A[x]}_{(2)}) \to L^h_*(\widehat{A}_{(2)})\right) \cong NL_*(A/2A),$$

where the latter isomorphism is Wall's reduction [**Wal73**, Thm. 6]. For example, the induced map $NL_n(\mathbf{Z}) \to NL_n(\mathbf{F}_2)$ is not an isomorphism for all $n \equiv 0, 3 \pmod 4$; see [**CR05, CD04, BR06**].

Now we establish a useful Mayer-Vietoris sequence for the excisive functor NL_*.

PROPOSITION 2.2.3. *Let R be a ring with involution of characteristic zero. Let K be a finite normal subgroup of any group G. Its* ***norm*** *is defined as*

$$\Sigma_K := \sum_{g \in K} g \in R[G].$$

(1) *There is a cartesian square of rings with involution:*

$$\begin{tikzcd} R[G] \arrow[r] \arrow[d] & R[G]/\Sigma_K \arrow[d] \\ R[G/K] \arrow[r] & \frac{R}{|K|}[G/K]. \end{tikzcd}$$

(2) *There is a Mayer-Vietoris exact sequence of 2-periodic Tate cohomology groups:*

$$\widehat{H}^j(R[G]) \longrightarrow \widehat{H}^j(R[G/K]) \oplus \widehat{H}^j(R[G]/\Sigma_K) \\ \longrightarrow \widehat{H}^j(\tfrac{R}{|K|}[G/K]) \xrightarrow{\partial} \widehat{H}^{j-1}(R[G]).$$

(3) *There is a Mayer-Vietoris exact sequence of 4-periodic quadratic NL-groups:*

$$NL_n(R[G]) \longrightarrow NL_n(R[G/K]) \oplus NL_n(R[G]/\Sigma_K) \\ \longrightarrow NL_n(\tfrac{R}{|K|}[G/K]) \xrightarrow{\partial} NL_{n-1}(R[G]).$$

Proof. Evidently, there is such a commutative square of quotient maps, which are denoted by [ ]. Let

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\varphi"'] & R[G]/\Sigma_K \arrow[d] \\ R[G/K] \arrow[r] & \frac{R}{|K|}[G/K]. \end{tikzcd}$$

be a commutative square of rings with involution for some S. We indicate the definition of a function

$$\chi : S \longrightarrow R[G].$$

It is routine to check that χ is in fact a well-defined morphism of rings with involution, and that χ is uniquely determined by the commutativity of its composite diagram.

Fix $r \in S$. There exists a natural number n, some K-coset representatives $\{ g_1, \ldots, g_n \}$, and integers a_i, b_i^h for all $1 \leq i \leq n$ and $h \in K$ such that

$$\varphi(r) = \left[\sum_{i=1}^{n} a_i g_i \right] \quad \text{and} \quad \psi(r) = \left[\sum_{i=1}^{n} \sum_{h \in K} b_i^h h g_i \right].$$

Then $[\varphi(r)] = [\psi(r)]$ implies that

$$\left[\sum_i \left(a_i - \sum_h b_i^h\right) g_i\right] = 0.$$

Since the g_i lie in different K-cosets, for each $1 \leq i \leq n$ there must exist integers c_i such that

$$a_i = c_i\,|K| + \sum_h b_i^h.$$

Now define the image element

$$\chi(r) := \sum_{i=1}^n \left(c_i \Sigma_K + \sum_{h \in K} b_i^h h\right) g_i \in R[G].$$

Thus the diagram of Part (1) is a pullback diagram of rings with involution. Since both φ and ψ are surjective, an argument with R-module bases shows that it is also a pushout. Therefore the diagram of Part (1) is cartesian.

In order to prove Part (2), given a finite group Γ, consider a short exact sequence of $\mathbf{Z}[\Gamma]$-modules:

$$0 \longrightarrow M_0 \longrightarrow M_- \oplus M_+ \longrightarrow M \longrightarrow 0.$$

Let C be a contractible complex of f.g. free Γ-modules such that

$$\mathrm{Cok}(\partial : C_1 \to C_0) = \mathbf{Z},$$

which is the trivial $\mathbf{Z}[\Gamma]$-module. Recall, for any coefficient $\mathbf{Z}[\Gamma]$-module N, the definition of Tate cohomology:

$$\widehat{H}^j(\Gamma; N) = H^j(\mathrm{Hom}_{\mathbf{Z}[\Gamma]}(C, N)).$$

Then we obtain the Bockstein sequence:

$$\cdots \xrightarrow{\partial} H^j(\Gamma; M_0) \longrightarrow H^j(\Gamma; M_-) \oplus H^j(\Gamma; M_+) \longrightarrow H^j(\Gamma; M) \xrightarrow{\partial} \cdots .$$

We obtain the desired result from substitution:

$$\Gamma = \mathbf{C}_2 \quad \text{and} \quad M_0 = R[G]$$

$$M_- = R[G/K] \quad \text{and} \quad M_+ = R[G]/\Sigma_K \quad \text{and} \quad M = \tfrac{R}{|K|}[G/K].$$

Either of the quotient maps [ ] is surjective. Therefore, by the Mayer-Vietoris sequence [**Ran81**, Prop. 6.3.1], and by naturality of the augmentation map aug_0, we obtain the exact sequence of Part (3). $\square$

2.3. Hyperelementary induction for NL

Our program for the determination of $L_*(\mathbf{Z}[V])$ for semitrivial type III virtually cyclic groups V is thus reduced by (2.1.2) to the computation of $N^u_\alpha L_*(\mathbf{Z}[F])$ for all finite groups F with certain (α, u). Recall for any prime p that a finite group H is **p-hyperelementary** if it is an extension

$$1 \longrightarrow \mathbf{C}_N \longrightarrow H \longrightarrow P \longrightarrow 1$$

with P a p-group and N coprime to p; the extension is necessarily split.

For the remainder of Part I, we shall restrict ourselves to the untwisted case; that is, we assume (α, u) is trivial.

For every group G and prime p, define a category $\mathcal{H}_p(G)$ as follows. It objects H are all the p-hyperelementary subgroups of G, and its morphisms are defined as conjugate-inclusions:

$$\varphi_{H,g,H'} : H \longrightarrow H'; \qquad x \longmapsto gxg^{-1},$$

for all possible p-hyperelementary subgroups H, H' and elements g of G. It is a subcategory of the category FINITE GROUPS of finite groups and monomorphisms. For any normal subgroup S of G, the category $\mathcal{H}_p(G)$ contains a full subcategory $\mathcal{H}_p(G) \cap S$ whose objects are all the p-hyperelementary subgroups of S. The inclusion map also induces a functor

$$\mathrm{incl}_* : \mathcal{H}_p(S) \longrightarrow \mathcal{H}_p(G) \cap S,$$

which is bijective on object sets and injective on morphism sets.

Definition 2.3.1. Fix $n \in \mathbf{Z}$. Define a pair $\mathscr{N} = (\mathscr{N}^*, \mathscr{N}_*)$ of functors by

$$\begin{aligned} \mathscr{N}^* &: \text{FINITE GROUPS}^{\mathrm{op}} \longrightarrow \text{ABELIAN GROUPS} \\ \mathscr{N}_* &: \text{FINITE GROUPS} \longrightarrow \text{ABELIAN GROUPS} \end{aligned}$$

$$\begin{aligned}
\mathscr{N}^*(F) = \mathscr{N}_*(F) &:= NL_n^{\langle -\infty \rangle}(\mathbf{Z}[F]) \\
\mathscr{N}^*\left(F \xrightarrow{\varphi} G\right) &:= \left(\mathscr{N}(G) \xrightarrow{\mathrm{incl}^*} \mathscr{N}(\varphi F) \xrightarrow{\varphi_*^{-1}} \mathscr{N}(F)\right) \\
\mathscr{N}_*\left(F \xrightarrow{\varphi} G\right) &:= \left(\mathscr{N}(F) \xrightarrow{\varphi_*} \mathscr{N}(\varphi F) \xrightarrow{\mathrm{incl}_*} \mathscr{N}(G)\right).
\end{aligned}$$

The morphism incl^* is the transfer map from G to the finite index subgroup φF. Since the two functors agree on objects, we write $\mathscr{N}(F)$ for the common value of $\mathscr{N}^*(F)$ and $\mathscr{N}_*(F)$.

The following reductions to colimits and coinvariants are consequences of Dress induction [**Dre75**] and Farrell's exponent theorem [**Far79**].

THEOREM 2.3.2. *Let F be a finite group and S a normal subgroup.*

(1) *The following induced map from the direct limit is an isomorphism:*

$$\mathrm{incl}_* : \operatorname*{colim}_{\mathcal{H}_2(F)} \mathscr{N} \longrightarrow \mathscr{N}(F).$$

(2) *The following map, from the group of coinvariants, is an induced isomorphism:*

$$\mathrm{incl}_* : \left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N}\right)_{F/S} \longrightarrow \operatorname*{colim}_{\mathcal{H}_2(F)\cap S} \mathscr{N}.$$

LEMMA 2.3.3. *The above $\mathscr{N}$ is a Mackey functor and moreover a module over the Green ring functor*

$$GW_0(-,\mathbf{Z}) : \text{FINITE GROUPS} \longrightarrow \text{COMMUTATIVE RINGS}.$$

PROOF. Observe that $\mathscr{N}$ transforms inner automorphisms into the identity, by Taylor's Lemma [**Tay73**, Cor. 1.1], since we are assuming that all our groups are equipped with trivial orientation character. Also, for any isomorphism $\varphi : F \to G = \varphi F$, by Definition 2.3.1, we have

$$\mathscr{N}^*(\varphi) = \left(\mathscr{N}(G) \xrightarrow{\varphi_*^{-1}} \mathscr{N}(F)\right) = \left(\mathscr{N}(F) \xrightarrow{\varphi_*} \mathscr{N}(G)\right)^{-1} = \mathscr{N}_*(\varphi)^{-1}.$$

Next we document the Mackey subgroup property (compare [**Bak78**, Thm. 4.1]). However, we shall more generally do so for the analogously defined (2.3.1) quadratic L-theory pair of functors

$$\mathscr{L} = (\mathscr{L}^*, \mathscr{L}_*) = L_n^{\langle -\infty \rangle}(R[-]) : \text{FINITE GROUPS} \longrightarrow \text{ABELIAN GROUPS}$$

for any ring R with involution. Let H, K be subgroups of a finite group F. Then we must show that the "double coset formula" holds, i.e. the following diagram commutes (we suppress labels for the inclusions):

$$\begin{tikzcd}
\mathscr{L}(H) \arrow[r, "\mathscr{L}_*"] \arrow[d, "\mathscr{L}^*"'] & \mathscr{L}(F) \arrow[r, "\mathscr{L}^*"] & \mathscr{L}(K). \\
\bigoplus_{KaH \in K\backslash G/H} \mathscr{L}(a^{-1}Ka \cap H) \arrow[rr, "(\mathrm{conj}^a_*)"] & & \bigoplus_{KaH \in K\backslash F/H} \mathscr{L}(K \cap aHa^{-1}) \arrow[u, "\mathscr{L}_*"']
\end{tikzcd}$$

Let P be an arbitrary left $R[H]$-module. Denote the inclusions $i : H \hookrightarrow F$ and $j : K \hookrightarrow F$. For all double cosets KaH with $a \in F$, denote inclusions

$$i_a : K \cap aHa^{-1} \hookrightarrow aHa^{-1} \quad \text{and} \quad j_a : aHa^{-1} \hookrightarrow K.$$

The Mackey Subgroup Theorem [**CR88**, Thm. 44.2] states, as an internal sum of $R[K]$-modules, that

$$j^* i_*(P) = \bigoplus_{KaH \in K\backslash F/H} j_{a*} i_a^*(a \otimes P).$$

Observe

$$\mathrm{conj}^a_*(P) = a \otimes P \subseteq R[F] \otimes_{R[H]} P,$$

where the $R[aHa^{-1}]$-module structure on the R-submodule $a \otimes P$ is given by

$$(axa^{-1}) \cdot (a \otimes p) = a \otimes xp.$$

Denote an inclusion

$$k_a : a^{-1}Ka \cap H \hookrightarrow H.$$

Since $(\mathrm{conj}^a)^* = (\mathrm{conj}^a_*)^{-1}$ and the following diagram commutes:

$$\begin{tikzcd}
a^{-1}Ka \cap H \arrow[r, "k_a"] \arrow[d, "\mathrm{conj}^a"', "\cong"] & H \arrow[d, "\mathrm{conj}^a"', "\cong"] \\
K \cap aHa^{-1} \arrow[r, "i_a"] & aHa^{-1}
\end{tikzcd}$$

we obtain

$$i_a^*(a \otimes P) = i_a^* \mathrm{conj}^a_*(P) = \mathrm{conj}^a_* k_a^*(P).$$

Hence the Mackey Subgroup Theorem is equivalent to the formula

$$j^*i_*(P) = \bigoplus_{KaH\in K\backslash F/H} j_{a*}\mathrm{conj}^a_* k^*_a(P),$$

and is functorial in left $R[H]$-modules P. Now consider its dual module

$$P^* := \mathrm{Hom}_{R[H]}(P, R[H])^t.$$

There is a functorial $R[K]$-module morphism, which is an isomorphism if P is finitely generated projective:

$$\Phi_a : j^*_a\mathrm{conj}^a_* k^*_a(P^*) \longrightarrow j^*_a\mathrm{conj}^a_* k^*_a(P)^*; \qquad \Phi_a(r\otimes a\otimes f) := (s\otimes a\otimes x \mapsto s\, k^!_a f(x)\,\overline{r}).$$

The trace

$$k^!_a : R[H] \to R[a^{-1}Ka\cap H]$$

is defined as projection onto the trivial coset (see [**HRT87**, 5.15]). Thus for all f.g. projective $R[H]$-modules P, we obtain a functorial isomorphism, which respects the above double coset decomposition:

$$\Phi := \prod_{KaH} \Phi_a : j^*i_*(P^*) \longrightarrow j^*i_*(P)^*.$$

Then there is a commutative diagram of algebraic bordism categories and their functors [**Ran92a**, §3]:

$$\begin{array}{ccccc}
\Lambda(R[H]) & \xrightarrow{i_*} & \Lambda(R[G]) & \xrightarrow{j^*} & \Lambda(R[K]) \\
\Big\downarrow{\scriptstyle \prod k^*_a} & & & & \Big\uparrow{\scriptstyle \prod j_{a*}} \\
\prod_{KaH}\Lambda(R[a^{-1}Ka\cap H]) & & \xrightarrow{\prod \mathrm{conj}^a_*} & & \prod_{KaH}\Lambda(R[K\cap aHa^{-1}])
\end{array}$$

with $\langle -\infty\rangle$ decorations. So the desired commutative diagram is induced [**Ran92a**, Prop. 3.8] on the level of $L^{\langle-\infty\rangle}_*$-groups. Therefore $\mathscr{L}$ (resp. $\mathscr{N}$) is a Mackey functor. The module structure on $\mathscr{L}$ (resp. $\mathscr{N}$) over the Green ring functor

$$GW_0(-, \mathbf{Z})$$

is defined (see [**Bak78**, p. 1452], resp. [**Far79**, p. 306]) using the diagonal F-action:

$$GW_0(F, \mathbf{Z}) \times \mathscr{L}(F) \longrightarrow \mathscr{L}(F); \qquad ([M,\lambda],[C,\psi]) \mapsto [M\otimes_{\mathbf{Z}} C, \mathrm{Ad}(\lambda)\otimes\psi].$$

This verifies all assertions for $\mathscr{N}$. □

Proof of Theorem 2.3.2(1). Since Lemma 2.3.3 shows that Dress Induction [**Dre75**, Thm. 1] is applicable in its covariant form [**Oli88**, Thm. 11.1], the functor $\mathbf{Z}_{(2)} \otimes \mathscr{N}$ is $\mathcal{H}_2$-computable. That is, the following induced map is an isomorphism:

$$\mathrm{incl}_* : \operatorname*{colim}_{\mathcal{H}_2(F)} \mathbf{Z}_{(2)} \otimes \mathscr{N} \longrightarrow \mathbf{Z}_{(2)} \otimes \mathscr{N}(F).$$

But Theorem 2.1.23 states for all groups G that $\mathscr{N}(G)$ has exponent 4. Hence the prime 2 localization map

$$\mathscr{N}(G) \longrightarrow \mathbf{Z}_{(2)} \otimes \mathscr{N}(G)$$

is an isomorphism. The result follows immediately. □

Proof of Theorem 2.3.2(2). For existence and surjectivity of the map, it suffices to show that the following commutative diagram exists:

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N} \arrow[r, two heads, "\mathrm{incl}_*"] \arrow[d, two heads, "\mathrm{quotient}"'] & \operatorname*{colim}_{\mathcal{H}_2(F)\cap S} \mathscr{N}. \\
\left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N}\right)_{F/S} \arrow[ur, dashed, "\mathrm{incl}_*"] &
\end{tikzcd}$$

The group F/S has a covariant action on the category $\mathcal{H}_2(S)$ defined by pushforward along conjugation:

$$\mathrm{conj} : F/S \longrightarrow \mathrm{Aut}(\mathcal{H}_2(S)).$$

Its **group of coinvariants** is defined by

$$\left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N}\right)_{F/S} := \left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N}\right) \Big/ \Big\langle x_H - \mathrm{conj}^g_*(x_H) \;\Big|\; x_H \in \mathscr{N}(H) \text{ and } gS \in F/S \Big\rangle.$$

Recall that the direct limit of a functor is defined in this case by

$$\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N} := \left(\prod_{H \in \mathrm{Ob}\,\mathcal{H}_2(S)} \mathscr{N}(H) \right) \Big/ \Big\langle x_H - \mathscr{N}_*(\varphi)(x_H) \;\Big|\; \varphi \in \mathrm{Mor}\,\mathcal{H}_2(S) \Big\rangle,$$

and similarly over the finite category $\mathcal{H}_2(F) \cap S$. This explains the terms in the above diagram.

In order to show that the induced map exists, let

$$z := x_H - \mathrm{conj}_*^g(x_H) \in \prod \mathscr{N}(H)$$

represent a generator of the kernel of the quotient map, where $x_H \in \mathscr{N}(H)$ and $g \in F$. But note

$$z = x_H - \mathscr{N}_*(\varphi_{H,g,gHg^{-1}})(x_H).$$

Hence it maps to zero in the direct limit over $\mathcal{H}_2(F) \cap S$. Thus the desired map exists and is surjective.

In order to show that the induced map is injective, suppose $[x_H]$ is an equivalence class in the coinvariants which maps to zero in the direct limit over $\mathcal{H}_2(F) \cap S$. Then there exists an expression

$$(x_H) = \sum_{i=1}^{r} (w_i - \mathscr{N}_*(\varphi_i)(w_i)) \in \prod_{H \in \mathrm{Ob}\, \mathcal{H}_2(S)} \mathscr{N}(H)$$

for some

$$H_1, \ldots, H_r \in \mathrm{Ob}\, \mathcal{H}_2(S) \quad \text{and} \quad w_i \in \mathscr{N}(H_i) \quad \text{and} \quad \varphi_i \in \mathrm{Mor}\, \mathcal{H}_2(F) \cap S.$$

But each monomorphism $\varphi_i = \varphi_{H_i, g_i, H_i'}$ admits a factorization

$$\varphi_i = \varphi_i' \circ \mathrm{conj}_*^{g_i}$$

into an isomorphism $\mathrm{conj}_*^{g_i}$ and an inclusion

$$\varphi_i' := \varphi_{g_i H_i g_i^{-1}, 1, H'} \in \mathrm{Mor}\, \mathcal{H}_2(S).$$

Then note

$$[w_i - \mathscr{N}_*(\varphi_i)(w_i)] = [w_i - \mathrm{conj}_*^{g_i}(w_i)] + [v_i - \mathscr{N}_*(\varphi_i')(v_i)] = 0 \in \left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N} \right)_{F/S}$$

where

$$v_i := \mathrm{conj}_*^{g_i}(w_i).$$

Hence $[x_H] = 0$ in the coinvariants. Thus the desired incl_* is injective. $\square$

Therefore we are reduced to the computation of

$$\mathscr{N}(H) = NL_*^{\langle -\infty \rangle}(\mathbf{Z}[H])$$

for all 2-hyperelementary groups.

THEOREM 2.3.4. *Suppose H is a 2-hyperelementary group:*

$$H = \mathbf{C}_N \rtimes_\tau P$$

Consider the ring $R := \mathbf{Z}[\frac{1}{N}]$. Then for all $n \in \mathbf{Z}$, there is an induced isomorphism

$$NL_n(\mathbf{Z}[H]) \longrightarrow NL_n\left((\bigoplus_{d|N} R[\zeta_d]) \circ_{\tau'} P \right).$$

Here the action τ' is induced by τ.

Moreover if τ is trivial, then it can be lifted to an induced isomorphism

$$NL_n(\mathbf{Z}[\mathbf{C}_N \times P]) \longrightarrow \bigoplus_{d|N} NL_n\left(\mathbf{Z}[\zeta_d][P]\right).$$

PROOF. For each divisor d of N, let ρ_d be the cyclotomic $\mathbf{Q}$-representation of $\mathbf{C}_N$ defined by $\rho_d(T) := \zeta_d$. The ρ_d represent all the distinct isomorphism classes of irreducible $\mathbf{Q}$-representations of the group $\mathbf{C}_N$. Recall the polynomial quotients

$$\mathbf{Q}[\mathbf{C}_N] = \mathbf{Q}[y]/(y^N - 1) \quad \text{and} \quad \mathbf{Q}[\zeta_d] = \mathbf{Q}[y]/\Phi_d(y),$$

where Φ_d is the d-th cyclotomic polynomial of degree $\phi(d)$, and ϕ is the Euler phi-function. Since

$$y^N - 1 = \prod_{d|N} \Phi_d(y)$$

is a product of pairwise comaximal elements (distinct and irreducible in the Euclidean domain $\mathbf{Q}[y]$ hence comaximal), by the chinese remainder theorem, we obtain an isomorphism of $\mathbf{Q}$-algebras with involution:

$$\rho := \bigoplus_{d|N} \rho_d : \mathbf{Q}[\mathbf{C}_N] \longrightarrow \bigoplus_{d|N} \mathbf{Q}[\zeta_d].$$

Now we claim that ρ restricts to an isomorphism of R-algebras with involution:

$$\rho' : R[\mathbf{C}_N] \longrightarrow \bigoplus_{d|N} R[\zeta_d].$$

It suffices to show that the R-algebra

$$A_\oplus := \rho^{-1}(\bigoplus_{d|N} R[\zeta_d])$$

is contained in $R[\mathbf{C}_N]$. Let $a \in A_\oplus$; we must prove for all $g \in \mathbf{C}_N$ that $ag^{-1} \in \mathbf{Q}[\mathbf{C}_N]$ has constant coefficient in R. Recall for any f.g. $\mathbf{Q}$-algebra A that the trace function

$\mathrm{Tr} : A \to \mathbf{Q}$ is defined by taking the trace of right multiplication in A considered as a f.g. free $\mathbf{Q}$-module. Then there is a restriction

$$\mathrm{Tr}' : R[\mathbf{C}_N] \to R,$$

since for all $g \in \mathbf{C}_N$ we have $\mathrm{Tr}(g) = 0$ if $g \neq 1$ and $\mathrm{Tr}(1) = N$. Therefore it suffices to show for all $b \in A_\oplus$ that $\mathrm{Tr}(b) \in R$, since $A_\oplus$ is invariant under $\mathbf{C}_N$ and $\frac{1}{N}R \subseteq R$. Observe

$$\mathrm{Tr} = \sum_{d|N} \mathrm{Tr}_d \circ \rho_d,$$

where each Tr_d is the trace function on $\mathbf{Q}[\zeta_d]$. Now, for all $b \in A_\oplus$ and $d|N$, it suffices to show that

$$(\mathrm{Tr}_d \circ \rho_d)(b) \in R.$$

But this is immediate from the definition of $A_\oplus$ and the existence (in number theory) of the restriction

$$\mathrm{Tr}'_d : R[\zeta_d] \to R.$$

Therefore we obtain a composition of isomorphisms (2.2.1):

$$NL_n(\mathbf{Z}[\mathbf{C}_N] \circ_\tau P) \longrightarrow NL_n\left(R[\mathbf{C}_N] \circ_\tau P\right) \xrightarrow{\rho'_*} NL_n\left((\bigoplus_{d|N} R[\zeta_d]) \circ_{\tau'} P \right).$$

The assertion for τ trivial follows from (2.2.1) and the additivity of L_* (hence NL_*) under finite products of rings with involution; compare [**HRT87**, Cor. 5.13]. $\square$

2.4. Generalized Banagl-Connolly-Ranicki theory of NQ

It turns out (see [**CR05**, Prop. 20]), in the case that R is a Dedekind domain or more generally a hereditary noetherian ring, that $NL^h_n(R)$ is naturally isomorphic to a homological gadget $NQ^h_{n+1}(R)$. If the NL-groups of R are obtained from a mixture of NQ-groups of certain component rings, we shall roughly say that R has *NL-groups of homological type*. In general, the NL-groups consist of cobordism classes, which are classically described as Witt type. Roughly speaking, one should think of homological type as describing linear objects, which are effectively computable, and of Witt type as describing quadratic objects. This simplification motivates our incursion into Michael Weiss's realm of "chain bundles."

2.4.1. Main result.

THEOREM 2.4.1. *Let A be a ring with involution such that there exist 1-dimensional finitely generated projective A-module resolutions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1^{(1)} & \xrightarrow{d^{(1)}} & C_0^{(1)} & \longrightarrow & \widehat{H}^1(A) & \longrightarrow & 0 \\ 0 & \longrightarrow & C_1^{(0)} & \xrightarrow{d^{(0)}} & C_0^{(0)} & \longrightarrow & \widehat{H}^0(A) & \longrightarrow & 0, \end{array} \tag{2.4.1.1}$$

e.g. A is (a polynomial extension of) a Dedekind domain. Define a 2-dimensional A-module chain bundle

$$(C,\gamma) := \mathscr{C}\left((d,0) : \left(\begin{array}{c} C_1^{(1)} \\ {\scriptstyle 0}\big\downarrow \\ C_1^{(0)} \end{array}, 0 \right) \to \left(\begin{array}{c} C_0^{(1)} \\ {\scriptstyle 0}\big\downarrow \\ C_0^{(0)} \end{array}, \delta \right) \right)$$

as the cone of the chain bundle map $(d,0)$ of 1-dimensional chain bundles, where $d_j := d^{(j)}$ for $j = 0,1$. Here the chain bundle δ corresponds (2.4.3) to the isomorphisms $\iota_j : H_0(C^{(j)}) \to \widehat{H}^j(A)$ for $j = 0,1$.

(1) *The following infinite-dimensional A-module chain bundle is universal:*
$$(B^A,\beta^A) := \bigoplus_{k\in\mathbf{Z}} \Sigma^{2k}(C,\gamma).$$
(2) *The 4-periodic twisted Q-groups of (B^A,β^A) fit into a commutative diagram (see Figure 2.4.1) with outer and inner dodecagons exact.*

COROLLARY 2.4.2 ([**BR06**, Proposition 56], see [**CR05**, §2.6] for the actual proof). *Suppose the hypotheses and notation of Theorem 2.4.1. Furthermore suppose $\widehat{H}^1(A) = 0$. Then the associated chain bundle is*

$$(C,\gamma) := \mathscr{C}\left((d,0) : \left(C_1^{(0)}, 0\right) \to \left(C_0^{(0)}, \delta\right)\right).$$

(1) *The following infinite-dimensional A-module chain bundle is universal:*
$$(B^A,\beta^A) := \bigoplus_{k\in\mathbf{Z}} \Sigma^{2k}(C,\gamma).$$

$$
\begin{tikzcd}
& Q_{-1}(C,\gamma) \oplus Q_3(C,\gamma) \arrow[r, "q"] \arrow[dr, "N_\gamma"] & Q_3(B^A,\beta^A) \arrow[r, "p"] & H_3(C \otimes_A \Sigma^2 C) \arrow[d, "k_C \circ \mathrm{incl}"'] \arrow[dr, "\partial"] & \\
\widehat{Q}^4(C) \arrow[ur, tail, "\partial"] & \widehat{Q}^0(C) \arrow[u, two heads, "\partial_\gamma"] & Q^3(C) \arrow[r, "J_\gamma"] & \widehat{Q}^3(C) \arrow[r, "\partial_\gamma"] & Q_2(C,\gamma) \arrow[dl, "N_\gamma"'] \arrow[d, "q"] \\
Q_0(B^A,\beta^A) \arrow[u, "0"] & Q^0(C) \arrow[u, "J_\gamma"] & & Q^2(C) \arrow[d, "J_\gamma"'] & Q_2(B^A,\beta^A) \arrow[d, "p"] \\
Q_0(C,\gamma) \arrow[u, "q"] \arrow[ur, "N_\gamma"] & \widehat{Q}^1(C) \arrow[l, "\partial_\gamma"'] & Q^1(C) \arrow[l, "J_\gamma"'] & \widehat{Q}^2(C) \arrow[d, "\partial_\gamma"'] & H_2(C \otimes_A \Sigma^2 C) \arrow[l, "k_C \circ \mathrm{incl}"'] \arrow[dl, "\partial"] \\
& 0 \arrow[u, "k_C \circ \mathrm{incl}"] \arrow[ul, "\partial"] & Q_1(B^A,\beta^A) \arrow[l, "p"'] & Q_1(C,\gamma). \arrow[l, "q"'] \arrow[ul, "N_\gamma"'] &
\end{tikzcd}
$$

FIGURE 2.4.1. The exact dodecagon of twisted Q-groups.

(2) *The 4-periodic twisted Q-groups of (B^A, β^A) are given by*

$$Q_n(B^A, \beta^A) \cong \begin{cases} Q_0(C, \gamma) & \text{if } n \equiv 0 \pmod 4 \\ \operatorname{Ker}(J^1_\gamma : Q^1(C) \to \widehat{Q}^1(C)) & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \equiv 2 \pmod 4 \\ Q_{-1}(C, \gamma) & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

2.4.2. Lemmas. We begin with a simplified version of [**BR06**, Proposition 36].

LEMMA 2.4.3. *Let A be a ring with involution. Suppose C is a f.g. projective A-module chain complex of the form:*

$$0 \longrightarrow C_1 \xrightarrow{\ 0\ } C_0 \longrightarrow 0.$$

Then the following map involving Wu classes is an isomorphism of abelian groups:

$$\left(\begin{smallmatrix} \widehat{v}_1 \\ \widehat{v}_0 \end{smallmatrix}\right) : \widehat{Q}^0(C^{0-*}) \longrightarrow \operatorname{Hom}_A(C_1, \widehat{H}^1(A)) \oplus \operatorname{Hom}_A(C_0, \widehat{H}^0(A)).$$

PROOF. This follows from the decomposition (cf. [**Ran81**, §1.1])

$$\widehat{Q}^0(C^{0-*}) = \widehat{Q}^0(C_1^{0-*}) \oplus \widehat{Q}^0(C_0^{0-*}),$$

where there are no cross terms (e.g. γ_{-1}) by hypothesis, and from [**BR06**, Proposition 35]. The proof of the latter uses the fact that each $\widehat{v}_k(\gamma)$ is an A-module morphism. ☐

REMARK 2.4.4. Recall, for a flat A-module chain complex C, that there is a Künneth spectral sequence [**McC01**]:

$$E^2_{p,q} = \bigoplus_{i+j=q} \operatorname{Tor}^p_A(H_i(C), H_j(C)) \quad \Longrightarrow \quad H_{p+q}(C \otimes_A C)$$

and Leray-Serre spectral spectral sequences (cf. [**CR05**, Proof 2.4(A)]):

$$E^2_{p,q} = H^{-p}(H_q(C \otimes_A C)) \quad \Longrightarrow \quad H_{p+q}(W^{\%}C) = Q^{p+q}(C)$$

$$\widehat{E}^2_{p,q} = \widehat{H}^{-p}(H_q(C \otimes_A C)) \quad \Longrightarrow \quad H_{p+q}(\widehat{W}^{\%}C) = \widehat{Q}^{p+q}(C).$$

Observe, for any chain bundle structure $\gamma \in \widehat{Q}^0(C^{-*})$, that the twisted map

$$J_\gamma : Q^n(C) \longrightarrow \widehat{Q}^n(C); \qquad \phi \longmapsto J(\phi) - (\widehat{\phi}_0)^{\%}(S^n\gamma)$$

is induced by a morphism J_γ of the corresponding spectral sequences.

We shall construct, in Theorem 2.4.1, a 2-dimensional f.g. projective A-module chain complex C such that $H_j(C) = \widehat{H}^j(A)$ for all $j = 0, 1$ and $H_j(C) = 0$ for all $j \neq 0, 1$. Consider the homology of $C \otimes_A C$.

LEMMA 2.4.5. *Let A be a ring with involution. Suppose C is a f.g. projective A-module chain complex such that $H_j(C) = 0$ for all $j \neq 0, 1$.*

(1) *For all $a, b, j \in \mathbf{Z}$, there are desuspension isomorphisms*

$$H_{j+a+b}(\Sigma^a C \otimes_A \Sigma^b C) \longrightarrow H_j(C \otimes_A C)$$

and vanishing homology groups

$$H_j(C \otimes_A C) = 0 \quad \textit{for all except} \quad 0 \leq j < 4.$$

(2) *There are exact sequences:*

$$0 \longrightarrow H_3(C \otimes_A C) \longrightarrow \mathrm{Tor}_A^1(H_1(C), H_1(C)) \longrightarrow 0$$

$$0 \longrightarrow H_1(C) \otimes_A H_1(C) \xrightarrow{\times} H_2(C \otimes_A C) \longrightarrow \mathrm{Tor}_A^1(H_1(C), H_0(C)) \oplus \mathrm{Tor}_A^1(H_0(C), H_1(C)) \longrightarrow 0$$

$$0 \longrightarrow H_1(C) \otimes_A H_0(C) \oplus H_0(C) \otimes_A H_1(C) \xrightarrow{\times} H_1(C \otimes_A C) \longrightarrow \mathrm{Tor}_A^1(H_0(C), H_0(C)) \longrightarrow 0$$

$$0 \longrightarrow H_0(C) \otimes_A H_0(C) \xrightarrow{\times} H_0(C \otimes_A C) \longrightarrow 0.$$

(3) *Suppose there is a bundle structure β^A on the chain complex*

$$B^A := \bigoplus_{k \in \mathbf{Z}} \Sigma^{2k} C$$

such that (B^A, β^A) is a universal bundle over A. Then for all $r \in \mathbf{Z}$, there is an isomorphism, dependent on β^A:

$$k_C : H_r\left(C \otimes \Sigma^{2\left[\frac{r}{2}\right]} C\right) \oplus H_r\left(C \otimes \Sigma^{2\left(\left[\frac{r}{2}\right]-1\right)} C\right) = H_r\left(C \otimes B^A\right) \longrightarrow \widehat{Q}^r(C).$$

PROOF. Immediate from Remark 2.4.4 and from [**CR05**, Statement (17)]. $\square$

The indeterminacy in the above homology groups of $C \otimes_A C$ is resolved by the following lemma.

LEMMA 2.4.6. *Suppose for each* $j = 0, 1$ *that* $C^{(j)}$ *is 1-dimensional f.g. free* A*-module complex whose differential* $d^{(j)}$ *equal to right multiplication by a non-zerodivisor* $\delta \in A$*. Consider the 2-dimensional mapping cone*

$$C := \mathscr{C}(\delta : C_1^{(*)} \to C_0^{(*)}) \quad \textit{where} \quad C_i^{(*)} := (0 : C_i^{(1)} \to C_i^{(0)}).$$

Then the group extensions $H_*(C \otimes_A C)$ *split as products in Lemma 2.4.5(2).*

PROOF. Observe that the 4-dimensional complex $C \otimes_A C$ equals

$$C_2 \otimes C_2 \xrightarrow{d_4^{\otimes}} C_2 \otimes C_1 \oplus C_1 \otimes C_2 \xrightarrow{d_3^{\otimes}} C_2 \otimes C_0 \oplus C_1 \otimes C_1 \oplus C_0 \otimes C_2$$
$$\xrightarrow{d_2^{\otimes}} C_1 \otimes C_0 \oplus C_0 \oplus C_1 \xrightarrow{d_1^{\otimes}} C_0 \otimes C_0.$$

Here the differentials d of C equal

$$d_1 := -\begin{pmatrix} \delta \\ 0 \end{pmatrix} \quad \text{and} \quad d_0 := \begin{pmatrix} 0 & \delta \end{pmatrix}.$$

So the differentials $d^{\otimes}$ of $C \otimes_A C$ equal

$$d_4^{\otimes} = \begin{pmatrix} 1 \otimes d_2 \\ d_2 \otimes 1 \end{pmatrix} \quad \text{and} \quad d_2^{\otimes} = \begin{pmatrix} d_2 \otimes 1 & 1 \otimes d_1 & 0 \\ 0 & -d_1 \otimes 1 & 1 \otimes d_2 \end{pmatrix}$$

$$d_3^{\otimes} = \begin{pmatrix} 1 \otimes d_1 & 0 \\ -d_2 \otimes 1 & 1 \otimes d_2 \\ 0 & d_1 \otimes 1 \end{pmatrix} \quad \text{and} \quad d_1^{\otimes} = \begin{pmatrix} d_1 \otimes 1 & 1 \otimes d_1 \end{pmatrix}.$$

The result now follows from direct computation, where the splitting map for the cross product $\times$ occurs naturally as a sum of restrictions of projections on the chain level. $\square$

2.4.3. Main proofs. Now we generalize [**BR06**, Proposition 56], which is recovered from $C^{(1)} = 0$. It is used in Theorem 3.2.1.

PROOF OF COROLLARY 2.4.2. Observe that [**BR06**, Prop. 56(i)] is exactly Theorem 2.4.1(1). Since $\widehat{H}^1(A) = 0$, we may take $C^{(1)} = 0$. Then, by Lemma

2.4.5(2) and [**CR05**, Rule 2.4(C)], we have

$$H_3(C \otimes_A C) = H_2(C \otimes_A C) = 0 \quad \text{and} \quad Q^4(C) = Q^3(C) = Q^2(C) = 0.$$

So we obtain isomorphisms

$$\partial_\gamma : \widehat{Q}^4(C) \longrightarrow Q_3(C,\gamma) \quad \text{and} \quad \partial_\gamma : H_3(C \otimes_A \Sigma^2 C) = \widehat{Q}^3(C) \longrightarrow Q_2(C,\gamma)$$

and a monomorphism

$$\partial_\gamma : H_2(C \otimes_A \Sigma^2 C) = \widehat{Q}^2(C) \longrightarrow Q_1(C,\gamma).$$

Therefore the outer dodecagon of Theorem 2.4.1(2) degenerates into exact sequences:

$$0 \longrightarrow Q_{-1}(C,\gamma) \longrightarrow Q_3(B^A,\beta^A) \longrightarrow 0$$

$$0 \longrightarrow Q_2(B^A,\beta^A) \longrightarrow 0$$

$$0 \longrightarrow \widehat{Q}^2(C) \xrightarrow{\partial_\gamma} Q_1(C,\gamma) \longrightarrow Q_1(B^A,\beta^A) \longrightarrow 0$$

$$0 \longrightarrow Q_0(C,\gamma) \longrightarrow Q_0(B^A,\beta^A) \longrightarrow 0.$$

Now [**BR06**, Proposition 56(ii)] follows. ☐

Proof of Theorem 2.4.1(1). One checks using [**BR06**, Definition 32] and (2.4.1.1) that $(d,0)$ is indeed a chain bundle map. For all $k \in \mathbf{Z}$ and $j = 0,1$, there is a commutative diagram

$$\begin{tikzcd} H_{2k+j}(B^A) \arrow[r, "\widehat{v}_{2k+j}(\beta^A)"] \arrow[d, "\cong"'] & \widehat{H}^{2k+j}(A) \arrow[d, "\cong"'] \\ H_j(C) \arrow[r, "\widehat{v}_j(\gamma)=\iota_j"] & \widehat{H}^j(A) \end{tikzcd}$$

so that for all $n \in \mathbf{Z}$, the Wu class is an isomorphism:

$$\widehat{v}_n(\beta^A) : H_n(B^A) \longrightarrow \widehat{H}^n(A)$$

is an isomorphism. Therefore the chain bundle (B^A, β^A) is universal, by M. Weiss's characterization [**BR06**, Prop. 55(ii)]. ☐

Proof of Theorem 2.4.1(2). Upon desuspension, there is a long exact sequence [**CR05**, Rule 2.4(B)]:

$$\cdots \xrightarrow{\partial} \bigoplus_{k\in\mathbf{Z}} Q_{n-4k}(C,\gamma) \xrightarrow{q} Q_n(B^A,\beta^A) \xrightarrow{p} \bigoplus_{\substack{k\in\mathbf{Z}\\0<m}} H_{n-4k}(C\otimes_A \Sigma^{2m}C) \xrightarrow{\partial} \cdots.$$

By Lemma 2.4.5(1), we have

$$H_j(C\otimes_A \Sigma^{2m}C) = 0 \quad \text{for all except} \quad 0\le j-2m<4.$$

Then, since (B^A,β^A) is universal by Part (1), we obtain an isomorphism for all $j\in\mathbf{Z}$, by Lemma 2.4.5(3):

$$k_C : \bigoplus_{0\le j-2m<4} H_j(C\otimes_A \Sigma^{2m}C) = H_j(C\otimes_A B^A) \longrightarrow \widehat{Q}^j(C).$$

So, by [**CR05**, Rule 2.4(A)], our sequence becomes:

$$\begin{aligned}\cdots \xrightarrow{\partial} \bigoplus_{-1\le n-4k} Q_{n-4k}(C,\gamma) &\xrightarrow{q} Q_n(B^A,\beta^A)\\ &\xrightarrow{p} \bigoplus_{n-4k=2,3} H_{n-4k}(C\otimes_A\Sigma^2 C) \bigoplus_{4\le n-4k} \widehat{Q}^{n-4k}(C) \xrightarrow{\partial} \cdots.\end{aligned}$$

Then, by [**CR05**, Rule 2.4(C)] with $H_2(C)=0$, we have

$$Q^j(C)=0 \quad \text{for all except} \quad 0\le j<4.$$

Hence we obtain an isomorphism

$$\partial_\gamma : \widehat{Q}^{j+1}(C) \to Q_j(C,\gamma) \quad \text{for all except} \quad -1\le j<4.$$

Therefore our sequence becomes:

$$\begin{aligned}\cdots \xrightarrow{\partial} \bigoplus_{-1\le n-4k<4} Q_{n-4k}(C,\gamma) &\xrightarrow{q} Q_n(B^A,\beta^A)\\ &\xrightarrow{p} \bigoplus_{n-4k=2,3} H_{n-4k}(C\otimes_A\Sigma^2 C) \bigoplus_{n-4k=4} \widehat{Q}^{n-4k}(C) \xrightarrow{\partial} \cdots.\end{aligned}$$

This establishes the outer exact dodecagon, where there is a monomorphism

$$\partial_\gamma^4 : \widehat{Q}^4(C) \longrightarrow Q_3(C,\gamma).$$

The inner exact sequence is [**CR05**, Sequence (18)], where by [**CR05**, Rule 2.4(A)], there is an epimorphism

$$\partial_\gamma^0 : \widehat{Q}^0(C) \longrightarrow Q_{-1}(C,\gamma).$$

$\square$

CHAPTER 3

L-theory of type III virtually cyclic groups: Reductions

Many standard techniques [**Wal76, HM80**] used to compute the quadratic L-theory of finite groups, namely: hyperelementary induction, the Mayer-Vietoris sequence for cartesian squares, nilradical quotients, maximal involuted orders, and Morita equivalence, along with our new technique of homological reduction (§3.1), are employed in combination to determine the quadratic NL-theory of certain finite groups (§3.2), up to extension issues addressed in Chapter 4.

3.1. Basic reductions

3.1.1. Orientable reduction.

PROPOSITION 3.1.1. *Suppose R is a ring with involution and G is a group with trivial orientation character. Then there is a natural decomposition*

$$NL_n(R[G]) = NL_n(R) \oplus \widetilde{NL}_n(R[G]),$$

where the ***reduced*** ***L-group*** *is defined by*

$$\widetilde{NL}_n(R[G]) := \operatorname{Ker}\left(\operatorname{aug}_1 : NL_n(R[G]) \to NL_n(R[1])\right).$$

PROOF. The covariant morphism on $NL_*(R[-])$-groups induced by the map $\mathrm{incl} : 1 \to G$ is a monomorphism split by the morphism of $NL_*(R[-])$-groups induced by the augmentation $\operatorname{aug}_1 : G \to 1$ of groups with orientation character. $\square$

3.1.2. Hyperelementary reduction. For simplicity, we shall re-use (see §2.3) the abbreviation $\mathscr{N}(G) := NL_n^{\langle -\infty \rangle}(\mathbf{Z}[G])$ for fixed $n \in \mathbf{Z}$.

THEOREM 3.1.2. *Let F be a finite group and S a normal subgroup. Suppose for all 2-hyperelementary subgroups H of F that the following induced map is an* *isomorphism:*

$$\mathrm{incl}_* : \mathscr{N}(H \cap S) \longrightarrow \mathscr{N}(H).$$

Then the following induced map, from the ***group of coinvariants****, is an isomorphism:*

$$\mathrm{incl}_* : \mathscr{N}(S)_{F/S} \longrightarrow \mathscr{N}(F).$$

PROOF. Observe that the following diagram commutes:

$$\begin{tikzcd}
\operatorname*{colim}_{\mathcal{H}_2(F)\cap S} \mathscr{N} \arrow[dr, "\mathrm{incl}_*"] & \\
\left(\operatorname*{colim}_{\mathcal{H}_2(S)} \mathscr{N}\right)_{F/S} \arrow[u, "\mathrm{incl}_*"] \arrow[r, "\mathrm{incl}_*"] \arrow[d, "\mathrm{incl}_*"'] & \operatorname*{colim}_{\mathcal{H}_2(F)} \mathscr{N} \arrow[d, "\mathrm{incl}_*"'] \\
\mathscr{N}(S)_{F/S} \arrow[r, "\mathrm{incl}_*"] & \mathscr{N}(F).
\end{tikzcd}$$

The vertical maps are isomorphisms by Hyperelementary Induction (2.3.2). It follows from the hypothesis that the diagonal map is an isomorphism. Therefore the bottom map is an isomorphism. □

3.1.3. Homological reduction.

THEOREM 3.1.3. *Let $f : R \to R'$ be a morphism of rings with involution such that the induced map $f_* : \widehat{H}^j(R) \to \widehat{H}^j(R')$ is an isomorphism for all $j = 0, 1$. Suppose R satisfies (2.4.1.1) and that R' is a flat R-module. Then for all $n \in \mathbf{Z}$, the induced map $f_* : NQ_n(R) \to NQ_n(R')$ is an isomorphism.*

We employ general principles and Theorem 2.4.1(1) alone.

PROOF. By Theorem 2.4.1(1) and naturality of null-augmentation, it is equivalent to show for all $n \in \mathbf{Z}$ that the following induced map is an isomorphism:

$$f_* : Q_n(B^{R[x]}, \beta^{R[x]}) \longrightarrow Q_n(B^{R'[x]}, \beta^{R'[x]}).$$

For all $j = 0, 1$, there exists a 1-dimensional f.g. projective $R[x]$-module resolution $C^{(j)}$ of

$$\widehat{H}^j(R[x]) = R[x]^2 \otimes_R \widehat{H}^j(R).$$

It is obtained via tensor product, since R satisfies (2.4.1.1), and since $R[x]$ is a flat R-module. For all $j = 0, 1$, define an $R'[x]$-module chain complex

$$C'^{(j)} := R'[x] \otimes_{R[x]} C^{(j)}.$$

It is a 1-dimensional f.g. projective $R'[x]$-module resolution of

$$\widehat{H}^j(R'[x]) = R'[x]^2 \otimes_{R'} \widehat{H}^j(R'),$$

since $f_* : \widehat{H}^j(R) \to \widehat{H}^j(R')$ is an isomorphism of R-modules, and since $R'[x]$ is a flat $R[x]$-module. Using the resolutions $C^{(j)}$, define the 2-dimensional $R[x]$-module chain bundle (C, γ) of Theorem 2.4.1. Similarly construct the chain bundle (C', γ') over $R'[x]$.

Consider the natural $R[x]$-module chain bundle map $g : (C, \gamma) \to (C', \gamma')$. It induces isomorphisms

$$g_* = \iota'_* \circ f_* \circ \iota_*^{-1} : H_*(C) \longrightarrow H_*(C').$$

There is the commutative ladder of Figure 3.1.1 with long exact rows [**BR06**, Prop. 38(ii)]. Therefore, by the five-lemma, it suffices to show that the induced component maps are isomorphisms:

(3.1.3.1) $$g_* : Q_n(\Sigma^{2k}(C, \gamma)) \longrightarrow Q_n(\Sigma^{2k}(C', \gamma'))$$

(3.1.3.2) $$g_* : H_n(\Sigma^{2k}(C) \otimes_{R[x]} \Sigma^{2\ell}(C)) \longrightarrow H_n(\Sigma^{2k}(C') \otimes_{R'[x]} \Sigma^{2\ell}(C')).$$

The latter (3.1.3.2) follows from the Künneth spectral sequence (see Remark 2.4.4) for C and C' over $R[x]$ and $R'[x]$. This is since $g_* : C \to C'$ induces an isomorphism in homology groups H_*. Hence it induces an isomorphism in Tor groups, by the flatness hypothesis. Moreover, the same argument shows that the latter map (3.1.3.2) is $\mathbf{C}_2$-equivariant with respect to switching the tensor factors if $k = \ell$.

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{k\in\mathbf{Z}} Q_n(\Sigma^{2k}(C,\gamma)) \arrow[r] \arrow[d, "g_*"] & Q_n(B^{R[x]}, \beta^{R[x]}) \arrow[r] \arrow[d, "g_*"] & \bigoplus_{k<\ell} H_n(\Sigma^{2k}(C) \otimes_{R[x]} \Sigma^{2\ell}(C)) \arrow[r] \arrow[d, "g_*"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{k\in\mathbf{Z}} Q_n(\Sigma^{2k}(C',\gamma')) \arrow[r] & Q_n(B^{R'[x]}, \beta^{R'[x]}) \arrow[r] & \bigoplus_{k<\ell} H_n(\Sigma^{2k}(C') \otimes_{R'[x]} \Sigma^{2\ell}(C')) \arrow[r] & \cdots .
\end{tikzcd}$$

FIGURE 3.1.1. Five-lemma argument for the universal twisted Q-groups.

Now the former (3.1.3.1) follows from the commutative ladder with exact rows [**BR06**, Prop. 38(i)]:

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{Q}^{n+1}(C) \arrow[r] \arrow[d, "g_*"] & Q_n(C,\gamma) \arrow[r] \arrow[d, "g_*"] & Q^n(C) \arrow[r] \arrow[d, "g_*"] & \widehat{Q}^n(C) \arrow[r] \arrow[d, "g_*"] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(C') \arrow[r] & Q_n(C',\gamma') \arrow[r] & Q^n(C') \arrow[r] & \widehat{Q}^n(C') \arrow[r] & \cdots ,
\end{tikzcd}$$

and the five-lemma (we desuspend to $k = 0$ for simplicity). This is since the $\mathbf{C}_2$-equivariant map

$$g_* : H_*(C \otimes_{R[x]} C) \longrightarrow H_*(C' \otimes_{R'[x]} C')$$

is an isomorphism, by the above argument for (3.1.3.2) with $k = \ell = 0$ substituted, and by the Leray-Serre spectral sequences (2.4.4) for C and C' over $R[x]$ and $R'[x]$. The desired result now follows. $\square$

3.1.4. Nilpotent reduction. The following is a special case of Wall's reduction [**Wal73**, Thm. 6], which was applied extensively in [**Wal76**]. In the classical L-theory of finite groups, Wall's reduction theorem is applied to the Jacobson radical [**Wal73**, §3] of the 2-adic integral group ring of a finite 2-group [**Wal76**, §5.2].

In our case, a theorem of Amitsur [**Ami56**, Thm. 1] states that the Jacobson radical of $R[x]$ for any ring R is a two-sided ideal $N[x]$, where N is a nil ideal of R containing the locally nilpotent radical. Recall for left artinian rings R that its locally nilpotent radical, nilradical, and Jacobson radical all coincide. Below we limit ourselves to rings of nonzero characteristic (cf. Remark 2.2.2).

PROPOSITION 3.1.4. *Let R be a ring with involution.*

(1) *Suppose that I is a nilpotent, involution-invariant, two-sided ideal of R. Then for all $n \in \mathbf{Z}$, the map induced by the quotient map $\pi : R \to R/I$ is an isomorphism:*

$$\pi_* : NL^h_n(R) \longrightarrow NL^h_n(R/I).$$

(2) *Suppose for some prime p that $\mathbf{F}$ is a finite field of characteristic p and P is a finite p-group. Then for all $n \in \mathbf{Z}$, the following induced map is an isomorphism:*

$$\mathrm{aug}_1 : NL^h_n(\mathbf{F}[P] \otimes_{\mathbf{Z}} R) \longrightarrow NL^h_n(\mathbf{F} \otimes_{\mathbf{Z}} R).$$

PROOF. For Part (1), observe that $I[x]$ is nilpotent implies that the map

$$R[x] \to \widehat{R[x]}_{I[x]},$$

to the $I[x]$-adic completion, is an isomorphism of rings with involution. So we are done by [**Wal73**, Theorem 6].

For Part (2), observe that the involution-invariant, two-sided ideal

$$J := (\{ g - 1 \mid g \in P \})$$

of $\mathbf{F}[P]$ is its Jacobson radical. Since $\mathbf{F}[P]$ is finite hence left artinian, we must have that J is nilpotent. So we are done by Part (1) using $I = J \otimes 1_R$. $\square$

3.2. Finite groups with Sylow 2-subgroup normal abelian

3.2.1. Statement of results. Our main theorem reduces the computation of UNil for certain finite groups F to those of finite 2-groups. Necessarily [**Bro94**, Cor. 3.13], these certain groups are of the form of semidirect products

$$F = S \rtimes E,$$

for some finite abelian 2-group S and odd order group E. A consequence is that our results combine with those of A. Bak in classical L-theory [**Bak76**] to yield information about the polynomial L-groups $L_*(\mathbf{Z}[F][x])$, for the following groups F.

THEOREM 3.2.1. *Suppose F is a finite group that contains a normal abelian Sylow 2-subgroup S. Then for all $n \in \mathbf{Z}$, the following induced map, from the group of coinvariants, is an isomorphism:*

$$\mathrm{incl}_* : NL_n^{\langle -\infty \rangle}(\mathbf{Z}[S])_{F/S} \longrightarrow NL_n^{\langle -\infty \rangle}(\mathbf{Z}[F]).$$

The Connolly-Ranicki isomorphism (2.1.2) yields a complete computation in the following case.

COROLLARY 3.2.2. *Suppose F is a finite group of odd order. Then the induced map is an isomorphism:*

$$\mathrm{incl}_* : \mathrm{UNil}_*(\mathbf{Z}; \mathbf{Z}, \mathbf{Z}) \longrightarrow \mathrm{UNil}_*(\mathbf{Z}[F]; \mathbf{Z}[F], \mathbf{Z}[F]).$$

Moreover, a complete set of invariants for the latter, which is a summand of the surgery group $L_*(\mathbf{Z}[F \times \mathbf{D}_\infty])$, are obtained first by transfer to the trivial subgroup *and then from the obstruction theory of Connolly-Davis* [**CD04**] *(compare* [**CR05, BR06**]*).* □

REMARK 3.2.3. The classical L-groups $L^h_*(\mathbf{Z}[F])$ are computed by Bak and Wall in [**Wal76**, Corollary 2.4.3] and extended to the colimit $L^{\langle -\infty\rangle}_*(\mathbf{Z}[F])$ by Madsen and Rothenberg [**MR88**]. Therefore, the following L-groups are calculable:

$$L^{\langle -\infty\rangle}_*(\mathbf{Z}[F][x]) \quad \text{and} \quad L^{\langle -\infty\rangle}_*(\mathbf{Z}[F \times \mathbf{D}_\infty]).$$

The first lemma is both a vanishing result and a reduction to the Gaussian integers.

LEMMA 3.2.4. *Let $\zeta_r := e^{2\pi\sqrt{-1}/r}$ be a primitive r-th root of unity for some $r > 0$. Write $r = d\,2^e$ for some $d > 0$ odd and $e \geq 0$. Note that*

$$\mathbf{Z}[\zeta_r] = \mathbf{Z}[\zeta_d, \zeta_{2^e}]$$

as rings whose involution is complex conjugation. Then for all $n \in \mathbf{Z}$, either:

(1) *$NL_n(\mathbf{Z}[\zeta_r]) = NL_n(\mathbf{Z})$ if $d = 1$ and $e = 0, 1$, or*
(2) *$NL_n(\mathbf{Z}[\zeta_r]) = 0$ if $d > 1$, or*
(3) *$\mathrm{incl}_* : NL_n(\mathbf{Z}[i]) \to NL_n(\mathbf{Z}[\zeta_r])$ is an isomorphism if $d = 1$ and $e > 1$.*

Its analogue in characteristic two is the following lemma.

LEMMA 3.2.5. *Let P be a finite 2-group, and let $d > 0$ be odd. Consider the ring $R = \mathbf{F}_2[P] \otimes \mathbf{Z}[\zeta_d]$ with involution. If $d = 1$, then for all $n \in \mathbf{Z}$, the induced map $NL_n(R) \to NL_n(\mathbf{F}_2)$ is an isomorphism. Otherwise if $d > 1$, then the groups $NL_*(R)$ vanish.*

REMARK 3.2.6. It seems appropriate to mention here that the techniques of Connolly-Ranicki [**CR05**, Lem. 21, Eqn. (27)] and of Connolly-Davis [**CD04**, Lems. 4.6(2), 4.3] can be used to generalize their computations of $\mathrm{UNil}_*(\mathbf{F}_2)$. Namely, let $\mathbf{F}$ be a perfect field of characteristic two with identity involution. Here perfect means that the squaring endomorphism is surjective. For example, any finite field $\mathbf{F}_{2^e}$ of characteristic two is perfect. If n is odd then $NL_n(\mathbf{F}, \mathrm{id})$ vanishes. Otherwise if n is even, then the Arf invariant of symplectic forms over the function field $\mathbf{F}(x)$ defines an isomorphism

$$\mathrm{Arf} : NL_n(\mathbf{F}, \mathrm{id}) \longrightarrow \mathrm{Cok}(\psi - \mathbf{1}) = \bigoplus_{d \text{ odd}} x^d\, \mathbf{F},$$

where the Frobenius automorphism ψ is defined as

$$\psi : \mathbf{F}[x]/\mathbf{F} \longrightarrow \mathbf{F}[x]/\mathbf{F}; \qquad f \longmapsto f^2.$$

The next lemma is a vanishing result for cyclic 2-groups C.

LEMMA 3.2.7. *Let C be a cyclic 2-group, and let $d > 1$ be odd. Consider the ring $R = \mathbf{Z}[\zeta_d]$ whose involution is complex conjugation. Then the groups $NL_*(R[C])$ vanish.*

Now we use induction to generalize this vanishing result from cyclic 2-groups C to finite abelian 2-groups P.

LEMMA 3.2.8. *Let R be a ring with involution, and let P be a finite abelian 2-group. If the groups $NL_*(R[C])$ vanish for all cyclic 2-groups C of exponent $e(C) \leq e(P)$, then the groups $NL_*(R[P])$ vanish.*

The last lemma reduces our computation from abelian 2-hyperelementary groups H to abelian 2-groups P.

LEMMA 3.2.9. *Consider any abelian 2-hyperelementary group $H = \mathbf{C}_N \times P$. Then for all $n \in \mathbf{Z}$, the following induced map is an isomorphism:*

$$\mathrm{incl}_* : NL_n(\mathbf{Z}[P]) \longrightarrow NL_n(\mathbf{Z}[H]).$$

3.2.2. Proofs.

PROOF OF LEMMA 3.2.4. Part (1) is immediate, since in this case $\mathbf{Z}[\zeta_r] = \mathbf{Z}$. So Parts (2) and (3) remain.

Write $R := \mathbf{Z}[\zeta_{2^e}]$, and consider d as a divisor of some odd $N > 0$. By the ring decomposition ρ' of Proof 2.3.4 and the isomorphisms of Theorem 2.2.1, the following induced upper map is an isomorphism:

$$\begin{tikzcd}
\widehat{H}^j(R[\mathbf{C}_N]) \arrow[r] \arrow[d] & \bigoplus_{d|N} \widehat{H}^j(R[\zeta_d]) \arrow[d] \\
\widehat{H}^j(R[\tfrac{1}{N}][\mathbf{C}_N]) \arrow[r, "\rho'_*"] & \bigoplus_{d|N} \widehat{H}^j(R[\tfrac{1}{N}][\zeta_d]).
\end{tikzcd}$$

But a direct computation shows that the upper map has image in the $d = 1$ factor. That is, the following induced map is an isomorphism:

$$\mathrm{incl}_* : \widehat{H}^j(R[1]) \longrightarrow \widehat{H}^j(R[\mathbf{C}_N]).$$

Moreover, for all $d > 1$, the corresponding Tate cohomology groups vanish:

$$\widehat{H}^j(R[\zeta_d]) = 0.$$

If $e > 1$, then another direct computation using the $\mathbf{Z}$-basis

$$\left\{ (\zeta_{2^e})^k \mid -2^{e-2} < k \leq 2^{e-2} \right\}$$

for R shows that the following induced map is an isomorphism:

$$\mathrm{incl}_* : \widehat{H}^j(\mathbf{Z}[i]) \longrightarrow \widehat{H}^j(R).$$

So by Homological Reduction (3.1.3), we obtain Parts (2) and (3) on the level of NQ-groups.

Recall for any Dedekind domain A with involution that its symmetric NL-groups vanish [**CR05**, Props. 9(iii), 11]. Hence the Weiss boundary map ∂ of the symmetric-quadratic-hyperquadratic sequence [**CR05**, Intro.] is an isomorphism:

$$\partial : NQ_{n+1}(A) \longrightarrow NL_n(A).$$

Therefore we obtain Parts (2) and (3) on the level of quadratic NL-groups, since $\mathbf{Z}[i]$ and R and $R[\zeta_d]$ are in fact Dedekind domains, whose involutions are given by complex conjugation. $\square$

Proof of Lemma 3.2.5. By Nilpotent Reduction (3.1.4), the following induced map is an isomorphism:

$$NL_n(R) \longrightarrow NL_n(\mathbf{F}_2 \otimes \mathbf{Z}[\zeta_d]).$$

Recall, in terms of the d-th cyclotomic polynomial $\Phi_d(x) \in \mathbf{Z}[x]$, that

$$\mathbf{F}_2 \otimes \mathbf{Z}[\zeta_d] = \mathbf{F}_2[x]/(\Phi_d(x)).$$

Note, by taking formal derivative of $x^d - 1$ with d odd, that $\Phi_d(x)$ is separable over $\mathbf{F}_2$. Then, by the chinese remainder theorem, the ring $\mathbf{F}_2 \otimes \mathbf{Z}[\zeta_d]$ is a finite product of fields[1] hence is 0-dimensional.

Therefore, by the argument of Proof 3.2.4, it suffices to show that its Tate cohomology vanishes. But, as in the previous proof, this follows from the direct computation that

$$\mathrm{incl}_* : \widehat{H}^*(\mathbf{F}_2[1]) \longrightarrow \widehat{H}^*(\mathbf{F}_2[\mathbf{C}_N])$$

is an isomorphism for all odd N. ☐

Proof of Lemma 3.2.7. We induct on the exponent of C. If $e(C) = 1$ then

$$NL_*(R[C]) = NL_*(R) = 0,$$

by Lemma 3.2.4(2). Otherwise suppose the lemma is true for all cyclic 2-groups C' with $e(C') < e(C)$. Then we may define a ring extension R' of R and a group quotient C' of C by

$$R' := R[\zeta_{e(C)}] \quad \text{and} \quad C' := \mathbf{C}_{e(C)/2},$$

and there is a cartesian square (2.2.3) of rings with involution:

$$\begin{tikzcd} R[C] \arrow[r] \arrow[d] & R' \arrow[d] \\ R[C'] \arrow[r] & \mathbf{F}_2[C'] \otimes R. \end{tikzcd}$$

Note, by Lemma 3.2.4(2) and Lemma 3.2.5, that

$$NL_*(R') = 0 \quad \text{and} \quad NL_*(\mathbf{F}_2[C'] \otimes R) = 0.$$

[1]It is in fact a product of $\phi(d)/n(d)$ copies of the finite field $\mathbf{F}_{2^{n(d)}}$, where $\phi(d)$ is the Euler ϕ-function and $n(d) > 0$ is minimal with respect to the congruence $2^{n(d)} \equiv 1 \pmod{d}$.

Therefore, by the Mayer-Vietoris sequence (2.2.3), the map induced by the left column is an isomorphism:

$$NL_*(R[C]) \xrightarrow{\cong} NL_*(R[C']).$$

But $NL_*(R[C']) = 0$ by inductive hypothesis. This concludes the argument. $\square$

Proof of Lemma 3.2.8. We induct on the order of P. If $|P| = 1$ then

$$NL_*(R[P]) = NL_*(R[1]) = 0,$$

by hypothesis. Otherwise suppose the lemma is true for all R and P'' with $|P''| < |P|$. Since P is a nontrivial abelian 2-group, we can write an internal direct product

$$P = P' \times \mathbf{C}_{e(P)}.$$

Then we can define a ring extension R' of R and a group quotient P_0 of P by

$$R' := R[\zeta_{e(P)}] = R[x]/\left(x^{e(P)/2} + 1\right) \quad \text{and} \quad P_0 := P' \times \mathbf{C}_{e(P)/2}.$$

Consider the cartesian square (2.2.3) of rings with involution:

$$\begin{tikzcd} R[P] \arrow[r] \arrow[d] & R'[P'] \arrow[d] \\ R[P_0] \arrow[r] & \mathbf{F}_2[P_0] \otimes R. \end{tikzcd}$$

Note, by Nilpotent Reduction (3.1.4) and by hypothesis using both 2-groups C with $e(C) \leq 2$, that

$$NL_*(\mathbf{F}_2[P_0] \otimes R) \xrightarrow{\cong} NL_*(\mathbf{F}_2 \otimes R) = 0.$$

Also $NL_*(R[P_0]) = 0$, by inductive hypothesis. Then, by the Mayer-Vietoris sequence (2.2.3), the map induced by the top row is an isomorphism:

$$NL_*(R[P]) \xrightarrow{\cong} NL_*(R'[P']).$$

We are done by induction if we can show that R' and P' also satisfy the hypothesis of the lemma.

Let C be any cyclic 2-group satisfying

$$1 \leq e(C) \leq e(P') \leq e(P).$$

We now induct on $e(C)$. If $e(C) = 1$ then

$$NL_*(R'[C]) = NL_*(R') = 0.$$

The latter follows from the Mayer-Vietoris sequence of the cartesian square of ring with involution:

$$\begin{array}{ccc} R[\mathbf{C}_{e(P)}] & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R[\mathbf{C}_{e(P)/2}] & \longrightarrow & \mathbf{F}_2[\mathbf{C}_{e(P)/2}] \otimes R \end{array}$$

and from Nilpotent Reduction, as in the above argument, using the hypothesis of the lemma.

Otherwise suppose $e(C) > 1$. Then we may define a quotient group

$$C' := \mathbf{C}_{e(C)/2}$$

of C, and there is a cartesian square of rings with involution:

$$\begin{array}{ccc} R'[C] & \longrightarrow & R'[\zeta_{e(C)}] \\ \downarrow & & \downarrow \\ R'[C'] & \longrightarrow & \mathbf{F}_2[C'] \otimes R'. \end{array}$$

We are again done by Nilpotent Reduction and induction on $e(C)$ if we show that

$$NL_*(R'[\zeta_{e(C)}]) = 0.$$

Consider the primitive root of unity:

$$\omega := (\zeta_{e(P)})^{e(P)/e(C)} \in R'.$$

Observe the quotient and factorization

$$R'[\zeta_{e(C)}] = R[\zeta_{e(P)}][x]/\left(x^{e(C)/2} + 1\right) \quad \text{and} \quad x^{e(C)/2} + 1 = \prod_{\text{odd } d=1}^{e(C)-1} (x - \omega^d).$$

Then, by the chinese remainder theorem, we obtain an isomorphism of rings with involution:

$$R'[\zeta_{e(C)}] \xrightarrow{\cong} \prod_{\text{odd } d=1}^{e(C)-1} R'.$$

Hence it induces an isomorphism

$$NL_*(R'[\zeta_{e(C)}]) \xrightarrow{\cong} \bigoplus_{\text{odd } d=1}^{e(C)-1} NL_*(R').$$

But we have already shown that $NL_*(R') = 0$. This concludes the induction on both $e(C)$ and $|P|$. $\square$

Proof of Lemma 3.2.9. Recall, by Theorem 2.3.4 and additivity of L-groups, that the following induced map is an isomorphism:

$$NL_n(\mathbf{Z}[H]) \longrightarrow \bigoplus_{d|N} NL_n(\mathbf{Z}[\zeta_d][P]).$$

But all the $d \neq 1$ factors vanish by Lemmas 3.2.7 and 3.2.8. The result now follows.

$\square$

Proof of Theorem 3.2.1. Let H be a 2-hyperelementary subgroup of F. Since S is normal abelian, the group H is abelian. Then we can write

$$H = \mathbf{C}_N \times P$$

for some odd N, and $P = H \cap S$ a finite abelian 2-group. So, by Lemma 3.2.9, the following map is an isomorphism:

$$\mathrm{incl}_* : \mathscr{N}(H \cap S) \longrightarrow \mathscr{N}(H).$$

Therefore, by Hyperelementary Reduction (3.1.2), the following induced map is an isomorphism:

$$\mathrm{incl}_* : \mathscr{N}(S)_{F/S} \longrightarrow \mathscr{N}(F).$$

$\square$

3.3. Abelian 2-groups

The following result shows that the NL-theory of abelian 2-groups is determined up to iterated extensions from the Dedekind domains:

$$\mathbf{Z}[i] \quad \text{and} \quad \mathbf{Z} \quad \text{and} \quad \mathbf{F}_2.$$

The latter two have identity involution and have been calculated [**CK95, CR05, CD04, BR06**], whereas the involution of the former is complex conjugation, and whose UNil-groups shall be calculated in another paper.

PROPOSITION 3.3.1. *Let P be a nontrivial abelian 2-group, and let $n \in \mathbf{Z}$. Write*

$$P = P' \times \mathbf{C}_{e(P)} \quad \text{and} \quad P_0 := P' \times \mathbf{C}_{e(P)/2}.$$

(1) *The Weiss boundary map is an isomorphism:*

$$NQ_{n+1}(\mathbf{Z}[P]) \xrightarrow{\partial} NL_n(\mathbf{Z}[P]).$$

(2) *There is an exact sequence*

$$\cdots \longrightarrow NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[P]) \longrightarrow NL_n(\mathbf{Z}[P_0]) \oplus A \longrightarrow NL_n(\mathbf{F}_2) \xrightarrow{\partial} \cdots$$

where

$$A := \begin{cases} NL_n(R_0[P']) & \text{if } P' \neq 1 \\ NL_n(\mathbf{Z}[i]) & \text{if } P' = 1, e(P) > 2 \\ NL_n(\mathbf{Z}) & \text{if } P' = 1, e(P) = 2 \end{cases} \quad \text{and} \quad R_0 := \mathbf{Z}[\zeta_{e(P)}].$$

(3) *Suppose R is of the form*

$$R = \mathbf{Z}[\zeta_e]$$

for some $e \geq e(P)$ a power of 2. There is an exact sequence

$$\cdots \longrightarrow NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(R[P]) \longrightarrow \bigoplus_2 NL_n(R[P_0]) \longrightarrow NL_n(\mathbf{F}_2) \xrightarrow{\partial} \cdots .$$

REMARK 3.3.2. One should beware that Part (1) merely states that all finite abelian 2-groups have NL-groups of homological type, which means they are classified theoretically by the generalized Arf invariant [**BR06**]. The statement itself is not useful for their computation. For example, the Tate cohomology of the group ring $\mathbf{Z}[\mathbf{C}_2]$ with involution vanishes in odd dimensions and has a minimal f.g. projective resolution of length two in even dimensions. Hence our generalization of Banagl-Connolly-Ranicki theory (§2.4) is not applicable to compute its NQ-groups.

Proof. The above sequences are derived from the Mayer-Vietoris exact sequences (2.2.3) of the cartesian squares

$$\begin{tikzcd} \mathbf{Z}[P] \arrow[r] \arrow[d] & R_0[P'] \arrow[d] \\ \mathbf{Z}[P_0] \arrow[r] & \mathbf{F}_2[P_0] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R[P] \arrow[r] \arrow[d] & R[P'][\zeta_{e(P)}] \arrow[d] \\ R[P_0] \arrow[r] & \mathbf{F}_2[P_0]. \end{tikzcd}$$

Part (1) follows by induction on $|P|$ and the five-lemma using these exact sequences, along with a similar exact sequence for $\mathbf{F}_2[P]$. Since $\mathbf{Z}[\zeta]$ and $\mathbf{F}_2$ are Dedekind domains with involution, the basic cases for the induction are isomorphisms

$$NQ_{n+1}(\mathbf{Z}[\zeta]) \xrightarrow[\cong]{\partial} NL_n(\mathbf{Z}[\zeta]) \qquad \text{and} \qquad NQ_{n+1}(\mathbf{F}_2) \xrightarrow[\cong]{\partial} NL_n(\mathbf{F}_2).$$

For Parts (2) and (3), recall Nilpotent Reduction (3.1.4) shows that the following induced map is an isomorphism:

$$NL_n(\mathbf{F}_2[P_0]) \longrightarrow NL_n(\mathbf{F}_2).$$

If $P' = 1$ and $e(P) > 2$, then Lemma 3.2.4 shows that the following induced map is an isomorphism:

$$\mathrm{incl}_* : NL_n(\mathbf{Z}[i]) \longrightarrow NL_n(R_0).$$

Finally, since $\zeta_e \in R$, observe that there exists an isomorphism of rings with involution:

$$f : R[\zeta_{e(P)}] \longrightarrow R[\mathbf{C}_{e(P)}]; \qquad \zeta_{e(P)} \longmapsto (\zeta_e)^{e/e(P)} T,$$

where T is a generator of the cyclic group $\mathbf{C}_{e(P)}$. Therefore we obtain an induced isomorphism

$$f_* : NL_n(R[P'][\zeta_{e(P)}]) \longrightarrow NL_n(R[P_0]).$$

□

3.4. Special 2-groups

A finite group is **special** if every normal abelian subgroup is cyclic.

PROPOSITION 3.4.1 ([**HTW84**, 2.2.1]). *A finite 2-group P is special if and only if it is either:*

(0) *for some $e \geq 0$, **cyclic***

$$C_e := \langle T \mid T^{2^e} = 1 \rangle$$

(1) *for some $e > 3$, **dihedral***

$$D_e := \left\langle T, R \mid T^{2^{e-1}} = 1 = R^2, RTR^{-1} = T^{-1} \right\rangle$$

(2) *for some $e > 3$, **semidihedral***

$$SD_e := \left\langle T, R \mid T^{2^{e-1}} = 1 = R^2, RTR^{-1} = T^{2^{e-2}-1} \right\rangle$$

(3) *for some $e \geq 3$, **quaternionic***

$$Q_e := \left\langle T, R \mid T^{2^{e-1}} = 1, R^2 = T^{2^{e-2}}, RTR^{-1} = T^{-1} \right\rangle.$$

The Mayer-Vietoris exact sequence for cyclic 2-groups C_e is provided in the previous section; the Mayer-Vietoris exact sequence for the other special 2-groups $P \in \{ D_e, SD_e, Q_e \}$ is provided in the following proposition. Its main ingredient is the fact that P has an index two dihedral quotient D_{e-1}.

PROPOSITION 3.4.2. *Consider any noncyclic special 2-group P, and let $n \in \mathbf{Z}$. Let $e \geq 3$, and write $\zeta := \zeta_{2^{e-1}}$ a dyadic root of unity. Denote $\circ_{\pm c}$ as twisting a quadratic extension by $\pm$ complex conjugation. Then there are the long exact sequences of Figure 3.4.1.*

In the next subsection we shall take a closer look at the twisted quadratic extensions for the first two cases.

PROOF. By Proposition 2.2.3, we obtain the Mayer-Vietoris exact sequence

$$NL_{n+1}(\mathbf{F}_2[G/K]) \xrightarrow{\partial} NL_n(\mathbf{Z}[G])$$
$$\longrightarrow NL_n(\mathbf{Z}[G/K]) \oplus NL_n(\mathbf{Z}[G]/\Sigma_K) \longrightarrow NL_n(\mathbf{F}_2[G/K])$$

using the following order two subgroup K of the group $G = D_e, SD_e, Q_e$:

$$K = \left\langle T^{2^{e-2}} \right\rangle.$$

$$NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[D_e]) \longrightarrow NL_n(\mathbf{Z}[D_{e-1}]) \oplus NL_n(\mathbf{Z}[\zeta] \circ_c \mathbf{C}_2) \longrightarrow NL_n(\mathbf{F}_2)$$

$$NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[SD_e]) \longrightarrow NL_n(\mathbf{Z}[D_{e-1}]) \oplus NL_n(\mathbf{Z}[\zeta] \circ_{-c} \mathbf{C}_2) \longrightarrow NL_n(\mathbf{F}_2)$$

$$NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[Q_e]) \longrightarrow NL_n(\mathbf{Z}[D_{e-1}]) \oplus NL_n(\mathbf{Z}[\zeta] \circ_c [i]) \longrightarrow NL_n(\mathbf{F}_2)$$

FIGURE 3.4.1. Exact sequences for $P = D_e, SD_e, Q_e$.

Observe that

$$G/K = D_{e-1} \quad \text{and} \quad \mathbf{Z}[D_e]/\Sigma_K = \mathbf{Z}[\zeta] \circ_c \mathbf{C}_2$$

$$\mathbf{Z}[SD_e]/\Sigma_K = \mathbf{Z}[\zeta] \circ_{-c} \mathbf{C}_2 \quad \text{and} \quad \mathbf{Z}[Q_e]/\Sigma_K = \mathbf{Z}[\zeta] \circ_c [i].$$

Finally, Nilpotent Reduction (3.1.4) shows that

$$NL_*(\mathbf{F}_2[D_{e-1}]) \longrightarrow NL_*(\mathbf{F}_2)$$

is an isomorphism. $\square$

3.4.1. Dihedral and semidihedral 2-groups. We now show that the above twisted quadratic extensions for the dihedral and semidihedral cases can be computed definitely from the NQ-groups of the Dedekind domains $\mathbf{Z}[\zeta \pm \zeta^{-1}]$ with involution.

PROPOSITION 3.4.3. *Let $e > 3$ and write $\zeta := \zeta_{2^e}$. Then for all $n \in \mathbf{Z}$, there is an isomorphism*

$$NL_n(\mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2) \longrightarrow NL_n(\mathbf{Z}[\zeta \pm \zeta^{-1}]) \, \oplus \, NL_n\left(\mathbf{Z}[\zeta \pm \zeta^{-1}] \to \mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2\right).$$

PROOF. There is an exact sequence of a pair:

$$\cdots \xrightarrow{\partial} NL_n(\mathbf{Z}[\zeta \pm \zeta^{-1}]) \longrightarrow NL_n(\mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2) \\ \longrightarrow NL_n\left(\mathbf{Z}[\zeta \pm \zeta^{-1}] \to \mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2\right) \xrightarrow{\partial} \cdots .$$

William Pardon [**Par82**, Proof 4.14] constructs an embedding of rings with involution:

$$f' : \mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2 \longrightarrow M_2(\mathbf{Z}[\zeta \pm \zeta^{-1}])$$

whose restriction to the center $\mathbf{Z}[\zeta \pm \zeta^{-1}]$ is the diagonal embedding. Here, the standard involution on the matrix ring is defined as the conjugate transpose. Then there is a commutative triangle

$$\begin{tikzcd} NL_n(\mathbf{Z}[\zeta \pm \zeta^{-1}]) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "\mathrm{diag}_*"'] & NL_n(\mathbf{Z}[\zeta] \circ_{\pm c} \mathbf{C}_2) \arrow[dl, "f'_*"] \\ NL_n(M_2(\mathbf{Z}[\zeta \pm \zeta^{-1}])). & \end{tikzcd}$$

The vertical map is an isomorphism, by quadratic Morita equivalence (see [**HRT87**]). Therefore we obtain the desired left-split short exact sequence. $\square$

CHAPTER 4

The cyclic group of order two

Consider the simplest nontrivial 2-group

$$P = \mathbf{C}_2 = \left\langle T \mid T^2 = 1 \right\rangle .$$

Recall (2.2.3) that its integral group ring fits into Rim's cartesian square

$$\begin{tikzcd} \mathbf{Z}[\mathbf{C}_2] \arrow[r, "i^-"] \arrow[d, "i^+"'] & \mathbf{Z} \arrow[d, "j^-"] \\ \mathbf{Z} \arrow[r, "j^+"] & \mathbf{F}_2 \end{tikzcd}$$

of rings with involution, where $i^{\pm}(T) = \pm 1$. We focus on piecing together its NL-data from the component rings $\mathbf{Z}$ and $\mathbf{F}_2$, using the Mayer-Vietoris sequence (2.2.3). The additional structure on

$$\mathrm{UNil}_*(R; R, R) \cong NL_*(R)$$

computed below is its covariant (pushforward) module structure over the **Verschiebung algebra**

$$\mathcal{V} := \mathbf{Z}[V_n \mid n > 0] = \mathbf{Z}[V_p \mid p \text{ prime}]$$

of n-th power operators

$$V_n := (x \mapsto x^n)$$

on any polynomial ring $R[x]$. An analogous structure in K-theory has been studied by J. Grunewald [**Gru05**, Ch. 3] for the Bass Nil-groups

$$\mathrm{Nil}_*(R) = NK_*(R).$$

4.1. Statement of results

The main computations of this section are Theorems 4.1.2 and 4.1.3.

REMARK 4.1.1. According to Connolly-Koźniewski [**CK95**], Connolly-Ranicki [**CR05**], and Connolly-Davis [**CD04**], the group $NL_{\mathrm{odd}}(\mathbf{F}_2)$ vanishes, and the Arf invariant is an isomorphism

$$\mathrm{Arf} : NL_{\mathrm{even}}(\mathbf{F}_2) \longrightarrow \frac{x\mathbf{F}_2[x]}{(f^2 - f)}.$$

The inverse of Arf is given by the map

$$q \longmapsto P_{q,1},$$

where for all $p, g \in \mathbf{Z}[x]$ the symplectic form $P_{p,g}$ is defined by

$$P_{p,g} := \left(\bigoplus_2 \mathbf{F}_2[x], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} p \\ g \end{pmatrix} \right).$$

Also the group $NL_n(\mathbf{Z})$ vanishes if $n \equiv 0, 1 \pmod 4$, the induced map to $NL_2(\mathbf{F}_2)$ is an isomorphism if $n \equiv 2 \pmod 4$, and there is a two-stage obstruction theory [**CD04**, Proof 1.7] if $n \equiv 3 \pmod 4$:

$$0 \longrightarrow \frac{x\mathbf{F}_2[x]}{(f^2 - f)} \xrightarrow{\mathcal{P}} NL_3(\mathbf{Z}) \xrightarrow{B} x\mathbf{F}_2[x] \times x\mathbf{F}_2[x] \longrightarrow 0.$$

It is given primarily by certain characteristic numbers B in Wu classes of (-1)-quadratic linking forms over $(\mathbf{Z}[x], 2)$, and secondarily by the Arf invariant, of even linking forms $\mathcal{P}$, over the function field $\mathbf{F}_2(x)$.

A general vanishing result and isomorphism are given by the following theorem.

THEOREM 4.1.2. *Suppose F is a finite group that contains a normal Sylow 2-subgroup of exponent two. If $n \equiv 0, 1 \pmod 4$, then the following abelian group vanishes:*

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{Z}[F]) = 0.$$

Furthermore, if $n \equiv 2 \pmod 4$, then the following induced map is an isomorphism:

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{Z}[F]) \longrightarrow \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{F}_2) \xrightarrow{r\cong} NL_n^{\langle -\infty \rangle}(\mathbf{F}_2) \xrightarrow{\mathrm{Arf}\cong} x\mathbf{F}_2[x]/(f^2 - f).$$

Now we take a closer look at non-vanishing in the remaining dimensions.

THEOREM 4.1.3. *If $n \equiv 3 \pmod 4$, then there exists a decomposition*

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{Z}[\mathbf{C}_2]) \cong \mathrm{UNil}_{n+1}^{\langle -\infty \rangle}(\mathbf{F}_2) \oplus \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{Z}) \oplus \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbf{Z}).$$

PROOF. Immediate from Theorems 4.1.4 and 4.1.11. $\square$

THEOREM 4.1.4. *Consider $P = \mathbf{C}_2$ with trivial orientation character. Then, as Verschiebung modules, there is a decomposition*

$$NL_3(\mathbf{Z}[\mathbf{C}_2]) = NL_3(\mathbf{Z}) \oplus \widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$$

and there is an exact sequence (a three-stage obstruction theory):

$$0 \longrightarrow NL_0(\mathbf{F}_2) \xrightarrow{\widetilde{\partial}} \widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2]) \xrightarrow{i^-} NL_3(\mathbf{Z}) \longrightarrow 0.$$

Ingredients for the next theorem are as follows. Given a ring A with involution and $\epsilon = \pm 1$, there is an identification [**Ran81**, Prop. 1.6.4] between split ϵ-quadratic formations over A and connected 1-dimensional ϵ-quadratic complexes over A. The identification between $(-\epsilon)$-quadratic linking forms over $(A, (2)^\infty)$ and resolutions by $(2)^\infty$-acyclic 1-dimensional ϵ-quadratic complexes over A is given by [**Ran81**, Proposition 3.4.1].

The determination of the above extension 4.1.4(3) of abelian groups involves algebraic gluing of quadratic complexes [**Ran81**, §1.7], given below (4.1.7) by a choice $\mathcal{M}$ of set-wise section. Recall from group cohomology that an extension of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and a choice of set-wise section $s : C \to B$ determine a factorset

$$f : C \times C \longrightarrow A; \qquad (c, c') \mapsto s(c) + s(c') - s(c + c').$$

Our main concern is the computation of such a function f, via generators of C and an invariant for A in the above sequence 4.1.4(3) of abelian groups.

REMARK 4.1.5. The Connolly-Davis computation of $NL_3(\mathbf{Z}) \cong NL_4(\mathbf{Z}, (2)^\infty)$ involves generators $\mathcal{N}_{p,g}$ indexed by polynomials $p, g \in \mathbf{Z}[x]$. Either p or g must have zero constant coefficient, and each generator is defined as the nonsingular $(+1)$-quadratic linking form

$$\mathcal{N}_{p,g} := \left(\bigoplus_2 \mathbf{Z}[x]/2, \begin{pmatrix} p/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} p/2 \\ g \end{pmatrix} \right)$$

over $(\mathbf{Z}[x],(2)^\infty)$ of exponent two, see [**CD04**, Dfn. 1.6 and p. 1057]. For our computation, we identify it with a choice of resolution by a nonsingular split (-1)-quadratic formation

$$\mathcal{N}_{p,g} = \left(\bigoplus_2 \mathbf{Z}[x], (\begin{pmatrix} p & 1 \\ 1 & 2g \\ 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} p & 1 \\ 1 & 2g \end{pmatrix}) \bigoplus_2 \mathbf{Z}[x] \right).$$

DEFINITION 4.1.6 ([**Ran81**, p. 69]). Let R be a ring with involution, and let F, G be finitely generated projective R-modules. A **nonsingular split ϵ-quadratic formation** $\left(F, (\begin{pmatrix}\gamma \\ \mu\end{pmatrix}, \theta)G\right)$ **over** R consists of the hyperbolic ϵ-quadratic form

$$\mathscr{H}_\epsilon(F) := \left(F \oplus F^*, \begin{pmatrix} 0 & \mathbf{1}_F \\ 0 & 0 \end{pmatrix} \right)$$

along with the standard lagrangian $F \oplus 0$, a second lagrangian

$$\mathrm{Im}(\begin{pmatrix}\gamma \\ \mu\end{pmatrix} : G \to F \oplus F^*),$$

and a hessian

$$\theta : G \longrightarrow G^*,$$

which is a choice of de-symmetrization of the pullback form:

$$\theta - \epsilon\theta^* = \begin{pmatrix}\gamma \\ \mu\end{pmatrix}^* \begin{pmatrix} 0 & \mathbf{1}_F \\ 0 & 0 \end{pmatrix} = \gamma^* \circ \mu : G \to G^*.$$

□

DEFINITION 4.1.7. For any polynomial $q \in x\mathbf{Z}[x]$, define the nonsingular split (-1)-quadratic formation $\mathcal{Q}_q$ over $\mathbf{Z}[\mathbf{C}_2][x]$, where $\hat{q} := 2(1-T)q$, by

$$\mathcal{Q}_q := \left(\bigoplus_2 \mathbf{Z}[\mathbf{C}_2][x], (\begin{pmatrix} 0 & \hat{q} \\ \hat{q} & 0 \\ 1 & (1-T)q \\ (1-T) & 1 \end{pmatrix}, \begin{pmatrix} \hat{q} & 0 \\ \hat{q} & q\hat{q} \end{pmatrix}) \bigoplus_2 \mathbf{Z}[\mathbf{C}_2][x] \right).$$

For any polynomials $p, g \in \mathbf{Z}[x]$ with $pg \in x\mathbf{Z}[x]$, define the nonsingular split (-1)-quadratic formation $\mathcal{M}_{p,g}$ over $\mathbf{Z}[\mathbf{C}_2][x]$ by

$$\mathcal{M}_{p,g} := \left(\bigoplus_2 \mathbf{Z}[\mathbf{C}_2][x], (\begin{pmatrix} p & 1 \\ 1 & (1-T)g \\ 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} p & 1 \\ 1 & (1-T)g \end{pmatrix}) \bigoplus_2 \mathbf{Z}[\mathbf{C}_2][x] \right).$$

□

Indeed each of these (-1)-quadratic formations consists of lagrangian summands, since the associated 1-dimensional (-1)-quadratic complex over $\mathbf{Z}[\mathbf{C}_2][x]$ is connected [**Ran80a**, Proof 2.3] and in fact Poincaré: the Poincaré duality map on the level of projective modules induces isomorphisms on its homology groups. For example in $\mathcal{M}_{p,g}$, the nontrivial homological Poincaré duality map is

$$\left(\begin{smallmatrix} p & 1 \\ 1 & (1-T)g \end{smallmatrix}\right) : H^0(C) \to H_1(C), \quad \text{where} \quad H^0(C) = H_1(C) = \mathbf{Z}[\mathbf{C}_2][x]/2.$$

Its determinant $(1-T)pg - 1$ is a unit mod 2 in the commutative ring $\mathbf{Z}[\mathbf{C}_2][x]$, since

$$((1-T)pg - 1)^2 = 2(1-T)(pg)^2 - 2(1-T)pg + 1 \equiv 1 \pmod 2.$$

Therefore the Poincaré duality map for $\mathcal{M}_{p,g}$ is a homology isomorphism. Also, the formation $\mathcal{Q}_q$ is obtained as a pullback of a nonsingular formation, cf Proof 4.1.8(1).

Proposition 4.1.8. *The following formulas are satisfied for cobordism classes in the reduced group* $\widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$.

(1) *Boundary map:* $\widetilde{\partial}[P_{q,1}] = [\mathcal{Q}_q]$
(2) *Lifts:* $i^-[\mathcal{M}_{p,g}] = [\mathcal{N}_{p,g}]$ *and* $i^+[\mathcal{M}_{p,g}] = 0$

Now we state the basic relations between our generators $\mathcal{Q}$ and $\mathcal{M}$, established by algebraic surgery. Their inspiration is the statement and proof of [**CD04**, Lemma 4.3], but they are proven independently.

Proposition 4.1.9. *The following formulas are satisfied for cobordism classes in the reduced group* $\widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$.

(1) *Additivity:* $[\mathcal{M}_{p_1,g}] + [\mathcal{M}_{p_2,g}] = [\mathcal{M}_{p_1+p_2,g}] + [\mathcal{Q}_q]$ *where* $q := (p_1 g)(p_2 g)$

(2) *Symmetry:* $[\mathcal{M}_{2p,g}] = [\mathcal{M}_{2g,p}]$

(3) *Square associativity:* $[\mathcal{M}_{x^2 p,g}] = [\mathcal{M}_{p,x^2 g}]$

(4) *Square root:* $[\mathcal{M}_{2p^2 g,g}] = [\mathcal{M}_{2p,g}]$

Here are some useful formal consequences, which do not require the technique of algebraic surgery.

COROLLARY 4.1.10. *The following formulas are satisfied for cobordism classes in the reduced group* $\widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$.

(1) *Exponent four:* $4 \cdot [\mathcal{M}_{p,g}] = 0$

(2) *Idempotence:* $2(V_2 - 1) \cdot [\mathcal{M}_{p,1}] = 0$

(3) *Exponent two:* $2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0$

(4) *Nilpotence:* $V_2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0$

Finally we conclude with a determination of the Verschiebung module extension $\mathrm{UNil}_3(\mathbf{Z}[\mathbf{C}_2])$, through the eyes of the Connolly-Ranicki isomorphism (2.1.2).

THEOREM 4.1.11. *The extension of* $\mathcal{V}$*-modules in Theorem 4.1.4(3) is trivial.*

(1) *A decomposition into irreducible* $\mathcal{V}$*-modules is given by the isomorphism*

$$\Phi := \begin{pmatrix} \Phi^{\mathcal{Q}} & \Phi_1^{\mathcal{M}} & \Phi_2^{\mathcal{M}} & \Phi_1^{\mathcal{N}} & \Phi_2^{\mathcal{N}} \end{pmatrix} : \mathrm{Domain}(\Phi) \longrightarrow NL_3(\mathbf{Z}[\mathbf{C}_2]).$$

The component maps are

$$\begin{array}{rclcl}
\Phi^{\mathcal{Q}} & : & \dfrac{\mathcal{V}}{(2, V_2 - 1)} & \longrightarrow & NL_3(\mathbf{Z}[\mathbf{C}_2]); \quad [1] \quad \longmapsto \quad [\mathcal{Q}_x] \\[2em]
\Phi_1^{\mathcal{M}} & : & \dfrac{\mathcal{V}}{(4, 2(V_2 - 1))} & \longrightarrow & NL_3(\mathbf{Z}[\mathbf{C}_2]); \quad [1] \quad \longmapsto \quad [\mathcal{M}_{x,1}] \\[2em]
\Phi_2^{\mathcal{M}} & : & \displaystyle\bigoplus_{e \in \mathbf{Z}_{\geq 0}} \frac{\mathcal{V}}{(2, V_2)} & \longrightarrow & NL_3(\mathbf{Z}[\mathbf{C}_2]); \quad [1]_e \quad \longmapsto \quad [\mathcal{M}_{x,x^p}] - [\mathcal{M}_{1,xx^p}] \\[2em]
\Phi_1^{\mathcal{N}} & : & \dfrac{\mathcal{V}}{(4, 2(V_2 - 1))} & \longrightarrow & NL_3(\mathbf{Z}[\mathbf{C}_2]); \quad [1] \quad \longmapsto \quad \iota([\mathcal{N}_{x,1}]) \\[2em]
\Phi_2^{\mathcal{N}} & : & \displaystyle\bigoplus_{e \in \mathbf{Z}_{\geq 0}} \frac{\mathcal{V}}{(2, V_2)} & \longrightarrow & NL_3(\mathbf{Z}[\mathbf{C}_2]); \quad [1]_e \quad \longmapsto \quad \iota([\mathcal{N}_{x,x^p}] - [\mathcal{N}_{1,xx^p}]).
\end{array}$$

Here $p := 2^e$ is a power of two and $\iota : 1 \to \mathbf{C}_2$ is the inclusion of groups with trivial involutions.

(2) *Consider the $\mathcal{V}$-module morphism*

$$s : NL_3(\mathbf{Z}) \longrightarrow \widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$$

given additively by

$$V_n \cdot [\mathcal{N}_{x,1}] \longmapsto V_n \cdot [\mathcal{M}_{x,1}]$$
$$V_n \cdot ([\mathcal{N}_{x,x^p}] - [\mathcal{N}_{1,xx^p}]) \longmapsto V_n \cdot ([\mathcal{M}_{x,x^p}] - [\mathcal{M}_{1,xx^p}]).$$

It is a splitting of the surjection of the short exact sequence of Theorem 4.1.4(3). Let $p := p_1 x + \cdots + p_n x^n \in x\mathbf{Z}[x]$ be a polynomial with null-augmentation. Define polynomials $p', p'' \in x\mathbf{Z}[x]$ by

$$\begin{aligned} p' &:= \sum_i \kappa(p_i) x^i \\ p'' &:= \sum_{i<j} p_i p_j x^{i+j} \end{aligned}$$

where

$$\kappa(c) := \begin{cases} 0 & \text{if } c \equiv 0, 1 \pmod 4 \\ 1 & \text{if } c \equiv 2, 3 \pmod 4. \end{cases}$$

Then s satisfies the formula

$$s([\mathcal{N}_{p,1}]) = [\mathcal{M}_{p,1}] + [\mathcal{Q}_{p'+p''}].$$

4.2. Proof of main results

Proof of Theorem 4.1.2. Denote S as the Sylow 2-subgroup of F. Since S is normal and abelian, by the reduction isomorphism of Theorem 3.2.1 and the Connolly-Ranicki isomorphism r of Corollary 2.1.3, it suffices to show that:

$$NL^h_n(\mathbf{Z}[S]) = 0 \quad \text{if } n \equiv 0, 1 \pmod 4$$

and the following induced map is an isomorphism:

$$NL^h_2(\mathbf{Z}[S]) \longrightarrow NL^h_2(\mathbf{F}_2).$$

We induct on the order of S. If $|S| = 1$, then recall from Remark 4.1.1 that

$$NL_n(\mathbf{Z}[S]) = NL_n(\mathbf{Z}[1]) = 0 \quad \text{if } n \equiv 0, 1 \pmod 4$$

and the following induced map is an isomorphism:

$$NL_2(\mathbf{Z}[S]) = NL_2(\mathbf{Z}[1]) \longrightarrow NL_2(\mathbf{F}_2).$$

Otherwise suppose $|S| > 1$. Since S has exponent two, there is a decomposition

$$S = S' \times \mathbf{C}_2$$

as an internal direct product of groups of exponent two. Then the Mayer-Vietoris sequence of Proposition 3.3.1 specializes to:

$$\cdots NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[S]) \longrightarrow \bigoplus_2 NL_n(\mathbf{Z}[S']) \longrightarrow NL_n(\mathbf{F}_2) \xrightarrow{\partial} \cdots .$$

Observe, by the inductive hypothesis and Remark 4.1.1, that

$$NL_n(\mathbf{Z}[S']) = 0 \quad \text{and} \quad NL_n(\mathbf{F}_2) = 0 \quad \text{for all } n \equiv 0, 1 \pmod 4.$$

So we obtain

$$NL_0(\mathbf{Z}[S]) = 0$$

and an exact sequence

$$0 \longrightarrow NL_2(\mathbf{Z}[S]) \longrightarrow \bigoplus_2 NL_2(\mathbf{Z}[S']) \longrightarrow NL_2(\mathbf{F}_2) \xrightarrow{\partial} NL_1(\mathbf{Z}[S]) \longrightarrow 0.$$

But the following induced map is an isomorphism, by inductive hypothesis:

$$NL_2(\mathbf{Z}[S']) \longrightarrow NL_2(\mathbf{F}_2).$$

Therefore

$$NL_1(\mathbf{Z}[S]) = 0$$

and the following composite of induced maps is an isomorphism:

$$NL_2(\mathbf{Z}[S]) \longrightarrow NL_2(\mathbf{Z}[S']) \longrightarrow NL_2(\mathbf{F}_2).$$

This concludes the induction on $|S|$. $\square$

Proof of Theorem 4.1.4. The exact sequence of Proposition 3.3.1 becomes

$$NL_{n+1}(\mathbf{F}_2) \xrightarrow{\partial} NL_n(\mathbf{Z}[\mathbf{C}_2]) \xrightarrow{\binom{i^-}{i^+}} NL_n(\mathbf{Z}) \oplus NL_n(\mathbf{Z}) \xrightarrow{(j^- \; -j^+)} NL_n(\mathbf{F}_2).$$

Since this sequence is functorial, it must consist of $\mathcal{V}$-module morphisms. It follows by Orientable Reduction (3.1.1) that there is the commutative diagram of Figure 4.2.1 with top row exact, where $\varepsilon : \mathbf{C}_2 \to \mathbf{C}_2$ is the trivial map and

$$\widetilde{\partial} := (\mathbf{1} - \varepsilon) \circ \partial.$$

The map $\widetilde{\partial}$ is a monomorphism, since $i^+ \circ \partial = 0$ and the left square commutes. The map i^- is an epimorphism, since the projection $\mathrm{proj}_{\text{skew-diag}}$ onto the skew-diagonal is surjective and the right square commutes. Exactness at $\widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$ follows from its definition and exactness of the top row at $NL_3(\mathbf{Z}[\mathbf{C}_2])$. Thus the bottom row exists and is an exact sequence of $\mathcal{V}$-modules. $\square$

Proof of Proposition 4.1.8(1). According to [**Ran81**, pp. 517–519], the boundary map

$$\partial = \partial_{i^-} \circ \delta : L_4(\mathbf{F}_2[x]) \to L_3(\mathbf{Z}[\mathbf{C}_2][x])$$

for our cartesian square is defined in general in terms of pullback modules by

$$(A'^r, \psi') \longmapsto$$

$$\left((B^r, \mathbf{1}, B'^r), (\begin{pmatrix} (\mathbf{1} - (\chi + \chi^*) \circ \phi, \mathbf{0}) \\ (\phi, \mathbf{1}) \end{pmatrix}, (\psi - \phi \circ \chi \circ \phi, \mathbf{0}))(B^r, \phi', B'^r) \right).$$

$$\begin{tikzcd}
0 \arrow[r] & NL_4(\mathbf{F}_2) \arrow[r, "\partial"] \arrow[d, equal] & NL_3(\mathbf{Z}[\mathbf{C}_2]) \arrow[r, "\binom{i^-}{i^+}"] \arrow[d, "\mathbf{1}-\varepsilon"] \arrow[dr, "i^- - i^+"] & NL_3(\mathbf{Z}) \oplus NL_3(\mathbf{Z}) \arrow[r] \arrow[d, "\mathrm{proj}_{\text{skew-diag}}"] & 0 \\
0 \arrow[r] & NL_4(\mathbf{F}_2) \arrow[r, "\widetilde{\partial}"] & \widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2]) \arrow[r, "i^-"] & NL_3(\mathbf{Z}) \arrow[r] & 0
\end{tikzcd}$$

FIGURE 4.2.1

It sends a Witt class of a rank r nonsingular form over $A' = \mathbf{F}_2[x]$ to the Witt class of split formation over $A = \mathbf{Z}[\mathbf{C}_2][x]$ obtained by pullback of the boundary formation of the lifted form over $B = \mathbf{Z}[x]$ and of the hyperbolic formation over $B' = \mathbf{Z}[x]$. The form ψ over B lifts the input form ψ' over A'. Their symmetrizations are denoted

$$\phi := \psi + \psi^* : B^r \longrightarrow (B^r)^* \quad \text{and} \quad \phi' := \psi' + \psi'^* : B'^r \longrightarrow (B'^r)^*.$$

The morphism $\chi : (B'^r)^* \to B'^r$ lifts the map

$$\chi' := (\phi')^{-1} \circ \psi' \circ (\phi')^{-1} : (A'^r)^* \longrightarrow A'^r.$$

Now we compute these morphisms in our situation. Let $p \in x\mathbf{Z}[x]$. Recall (4.1.1) and take

$$(A'^r, \psi') = P_{q,1} = \left(\mathbf{F}_2[x]^2, \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Choose a lift

$$(B^r, \psi) = \left(\mathbf{Z}[x]^2, \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Then we obtain and select

$$\chi' = \begin{bmatrix} 1 & 0 \\ 1 & q \end{bmatrix} : \mathbf{F}_2[x]^2 \to \mathbf{F}_2[x]^2 \qquad\qquad \chi = \begin{bmatrix} -1 & 0 \\ 1 & -q \end{bmatrix} : \mathbf{Z}[x]^2 \to \mathbf{Z}[x]^2.$$

Using the pullback module structure [**Ran81**, p. 507]

$$\mathbf{Z}[\mathbf{C}_2][x] \xrightarrow{\cong} (\mathbf{Z}[x], \mathbf{1} : \mathbf{F}_2[x] \to \mathbf{F}_2[x], \mathbf{Z}[x]); \qquad (m + nT) \longmapsto (m - n, m + n),$$

the pullback formation is

$$\partial[P_{q,1}] = \left((\mathbf{Z}[x]^2, \mathbf{1}, \mathbf{Z}[x]^2), (\begin{pmatrix} (\begin{bmatrix} 4q & 0 \\ 0 & 4q \end{bmatrix}, \mathbf{0}) \\ (\begin{bmatrix} 2q & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{1}) \end{pmatrix}, (\begin{bmatrix} 4q^2 & 4q \\ 0 & 4q \end{bmatrix}, \mathbf{0}))(\mathbf{Z}[x]^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{Z}[x]^2) \right)$$

$$= \left((\mathbf{Z}[x]^2, \mathbf{1}, \mathbf{Z}[x]^2), (\begin{pmatrix} (\begin{bmatrix} 0 & 4q \\ 4q & 0 \end{bmatrix}, \mathbf{0}) \\ (\begin{bmatrix} 1 & 2q \\ 2 & 1 \end{bmatrix}, \mathbf{1}) \end{pmatrix}, (\begin{bmatrix} 4q & 0 \\ 4q & 4q^2 \end{bmatrix}, \mathbf{0}))(\mathbf{Z}[x]^2, \mathbf{1}, \mathbf{Z}[x]^2) \right)$$

$$= \left(\mathbf{Z}[\mathbf{C}_2][x]^2, (\begin{bmatrix} 0 & 2(1-T)q \\ 2(1-T)q & 0 \\ 1 & (1-T)q \\ (1-T) & 1 \end{bmatrix}, \begin{bmatrix} 2(1-T)q & 0 \\ 2(1-T)q & 2(1-T)q^2 \end{bmatrix})\mathbf{Z}[\mathbf{C}_2][x]^2 \right)$$

$$= [\mathcal{Q}_q].$$

□

Proof of Proposition 4.1.8(2). Clearly

$$i^-(\mathcal{M}_{p,g}) = \mathcal{N}_{p,g}.$$

Note that the second lagrangian G of $i^+(\mathcal{M}_{p,g})$ is

$$\operatorname{Im} \begin{pmatrix} p & 1 \\ 1 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \operatorname{Im} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2p & 2 \end{pmatrix} = \operatorname{Im} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 2 & -2p \end{pmatrix}.$$

Therefore $i^+(\mathcal{M}_{p,g})$ is a graph formation over $\mathbf{Z}[x]$, hence represents 0 in $NL_3(\mathbf{Z})$. □

PROOF OF COROLLARY 4.1.10(1). Note, by Proposition 4.1.9(1) and the relations (4.1.1) in $NL_4(\mathsf{F}_2)$, that

$$\begin{aligned}4\cdot[\mathcal{M}_{p,g}] &= 2\cdot[\mathcal{Q}_{pg}]+2\cdot[\mathcal{M}_{2p,g}]\\ &= 2\cdot[\mathcal{Q}_{pg}]+[\mathcal{Q}_{2pg}]+[\mathcal{M}_{4p,g}]\\ &= [\mathcal{M}_{4p,g}].\end{aligned}$$

There is an isomorphism

$$(\mathbf{1},\mathbf{1},\begin{pmatrix}p & 0\\ 0 & 0\end{pmatrix}):\mathcal{M}_{0,g}\longrightarrow\mathcal{M}_{4p,g}$$

of split (-1)-quadratic formations over $\mathsf{Z}[\mathsf{C}_2][x]$, see [**Ran81**, p. 69, defn.]. Therefore as cobordism classes in $NL_3(\mathsf{Z}[\mathsf{C}_2])$ we obtain

$$\begin{aligned}4\cdot[\mathcal{M}_{p,g}] &= [\mathcal{M}_{4p,g}]\\ &= [\mathcal{M}_{0,g}]\\ &= 0,\end{aligned}$$

by [**Ran81**, Proposition 1.6.4] and since $\mathcal{M}_{0,g}$ is a graph formation. □

PROOF OF COROLLARY 4.1.10(2). Note, by Proposition 4.1.9(1) and the relations (4.1.1) in $NL_4(\mathsf{F}_2)$, that

$$\begin{aligned}2(V_2-1)\cdot[\mathcal{M}_{p,1}] &= (V_2-1)\cdot([\mathcal{Q}_p]+[\mathcal{M}_{2p,1}])\\ &= [\mathcal{M}_{2V_2p,1}]-[\mathcal{M}_{2p,1}]\\ &= \sum_{k=1}^{n}p_k\cdot\big([\mathcal{M}_{2(x^k)^2,1}]-[\mathcal{M}_{2(x^k),1}]\big)\end{aligned}$$

where we write the polynomial

$$p=p_1x+\cdots+p_nx^n\in\mathrm{Ker}(\mathrm{aug}_0)$$

for some $n\in\mathsf{Z}_{\geq 0}$ and $p_1,\ldots,p_n\in\mathsf{Z}$. But by Proposition 4.1.9(4), we have

$$[\mathcal{M}_{2(x^k)^2,1}]-[\mathcal{M}_{2(x^k),1}]=0\quad\text{for all } k>0.$$

Therefore

$$2(V_2-1)\cdot[\mathcal{M}_{p,1}]=0.$$

□

PROOF OF COROLLARY 4.1.10(3). Note by Proposition 4.1.9(1,2) and the relations (4.1.1) in $NL_4(\mathbf{F}_2)$ that

$$\begin{aligned} 2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) &= ([\mathcal{Q}_{xg}] - [\mathcal{Q}_{xg}]) + ([\mathcal{M}_{2x,g}] - [\mathcal{M}_{2,xg}]) \\ &= [\mathcal{M}_{2g,x}] - [\mathcal{M}_{2xg,1}]. \end{aligned}$$

By Proposition 4.1.9(1) using the fact that $[\mathcal{Q}_q] = 0$ if q is a multiple of 2, and since g has $\mathbf{Z}$-coefficients, we may assume that $g = x^k$ for some $k \in \mathbf{Z}_{\geq 0}$ in order to show that the right-hand term vanishes. If $k = 2i$ is even, then by Proposition 4.1.9(3,2), note

$$[\mathcal{M}_{2g,x}] = [\mathcal{M}_{2(x^i)^2,x}] = [\mathcal{M}_{2,(x^i)^2x}] = [\mathcal{M}_{2,xg}] = [\mathcal{M}_{2xg,1}].$$

Otherwise suppose $k = 2i + 1$ is odd. Then by Proposition 4.1.9(4) twice and by induction on k, note

$$\begin{aligned} [\mathcal{M}_{2g,x}] &= [\mathcal{M}_{2(x^i)^2x,x}] \\ &= [\mathcal{M}_{2(x^i),x}] \\ &= [\mathcal{M}_{2(x^{i+1}),1}] \\ &= [\mathcal{M}_{2(x^{i+1})^2,1}] \\ &= [\mathcal{M}_{2xg,1}]. \end{aligned}$$

Therefore for all $g \in \mathbf{Z}[x]$ we obtain

$$2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0.$$

□

PROOF OF COROLLARY 4.1.10(4). Note by Proposition 4.1.9(3) that

$$\begin{aligned} V_2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) &= [\mathcal{M}_{x^2,V_2g}] - [\mathcal{M}_{1,x^2V_2g}] \\ &= 0. \end{aligned}$$

□

PROOF OF THEOREM 4.1.11(1). The section s is a well-defined $\mathcal{V}$-module morphism by Corollary 4.1.10(1,2,3,4). By Theorem 4.1.4(3), the asserted properties of Φ follow immediately from the explicit Connolly-Davis calculation of $NL_4(\mathbf{F}_2)$ (see

[**CD04**, Thms. 1.2, 1.3]) and of $NL_3(\mathbf{Z})$ (see [**CD04**, Prop. 1.4, Cor. 1.8]) as $\mathcal{V}$-modules. $\square$

PROOF OF THEOREM 4.1.11(2). We induct on the number of monomials. If $p = x^m$ then

$$\begin{aligned} s([\mathcal{N}_{x^m,1}]) &= s(V_m \cdot [\mathcal{N}_{x,1}]) \\ &= V_m \cdot [\mathcal{M}_{x,1}] \\ &= [\mathcal{M}_{x^m,1}] + [\mathcal{Q}_0]. \end{aligned}$$

Otherwise assume the formula holds for a particular $p \in x\mathbf{Z}[x]$. Then, by Proposition 4.1.9(1), note

$$\begin{aligned} s([\mathcal{N}_{p+x^m,1}]) &= s([\mathcal{N}_{p,1}] + [\mathcal{N}_{x^m,1}]) \\ &= ([\mathcal{M}_{p,1}] + [\mathcal{Q}_{p'+p''}]) + ([\mathcal{M}_{x^m,1}] + [\mathcal{Q}_0]) \\ &= [\mathcal{M}_{p+x^m}] + [\mathcal{Q}_q] \end{aligned}$$

where $q := p' + p'' + px^m$. Denote

$$q' := (p + x^m)' \quad \text{and} \quad q'' := (p + x^m)''.$$

Then observe

$$q' = \begin{cases} p' & \text{if } p_m \equiv 0 \pmod 2 \\ p' \pm x^m & \text{if } p_m \equiv 1 \pmod 2 \end{cases} \qquad \text{and} \qquad q'' = p'' + px^m - p_m x^{2m}$$

where $p_m = 0$ if $\deg(p) < m$. Therefore the Arf invariant is

$$[q] = [q' + q''] \in x\mathbf{F}_2[x]/(f^2 - f),$$

as desired. $\square$

4.3. Some algebraic surgery machines

The proofs of all parts of Proposition 4.1.9 are technical– algebraic surgery and gluing are required. The first machine has input certain quadratic formations and outputs certain quadratic forms.

LEMMA 4.3.1. *Suppose* (C, ψ) *is a 1-dimensional* (-1)*-quadratic Poincaré complex over* $\mathbf{Z}[\mathbf{C}_2][x]$ *satisfying the following hypotheses.*

(a) *The 1-dimensional chain complex* C *over* $\mathbf{Z}[\mathbf{C}_2][x]$ *has modules* $C_1 = C_0$ *and differential* $d_C = 2 \cdot \mathbf{1}$.

(b) *There is a null-cobordism*

$$(f : i^-(C) \to D, (\delta\psi, i^-(\psi)) \in W_\%(f, -1)_2)$$

over $\mathbf{Z}[x]$ *such that* $f_0 = \mathbf{1} : C_0 \to D_0$ *and* $\delta\psi_2 = 0 : D^0 \to D_0$.

(c) *The quadratic Poincaré complex* $i^+(C, \psi)$ *over* $\mathbf{Z}[x]$ *corresponds to a graph formation.*

Then we obtain the following conclusions.

(1) *There exists a 2-dimensional* (-1)*-quadratic Poincaré complex* (F, Ψ) *over* $\mathbf{F}_2[x]$ *such that*

$$\partial[\overline{S}(F, \Psi)] = [\overline{S}(C, \psi)].$$

Here,

$$\partial : L_4(\mathbf{F}_2[x]) \longrightarrow L_3(\mathbf{Z}[\mathbf{C}_2][x])$$

is the boundary map of the Mayer-Vietoris sequence of Rim's cartesian square, and $\overline{S}$ *is the skew-suspension isomorphism.*

(2) *The instant surgery obstruction* $\Omega(F, \Psi)$ *is Witt equivalent to the nonsingular* $(+1)$*-quadratic form* $j^-(D^1, \delta\psi_0)$ *over* $\mathbf{F}_2[x]$.

The next machine constructs inputs for the above one given a lagrangian of a certain linking form. It is obtained as a specialization of [**Ran81**, Proof 3.4.5(ii)][1]

LEMMA 4.3.2. *Suppose* (C, ψ) *is a 1-dimensional* (-1)*-quadratic Poincaré complex over* $\mathbf{Z}[\mathbf{C}_2][x]$ *satisfying the following hypotheses.*

(a) *The 1-dimensional chain complex* C *over* $\mathbf{Z}[\mathbf{C}_2][x]$ *has modules* $C_1 = C_0$ *and differential* $d_C = 2 \cdot \mathbf{1}$.

[1]See errata for the formulas at `http://www.maths.ed.ac.uk/~aar/books/exacterr.pdf`

(b) *There exists a lagrangian L of the nonsingular $(+1)$-quadratic linking form (N, b, q) over $(\mathbf{Z}[x], (2)^\infty)$ associated to $i^-(C, \psi)$.*

(c) *The evaluation $i^+(C, \psi)$ corresponds to a graph formation over $\mathbf{Z}[x]$.*

Choose a finitely generated projective module P over $\mathbf{Z}[\mathbf{C}_2][x]$ and morphisms

(i) $\pi : P \to C^1$ *monic with image*

$$i^-(\pi)(P) = e^{-1}(L),$$

where the quotient map e is

$$e : i^-(C^1) \longrightarrow N := \mathrm{Cok}\left(i^-(d_C^* : C^0 \to C^1)\right)$$

(ii) $\chi : P \to P^*$ *satisfying the de-symmetrization identity*

$$(\pi^{-1} \circ d_C^*)^* \circ (\chi + \chi^*) = (\widetilde{\psi}_0 - \psi_0^*) \circ \pi : P \to C_0.$$

Then we obtain the following conclusions.

(1) *Define a quadratic cycle $\widehat{\psi} \in W_\%(C, -1)_1$ by*

$$\widehat{\psi}_0 := \psi_0 : C^0 \longrightarrow C_1 \qquad \widetilde{\widehat{\psi}}_0 := \widetilde{\psi}_0 : C^1 \longrightarrow C_0$$

$$\widehat{\psi}_1 := (\pi^{-1} \circ d_C^*)^* \circ \chi \circ (\pi^{-1} \circ d_C^*) - \widetilde{\psi}_0 \circ d_C^* : C^0 \longrightarrow C_0.$$

Then the quadratic cycle $\widehat{\psi}$ is homologous to ψ in $W_\%(C, -1)_1$ over $\mathbf{Z}[\mathbf{C}_2][x]$.

(2) *Define a chain complex $D = \left\{ D_1 \xrightarrow{d_D} D_0 \right\}$ with modules*

$$D_1 := i^-(P^*) \quad \textit{and} \quad D_0 := i^-(C_0)$$

and with differential

$$d_D := i^-(\pi^{-1} \circ d_C^*)^*.$$

Define a chain map $f : i^-(C) \to D$ by

$$f_0 := \mathbf{1} : i^-(C_0) \longrightarrow D_0 \qquad f_1 := i^-(\pi^*) : i^-(C_1) \longrightarrow D_1.$$

Define a quadratic chain $\delta\psi \in W_\%(D, -1)_2$ by

$$\delta\psi_0 := -i^-(\chi)^* : D^1 \longrightarrow D_1 \qquad \delta\psi_1 := -i^-(\chi) \circ d_D^* : D^0 \longrightarrow D_1$$

$$\widetilde{\delta\psi}_1 := \widetilde{\psi}_0 \circ i^-(\pi) : D^1 \longrightarrow D_0 \qquad \delta\psi_2 := 0 : D^0 \longrightarrow D_0.$$

Then

$$\left(f : i^-(C) \to D, (\delta\psi, i^-(\widehat{\psi})) \in W_{\%}(f, -1)_2\right)$$

is a null-cobordism over $\mathbf{Z}[x]$.

(3) *The evaluation* $i^+(C, \widehat{\psi})$ *corresponds to a graph formation over* $\mathbf{Z}[x]$.

Composition of the lemmas yields immediately the following result.

PROPOSITION 4.3.3. *Suppose* (C, ψ) *satisfies Hypotheses (a,b,c) of Lemma 4.3.2, and choose* P, π, χ *accordingly. Then there exists a 2-dimensional* (-1)*-quadratic complex* (F, Ψ) *over* $\mathbf{F}_2[x]$ *such that*

$$\partial[(F, \Psi)] = [(C, \widehat{\psi})] = [(C, \psi)]$$

as cobordism classes in

$$L_1(\mathbf{Z}[\mathbf{C}_2][x], -1) \xrightarrow{\overline{S}\cong} L_3(\mathbf{Z}[\mathbf{C}_2][x]),$$

and that its instant surgery obstruction is

$$[\Omega(F, \Psi)] = [j^-(D^1, \delta\psi_0)] = [k(P, -\chi^*)]$$

as Witt classes of the nonsingular $(+1)$*-quadratic forms over* $\mathbf{F}_2[x]$. □

Now we show why these machines work.

PROOF OF LEMMA 4.3.1. We shall put together the information in the hypotheses using a technique called "algebraic gluing" [**Ran81**, §1.7]. The resultant object (F, Ψ) is a union [**Ran81**, pp. 77–78] over $\mathbf{F}_2[x]$. The more efficient "direct union" [**Ran81**, pp. 79–80] does not apply here since the null-cobordisms $(D, \delta\psi)$ and $(E, 0)$ are non-split in general.

First, define a chain complex $E = \{ E_1 \to 0 \}$ over $\mathbf{Z}[x]$ with module

$$E_1 := i^+(C_1),$$

and a chain map $g : i^+(C) \to E$ by

$$g_1 := \mathbf{1} : i^+(C_1) \to E_1.$$

Then the quadratic pair

$$\big(g : i^+(C) \to E, (0, i^+(\psi) \in W_\%(g, -1)_2)\big)$$

is the data for an algebraic surgery. Consider the 2-dimensional mapping cone

$$\mathscr{C}(g) = \left(i^+(C_1) \xrightarrow{\binom{-\mathbf{1}}{2\cdot\mathbf{1}}} E_1 \oplus i^+(C_0) \longrightarrow 0 \right).$$

Note that

$$H^2(E) = H_0(\mathscr{C}(g)) = 0 \quad \text{and} \quad H^0(E) = H_2(\mathscr{C}(g)) = 0.$$

Observe that

$$H^1(E) = E^1 \quad \text{and} \quad \mathrm{proj}_* : H_1(\mathscr{C}(g)) \xrightarrow{\cong} i^+(C_0).$$

Then the homological Poincaré duality map $H^1(E) \to H_1(\mathscr{C}(g))$ is given by

$$i^+(\widetilde{\psi}_0 - \psi_0^*) : E^1 \longrightarrow i^+(C_0).$$

By hypothesis, $i^+(C, \psi)$ represents a graph formation

$$\left(F, (\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta) G \right),$$

which means that $\gamma : G \to F$ an isomorphism. According to [**Ran80a**, Proof 2.5], the representation is given by

$$F = i^+(C_1) \quad \text{and} \quad G = i^+(C^0)$$

$$\gamma = i^+(\widetilde{\psi}_0^* - \psi_0) \quad \text{and} \quad \mu = i^+(d_C^*) \quad \text{and} \quad \theta = -i^+(\psi + d_C \circ \psi_0).$$

Thus the map $H^1(E) \to H_1(\mathscr{C}(g))$ is given by the isomorphism γ^*. Since the Poincaré duality map $E^{2-*} \to \mathscr{C}(g)$ of projective module chain complexes induces isomorphisms in homology, it must be a chain homotopy equivalence. Thus the following 2-dimensional (-1)-quadratic pair is Poincaré:

$$\big(g : i^+(C) \to E, (0, i^+(\psi))\big).$$

Next, define a 2-dimensional (-1)-quadratic Poincaré complex (F, Ψ) over $\mathbf{F}_2[x]$ as the union (see [**Ran81**, pp. 77–78])

$$(F, \Psi) := j^- \left(f : i^-(C) \to D, (-\delta\psi, -i^-(\psi))\right) \bigcup_{k(C,\psi)} j^+ \left(g : i^+(C) \to E, (0, i^+(\psi))\right),$$

where k is composite morphism of rings with involution:

$$k := j^- \circ i^- = j^+ \circ i^+ : \mathbf{Z}[\mathbf{C}_2] \longrightarrow \mathbf{F}_2.$$

By construction,

$$\partial[(F, \Psi)] = [(C, \psi)],$$

where the boundary map

$$\partial : L_4(\mathbf{F}_2[x]) \longrightarrow L_3(\mathbf{Z}[\mathbf{C}_2][x])$$

is defined in [**Ran81**, Props. 6.3.1, 6.1.3] for our cartesian square. For simplicity, we suppress the morphisms $i^\pm, j^\pm, k$ in the remainder of the proof.

The 2-dimensional chain complex F over $\mathbf{F}_2[x]$ has modules

$$F_2 = C_1 \qquad F_1 = D_1 \oplus C_0 \oplus E_1 \qquad F_0 = D_0$$

and differentials

$$d_F^2 = \begin{pmatrix} -f_1 \\ d_C \\ -g_1 \end{pmatrix} : F_2 \to F_1 \qquad d_F^1 = \begin{pmatrix} d_D & f_0 & 0 \end{pmatrix} : F_1 \to F_0.$$

The quadratic cycle $\Psi \in W_\%(F, -1)_2$ has components

$$\Psi_0^2 = \begin{pmatrix} -\psi_0 \circ f_0^* \end{pmatrix} : F^0 \to F_2 \qquad \Psi_0^1 = \begin{pmatrix} -\delta\psi_0 & 0 & 0 \\ \widetilde{\psi}_0 \circ f_1^* & \psi_1^* & 0 \\ 0 & g_1 \circ \psi_0 & 0 \end{pmatrix} : F^1 \to F_1$$

$$\Psi_0^0 = \begin{pmatrix} 0 \end{pmatrix} : F^2 \to F_0 \qquad \Psi_1^1 = \begin{pmatrix} -\delta\psi_1 \\ \psi_1 \circ f_0^* \\ 0 \end{pmatrix} : F^0 \to F_1$$

$$\Psi_1^0 = \begin{pmatrix} -\widetilde{\delta\psi}_1 & 0 & 0 \end{pmatrix} : F^1 \to F_0 \qquad \Psi_2^0 = \begin{pmatrix} 0 \end{pmatrix} : F^0 \to F_0.$$

The differential

$$(d_F^1)^* : F^0 \longrightarrow F^1$$

is a split monomorphism, since $f_0 = \mathbf{1} : C_0 \to D_0$. Hence the instant surgery obstruction [**Ran80a**, Prop. 4.3] is represented by

$$\Omega(F,\Psi) = \left(D^1 \oplus E^1 \oplus C_1, \begin{pmatrix} \delta\psi_0 & 0 & -f_1 \\ 0 & 0 & -\mathbf{1} \\ 0 & 0 & 0 \end{pmatrix} \right).$$

This is Witt equivalent to the (necessarily) nonsingular $(+1)$-quadratic form $(D^1, \delta\psi_0)$ over $\mathsf{F}_2[x]$. □

Proof of Lemma 4.3.2. Indeed $\widehat{\psi} \in W_\%(C,-1)_1$ is a quadratic cycle, since

$$\begin{aligned}
\widehat{\psi}_1 + \widehat{\psi}_1^* &= (\pi^{-1} \circ d_C^*)^* \circ (\chi + \chi^*) \circ (\pi^{-1} \circ d_C^*) - (\widetilde{\psi}_0 \circ d_C^* + d_C \circ \widetilde{\psi}_0^*) \\
&= (\widetilde{\psi}_0 - \psi_0^*) \circ d_C^* - \widetilde{\psi}_0 \circ d_C^* - d_C \circ \widetilde{\psi}_0^* \\
&= -(d_C \circ \psi_0 + \widetilde{\psi}_0 \circ d_C^*)^* \\
&= (\psi_1 + \psi_1^*)^* \\
&= \psi_1 + \psi_1^*.
\end{aligned}$$

A similar check shows that $f : i^-(C) \to D$ is a chain map and that

$$\left(f : i^-(C) \to D, (\delta\psi, i^-(\widehat{\psi})) \right)$$

is a 2-dimensional (-1)-quadratic pair over $\mathsf{Z}[x]$. It is Poincaré (see [**Ran81**, p. 259]), since it is the data for an algebraic surgery to a contractible complex, killing the lift $i^-(P)$ of the lagrangian L.

The quadratic cycles $\widehat{\psi}$ and ψ are homologous[2]: the differences

$$\widehat{\psi}_0 - \psi_0 \quad \text{and} \quad \widetilde{\widehat{\psi}}_0 - \widetilde{\psi}_0$$

are zero, and the difference

$$\widehat{\psi}_1 - \psi_1$$

is (-1)-symmetric (see above calculation). Therefore, the latter difference is (-1)-even since $\widehat{H}^0(\mathsf{Z}[x],-1) = 0$. Finally, $i^+(C,\widehat{\psi})$ corresponds to the same graph formation as $i^+(C,\psi)$, except that their hessians have difference $\widehat{\theta} - \theta = \psi_1 - \widehat{\psi}_1$. □

[2]In the generality of [**Ran81**, Prop. 3.4.5(ii)], we only have that $\widehat{\psi}$ and ψ are *quadratic* homotopy equivalent; see [**Ran81**, Defn., p. 71].

4.4. Remaining proofs

Using our machine (4.3.3), we grind out the primary relations (4.1.9) in $\widetilde{NL}_3(\mathbf{Z}[\mathbf{C}_2])$ as a $\mathcal{V}$-module.

PROOF OF PROPOSITION 4.1.9(1). Let (C, ψ) be a 1-dimensional (-1)-quadratic Poincaré complex associated to the following nonsingular split (-1)-quadratic formation over $\mathbf{Z}[\mathbf{C}_2][x]$:

$$\mathcal{M}_{p_1,g} \oplus \mathcal{M}_{p_2,g} \oplus \mathcal{M}_{p_1+p_2,g}.$$

In particular, it has modules $C_1 = C_0$ of rank 6 and differential $d_C = 2 \cdot \mathbf{1}$. Consider the exponent two linking form (N, b, q) over $(\mathbf{Z}[x], (2)^\infty)$ associated to the evaluation $i^-(C, \psi)$, defined as

$$(N, b, q) = \mathcal{N}_{p_1,g} \oplus \mathcal{N}_{p_2,g} \oplus -\mathcal{N}_{p_1+p_2,g}.$$

Define a lift $\pi : P \to C^1$ of a lagrangian L of (N, b, q) by

$$\pi := \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} : P = \bigoplus_6 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow C^1 = \bigoplus_6 \mathbf{Z}[\mathbf{C}_2][x].$$

Also define a morphism $\chi : P \to P^*$ by

$$\chi := \begin{bmatrix} g & 1 & g & 1 & 2g & 1 \\ 0 & 0 & 0 & p_1 & 1 & p_2 \\ 0 & 0 & 0 & 1 & 2g & 0 \\ 0 & 0 & 0 & p_1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2g & 0 \\ 0 & 0 & 0 & 0 & 0 & p_2 \end{bmatrix} : P = \bigoplus_6 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow P^* = \bigoplus_6 \mathbf{Z}[\mathbf{C}_2][x].$$

It is straightforward to verify that $i^-(\pi)(P)$ is the inverse image of a lagrangian L and that χ satisfies the de-symmetrization identity. Therefore, by Proposition 4.3.3, we obtain a 2-dimensional (-1)-quadratic Poincaré complex (F, Ψ) over $\mathbf{F}_2[x]$ such that

$$\partial[(F, \Psi)] = [(C, \widehat{\psi})] = [(C, \psi)] \qquad \text{and} \qquad [\Omega(F, \Psi)] = [k(P, -\chi^*)].$$

In classical notation, we have that (F, Ψ) is represented by the nonsingular $(+1)$-quadratic form

$$(M, \lambda, \mu) := \left(\bigoplus_6 \mathbf{F}_2[x], \begin{bmatrix} 0 & 1 & g & 1 & 0 & 1 \\ 1 & 0 & 0 & p_1 & 1 & p_2 \\ g & 0 & 0 & 1 & 0 & 0 \\ 1 & p_1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & p_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} g \\ 0 \\ 0 \\ p_1 \\ 0 \\ p_2 \end{bmatrix} \right).$$

Its pullback along the choice (see [**Wal99**, Proof 5.3]) of automorphism

$$\alpha := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & g & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & g & 0 \\ 0 & 0 & 0 & p_1 & 1 + p_1 g & p_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} : M \longrightarrow M$$

is the symplectic form

$$\alpha^*(M, \lambda, \mu) = \left(\bigoplus_6 \mathbf{F}_2[x], \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} g \\ 0 \\ 0 \\ p_1 \\ p_1 g^2 \\ p_2 \end{bmatrix} \right)$$

which has Arf invariant $[q]$, where

$$q := (p_1 g)(p_2 g).$$

So, by Remark 4.1.1, as cobordism classes in $L_4(\mathbf{F}_2[x])$, we must have

$$[(F,\Psi)] = [P_{q,1}]$$

Therefore, by Proposition 4.1.8(1), we obtain

$$\begin{aligned}
[\mathcal{M}_{p_1,g}] + [\mathcal{M}_{p_2,g}] - [\mathcal{M}_{p_1+p_2,g}] &= [(C,\psi)] = \partial[(F,\Psi)] \\
&= \partial[P_{q,1}] \\
&= \widetilde{\partial}[P_{q,1}] \\
&= [\mathcal{Q}_q].
\end{aligned}$$

□

Proof of Proposition 4.1.9(2). Let (C,ψ) be a 1-dimensional (-1)-quadratic Poincaré complex associated to the following nonsingular split (-1)-quadratic formation over $\mathbf{Z}[\mathbf{C}_2][x]$:

$$\mathcal{M}_{2p,g} \oplus -\mathcal{M}_{2g,p}.$$

In particular, it has modules $C_1 = C_0$ of rank 4 and differential $d_C = 2\cdot\mathbf{1}$. Consider the exponent two linking form (N,b,q) over $(\mathbf{Z}[x],(2)^\infty)$ associated to evaluation $i^-(C,\psi)$, defined as

$$(N,b,q) = \mathcal{N}_{2p,g} \oplus -\mathcal{N}_{2g,p}.$$

Define a lift $\pi : P \to C^1$ of a lagrangian L of (N,b,q) by

$$\pi := \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow C^1 = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

Also define a morphism $\chi : P \to P^*$ by

$$\chi := \begin{bmatrix} 0 & 0 & 2p & 1 \\ 0 & 0 & 1 & 2g \\ 0 & 0 & 2p & 2 \\ 0 & 0 & 0 & 2g \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow P^* = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

It is straightforward to verify that $i^{-}(\pi)(P)$ is the inverse image of a lagrangian L and that χ satisfies the de-symmetrization identity. Therefore, by Proposition 4.3.3, we obtain a 2-dimensional (-1)-quadratic Poincaré complex (F, Ψ) over $\mathbf{F}_2[x]$ such that

$$\partial[(F,\Psi)] = [(C,\widehat{\psi})] = [(C,\psi)] \qquad \text{and} \qquad [\Omega(F,\Psi)] = [k(P,-\chi^*)].$$

In classical notation, we have that (F,Ψ) is represented by the nonsingular $(+1)$-quadratic form

$$(M,\lambda,\mu) := \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right).$$

Its pullback along the choice (see [**Wal99**, Proof 5.3]) of automorphism

$$\alpha := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is the symplectic form

$$\alpha^*(M,\lambda,\mu) = \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

which has Arf invariant 0. So $[(F,\Psi)] = 0$ in $L_4(\mathbf{F}_2[x])$ hence in $NL_4(\mathbf{F}_2)$. Therefore

$$\begin{aligned} [\mathcal{M}_{2p,g}] - [\mathcal{M}_{2g,p}] &= [(C,\psi)] \\ &= \partial[(F,\Psi)] \\ &= 0. \end{aligned}$$

$\square$

Proof of Proposition 4.1.9(3). Let (C,ψ) be a 1-dimensional (-1)-quadratic Poincaré complex associated to the following nonsingular split (-1)-quadratic formation over $\mathbf{Z}[\mathbf{C}_2][x]$:

$$\mathcal{M}_{x^2p,g} \oplus -\mathcal{M}_{p,x^2g}.$$

In particular, it has modules $C_1 = C_0$ of rank 4 and differential $d_C = 2\cdot\mathbf{1}$. Consider the exponent two linking form (N,b,q) over $(\mathbf{Z}[x],(2)^\infty)$ associated to the evaluation $i^-(C,\psi)$, defined as

$$(N,b,q) = \mathcal{N}_{x^2p,g} \oplus -\mathcal{N}_{p,x^2g}.$$

Define a lift $\pi : P \to C^1$ of a lagrangian L of (N,b,q) by

$$\pi := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 2 \\ x & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow C^1 = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

Also define a morphism $\chi : P \to P^*$ by

$$\chi := \begin{bmatrix} 0 & 0 & -xp & 1 \\ 0 & 0 & -1 & 2xg \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 2g \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow P^* = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

It is straightforward to verify that $i^-(\pi)(P)$ is the inverse image of a lagrangian L and that χ satisfies the de-symmetrization identity. Therefore, by Proposition 4.3.3, we obtain a 2-dimensional (-1)-quadratic Poincaré complex (F,Ψ) over $\mathbf{F}_2[x]$ such that

$$\partial[(F,\Psi)] = [(C,\widehat{\psi})] = [(C,\psi)] \qquad \text{and} \qquad [\Omega(F,\Psi)] = [k(P,-\chi^*)].$$

In classical notation, we have that (F, Ψ) is represented by the nonsingular $(+1)$-quadratic form

$$(M, \lambda, \mu) := \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & 0 & xp & 1 \\ 0 & 0 & 1 & 0 \\ xp & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ p \\ 0 \end{bmatrix} \right).$$

Its pullback along the choice (see [**Wal99**, Proof 5.3]) of automorphism

$$\alpha := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -xp & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the symplectic form

$$\alpha^*(M, \lambda, \mu) = \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} p \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

which has Arf invariant 0. So $[(F, \Psi)] = 0$ in $L_4(\mathbf{F}_2[x])$ hence in $NL_4(\mathbf{F}_2)$. Therefore

$$\begin{aligned} [\mathcal{M}_{x^2p,g}] - [\mathcal{M}_{p,x^2g}] &= [(C, \psi)] \\ &= \partial[(F, \Psi)] \\ &= 0. \end{aligned}$$

□

Proof of Proposition 4.1.9(4). Let (C, ψ) be a 1-dimensional (-1)-quadratic Poincaré complex associated to the following nonsingular split (-1)-quadratic formation over $\mathbf{Z}[\mathbf{C}_2][x]$:

$$\mathcal{M}_{2p^2g,g} \oplus -\mathcal{M}_{2p,g}.$$

In particular, it has modules $C_1 = C_0$ of rank 4 and differential $d_C = 2 \cdot \mathbf{1}$. Consider the exponent two linking form (N, b, q) over $(\mathbf{Z}[x], (2)^\infty)$ associated to the evaluation

$i^-(C,\psi)$, defined as

$$(N,b,q) = \mathcal{N}_{2p^2g,g} \oplus -\mathcal{N}_{2p,g}.$$

Define a lift $\pi : P \to C^1$ of a lagrangian L of (N,b,q) by

$$\pi := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & p & 2 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 2 \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow C^1 = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

Also define a morphism $\chi : P \to P^*$ by

$$\chi := \begin{bmatrix} 0 & p^2g & 1 & -2pg \\ 0 & p^2g & 1+2pg & -1 \\ 0 & 0 & 2g & 0 \\ 0 & 0 & 0 & -2g \end{bmatrix} : P = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x] \longrightarrow P^* = \bigoplus_4 \mathbf{Z}[\mathbf{C}_2][x].$$

It is straightforward to verify that $i^-(\pi)(P)$ is the inverse image of a lagrangian L and that χ satisfies the de-symmetrization identity. Therefore, by Proposition 4.3.3, we obtain a 2-dimensional (-1)-quadratic Poincaré complex (F,Ψ) over $\mathbf{F}_2[x]$ such that

$$\partial[(F,\Psi)] = [(C,\widehat{\psi})] = [(C,\psi)] \qquad \text{and} \qquad [\Omega(F,\Psi)] = [k(P,-\chi^*)].$$

In classical notation, we have that (F,Ψ) is represented by the nonsingular $(+1)$-quadratic form

$$(M,\lambda,\mu) := \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & p^2g & 1 & 0 \\ p^2g & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ p^2g \\ 0 \\ 0 \end{bmatrix} \right).$$

Its pullback along the choice (see [**Wal99**, Proof 5.3]) of automorphism

$$\alpha := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & p^2g & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the symplectic form

$$\alpha^*(M,\lambda,\mu) = \left(\bigoplus_4 \mathbf{F}_2[x], \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

which has Arf invariant 0. So $[(F,\Psi)] = 0$ in $L_4(\mathbf{F}_2[x])$ hence in $NL_4(\mathbf{F}_2)$. Therefore

$$\begin{aligned} [\mathcal{M}_{2p^2g,g}] - [\mathcal{M}_{2p,g}] &= [(C,\psi)] \\ &= \partial[(F,\Psi)] \\ &= 0. \end{aligned}$$

□

PART II. TOPOLOGY: Connected sums of real projective spaces

CHAPTER 5

Preliminaries for the connected sum problem

Our purpose here is to study phenomena specific to the infinite dihedral group $\mathbf{D}_\infty$, with particular interest in the connected sum $\mathbf{RP}^n \# \mathbf{RP}^n$ of real projective spaces.

5.1. The infinite dihedral group, $\mathbf{D}_\infty$

The infinite dihedral group, defined below, is isomorphic to the fundamental group of the connected sum of two real projective n-spaces for all $n > 2$. Otherwise, note for $n = 2$ that $\mathbf{RP}^2 \# \mathbf{RP}^2$ is diffeomorphic to the Klein bottle with fundamental group $\mathbf{C}_\infty \rtimes \mathbf{C}_\infty$.

5.1.1. Action on the real line. Define the **infinite dihedral group**

$$\mathbf{D}_\infty := \mathrm{Isom}(\mathbf{Z}),$$

which is a crystallographic subgroup of the full isometry group

$$\mathrm{Isom}(\mathbf{R}) \cong \mathbf{R}^1 \rtimes O(1).$$

Thus it has a decomposition

$$\mathbf{D}_\infty \cong \mathbf{C}_\infty \rtimes \mathbf{C}_2 = \left\langle t, a \mid a^2 = 1, ata^{-1} = t^{-1} \right\rangle,$$

where a is reflection through 0 and t is translation by -1. Remarkably, it has another decomposition

$$\mathbf{D}_\infty \cong \mathbf{C}_2 * \mathbf{C}_2 = \left\langle a, b \mid a^2 = 1 = b^2 \right\rangle,$$

where b reflects through $1/2$. Hence $t = ab$.

5.1.2. Subgroups. In this section we classify all the subgroups of the infinite dihedral group. Consider the element

$$x_i := t^i a \in \mathbf{D}_\infty.$$

Observe, by the above decompositions, that every element of $\mathbf{D}_\infty$ is uniquely of the form x_i or t^i for some $i \in \mathbf{Z}$. For example, its generators can be expressed as

$$a = x_0 \quad \text{and} \quad b = x_{-1} \quad \text{and} \quad t = t^1.$$

Proposition 5.1.1.

(1) *Any subgroup has a generating set uniquely of the form:*
   (i) $\{ t^j \}$ *for some* $j \in \mathbf{Z}_{\geq 0}$, *or*
   (ii) $\{ x_i, t^j \}$ *for some* $i \in \mathbf{Z}, j \in \mathbf{Z}_{\geq 0}$ *such that if* $j \neq 0$ *then* $0 \leq i < j$.

(2) *Every subgroup of Form (i) is normal. A subgroup of Form (ii) is normal if and only if it is either* $\langle a, t\rangle$ *or* $\langle a, t^2\rangle$ *or* $\langle b, t^2\rangle$.

(3) *The quotient of* $\mathbf{D}_\infty$ *by the normal subgroup* $\langle t^j\rangle$ *is the* ***finite dihedral group*** $\mathbf{D}_j := \mathbf{C}_j \rtimes \mathbf{C}_2$. *The quotient of* $\mathbf{D}_\infty$ *by the normal subgroup* $\langle a, t^2\rangle$ *or* $\langle ta, t^2\rangle = \langle b, t^2\rangle$ *is the cyclic group* $\mathbf{C}_2$.

(4) *Let* $j > 2$ *and* $0 \leq i < j$. *The index* j, *non-normal subgroup* $\langle x_i, t^j\rangle$ *is conjugate to the subgroup*
$$\begin{cases} \langle a, t^j\rangle & \text{if } j \text{ is odd} \\ \langle a, t^j\rangle \text{ or } \langle ta, t^j\rangle & \text{if } j \text{ is even.} \end{cases}$$
If j *is odd, then the normalizer of* $\langle x_i, t^j\rangle$ *in* $\mathbf{D}_\infty$ *is itself. If* j *is even, then the normalizer of* $\langle x_i, t^j\rangle$ *in* $\mathbf{D}_\infty$ *is* $\left\langle x_i, t^{j/2}\right\rangle$ *with quotient* $\mathbf{C}_2$.

(5) *Let* $j = 0$ *and* $i \in \mathbf{Z}$. *The order* 2, *non-normal subgroup* $\langle x_i\rangle$ *is conjugate to the subgroup*
$$\begin{cases} \langle a\rangle & \text{if } i \text{ is even} \\ \langle ta\rangle \text{ or } \langle b\rangle & \text{if } i \text{ is odd.} \end{cases}$$
The normalizer of $\langle x_i\rangle$ *in* $\mathbf{D}_\infty$ *is itself.*

Proof (1). Let S be a subgroup of $\mathbf{D}_\infty$. There exist subsets $I, J \subseteq \mathbf{Z}$ such that

$$S = \{ x_i \}_{i \in I} \cup \{ t^j \}_{j \in J}.$$

Note that the set $\{ t^j \}_{j \in J}$ generates the same subgroup as the singleton

$$\left\{ t^{\gcd(J)} \right\}.$$

Also note for all $i_0 \in \mathbf{Z}$ that the set $\{ x_{i_0} \} \cup \{ x_i \}_{i \in I}$ generates the same subgroup as the set

$$\{ x_{i_0} \} \cup \left\{ x_i x_{i_0} = t^i a t^{i_0} a = t^{i-i_0} \right\}_{i \in I}.$$

Then S is generated by

$$\left\{ t^j \right\} \quad \text{or} \quad \left\{ x_i, t^j \right\} \quad \text{for some} \quad i \in \mathbf{Z}, j \in \mathbf{Z}_{\geq 0}.$$

In the case $j \neq 0$, we may write

$$i = qj + r \quad \text{for unique} \quad q \in \mathbf{Z}, 0 \leq r < j,$$

and so the subgroup S is generated by

$$\left\{ t^j \right\} \quad \text{or} \quad \left\{ x_r = (t^j)^{-q} x_i, t^j \right\}.$$

The former possibility occurs if $I = \varnothing$.

Let T be the maximal infinite cyclic subgroup of S:

$$T := \mathbf{C}_\infty \cap S,$$

where the subgroup $\mathbf{C}_\infty \subset \mathbf{D}_\infty$ is generated by t. Then observe that j is uniquely determined:

$$j = [S : T].$$

Suppose $j \neq 0$. Let $x_i, x_{i'} \in S$, where $0 \leq i \leq i' < j$. Then their product satisfies

$$x_{i'} x_i = t^{i'-i} \in T = \left\langle t^j \right\rangle.$$

Hence $i' - i$ is a multiple of j. But $0 \leq i' - i < j$, thus $i = i'$. Therefore i is uniquely determined also. □

Proof (2). It suffices to check normality by conjugation with the generators $\{ a, t \}$ of $\mathbf{D}_\infty$. Note

$$\begin{aligned} a t^j a^{-1} &= t^{-j} \\ t t^j t^{-1} &= t^j \\ a x_i a^{-1} &= a t^i = x_{-i} \\ t x_i t^{-1} &= t^{i+1} a t^{-1} = x_{i+2} a. \end{aligned}$$

Therefore $\langle t^j \rangle$ is normal and $\langle x_i \rangle$ is not normal.

Suppose $j \neq 0$ and $0 \leq i < j$. If $j > 2$ then

$$i \not\equiv i + 2 \pmod{j};$$

therefore $\langle x_i, t^j \rangle$ is not normal. If $j = 1, 2$ then

$$-i \equiv i \equiv i + 2 \pmod{j};$$

therefore we have three normal subgroups:

$$\left\langle x_0, t^1 \right\rangle = \langle a, t \rangle \qquad \left\langle x_1, t^2 \right\rangle = \left\langle ta, t^2 \right\rangle \qquad \left\langle x_0, t^2 \right\rangle = \left\langle a, t^2 \right\rangle.$$

□

PROOF (3). Note the relations

$$\begin{aligned} \mathbf{D}_\infty / \langle t^j \rangle &\cong \langle a, t \mid a^2 = 1, ata^{-1} = t^{-1}, t^j = 1 \rangle &&\cong \mathbf{D}_j \\ \mathbf{D}_\infty / \langle a, t^2 \rangle &\cong \langle a, t \mid a^2 = 1, ata^{-1} = t^{-1}, a = 1, t^2 = 1 \rangle &&\cong \mathbf{C}_2. \end{aligned}$$

The remaining result follows by symmetry:

$$\mathbf{D}_\infty / \left\langle b, t^2 \right\rangle \cong \mathbf{C}_2.$$

□

PROOF (4). Write $i = 2i_1 + i_0$ for unique $i_1 \in \mathbf{Z}, 0 \leq i_0 < 2$. Note

$$t^{-i_1} \left\langle x_i, t^j \right\rangle t^{i_1} = \left\langle x_{i_0}, t^j \right\rangle.$$

Hence the subgroup $\langle x_i, t^j \rangle$ is conjugate to

$$\left\langle x_1 = ta, t^j \right\rangle \quad \text{or} \quad \left\langle x_0 = a, t^j \right\rangle.$$

If j is odd, then a conjugates the former to $\langle x_{j-1}, t^j \rangle$, which conjugates to the latter. If j is even, then $ta = at^{-1}$ is not of the form $a(t^j)^m(t^2)^n$ for any $m, n \in \mathbf{Z}$, so the former is not conjugate to the latter.

Note the conjugations

$$\begin{aligned} x_k \left\langle x_i, t^j \right\rangle x_k^{-1} &= \left\langle x_{2k-i}, t^{-j} \right\rangle \\ t^l \left\langle x_i, t^j \right\rangle t^{-l} &= \left\langle x_{i+2l}, t^j \right\rangle. \end{aligned}$$

Then x_k and t^l normalize $\langle x_i, t^j\rangle$ if and only if

$$\begin{aligned} i &\equiv 2k - i \pmod{j} \\ i &\equiv i + 2l \pmod{j}. \end{aligned}$$

If j is odd, then they normalize if and only if

$$\begin{aligned} k &\equiv i \pmod{j} \\ l &\equiv 0 \pmod{j} \end{aligned}$$

if and only if

$$x_k, t^l \in \left\langle x_i, t^j \right\rangle .$$

If j is even, then they normalize if and only if

$$\begin{aligned} k &\equiv i \pmod{j/2} \\ l &\equiv 0 \pmod{j/2} \end{aligned}$$

if and only if

$$x_k, t^l \in \left\langle x_i, t^{j/2} \right\rangle .$$

Therefore the normalizer of $\langle x_i, t^j\rangle$ is itself if j is odd, and is $\left\langle x_i, t^{j/2}\right\rangle$ if j is even. $\square$

Proof (5). In the above argument for conjugacy classes, we did not use the condition $0 \leq i < j$. By taking $j = 0$, it shows that $\langle x_i\rangle$ is conjugate to the order two subgroup

$$\langle a\rangle \quad \text{or} \quad \langle ta\rangle .$$

The former occurs if $i \equiv i_0 \equiv 0 \pmod{2}$, and the latter occurs otherwise. The above argument for normalizers shows that x_k and t^l normalize $\langle x_i\rangle$ if and only if $k = i$ and $l = 0$. Therefore the normalizer of $\langle x_i\rangle$ is itself. $\square$

5.1.3. Whitehead group and growth. The infinite dihedral group $\mathbf{D}_\infty$ satisfies the following properties important to geometric topology.

Proposition 5.1.2.

(1) *The Whitehead torsion group* $\mathrm{Wh}(\mathbf{D}_\infty)$ *vanishes. In particular for all* $n > 2$, *any homotopy equivalence* $M^n \to \mathbf{RP}^n \# \mathbf{RP}^n$ *is simple.*

(2) *The group* $\mathbf{D}_\infty$ *is good ("small"), in the sense of Freedman-Quinn* [**FQ90**].

PROOF. Part (1) follows from Waldhausen's Mayer-Vietoris sequence in K-theory (1.3.4) for the amalgam

$$\mathbf{D}_\infty \cong \mathbf{C}_2 *_1 \mathbf{C}_2,$$

using

$$\mathrm{Wh}(1) = \widetilde{\mathrm{Nil}}_0(\mathbf{Z}; \mathbf{Z}, \mathbf{Z}) = \mathrm{Wh}(\mathbf{C}_2) = 0.$$

The vanishing of the Nil-group follows from [**Bas68**, Corollary XII.6.3], and the calculation of the Whitehead groups is given in [**Oli88**, Theorem 14.1(i)].

Part (2) follows from [**FQ90**, Theorem 5.1A] for the polycyclic-by-finite group

$$\mathbf{D}_\infty \cong \mathbf{C}_\infty \rtimes \mathbf{C}_2.$$

$\square$

5.2. TOP surgery theory: The real projective space, $\mathbf{RP}^n$

The purpose of this section is to compute the structure group $\mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^n)$ for all $n > 3$ in Theorem 5.2.10. It was initially López de Medrano [**LdM71**, Thm. IV.3.4, §IV.5] who computed $\mathcal{S}_{\mathrm{CAT}}(\mathbf{RP}^n)$ as a based set; an alternative PL classification in terms of surgery characteristic classes can be found in [**Wal99**, §14D]. However, we simplify some of his invariants and determine the abelian group structure on the structure group $\mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^n)$.

5.2.1. Normal invariants.

REMARK 5.2.1. The space $\mathbf{Z} \times \mathrm{G}/\mathrm{TOP}$ is a 4-periodic infinite loop space, hence the set of normal invariants

$$\mathcal{N}_{\mathrm{TOP}}(X) = [X, \mathrm{G}/\mathrm{TOP}]$$

has an abelian group structure for all connected compact TOP n-manifolds X, and similarly for the relative normal invariants

$$\mathcal{N}_{\mathrm{TOP}}(X \times (\Delta^1, \partial\Delta^1)) = [X, \Omega(\mathrm{G}/\mathrm{TOP})].$$

Moreover, the Quinn-Ranicki algebraic surgery sequence [**Ran92a**, §14] endows the structure set $\mathcal{S}_{\mathrm{TOP}}(X)$ with an abelian group structure, so that the maps ∂ and N

and the surgery obstruction map

$$\theta : \mathcal{N}_{\mathrm{TOP}}(X) \longrightarrow H_n(\pi; \mathbf{L}.^{\omega}) \xrightarrow{\text{asmb}} L_n(\mathbf{Z}[\pi^{\omega}])$$

are homomorphisms of groups (cf. Rem. 5.2.6). Here we abbreviate

$$\begin{aligned} \mathbf{L}. &:= \mathbf{L}.\langle 1 \rangle(\mathbf{Z}) \\ (\pi, \omega) &:= (\pi_1(X), w_1(X)). \end{aligned}$$

Proposition 5.2.2. *The topological normal invariants of real projective n-space are calculated by splitting invariants as follows for all $n > 3$.*

(1) *There is a group isomorphism*

$$\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n) \longrightarrow \bigoplus_{i=1}^{[n/2]} \mathbf{Z}_2.$$

It is defined on degree one TOP *normal maps f by the formula*

$$(f : M^n \to \mathbf{RP}^n) \quad \longmapsto \quad \bigoplus_{i=1}^{[n/2]} c\left(f|_{f^{-1}(\mathbf{RP}^{2i})}\right),$$

where c is the Kervaire-Arf invariant.

(2) *Also there is a group isomorphism*

$$\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n \times (\Delta^1, \partial\Delta^1)) \longrightarrow \bigoplus_{j=0}^{[(n-1)/4]} \mathbf{Z}_2 \bigoplus_{k=1}^{[(n+1)/4]} \mathbf{Z}.$$

It is defined on degree one TOP *normal bordisms F, with ∂F a homeomorphism, by the formula*

$$\begin{aligned} &\left(F : (M^n, \partial M) \to \mathbf{RP}^n \times (\Delta^1, \partial\Delta^1)\right) \\ &\qquad \longmapsto \bigoplus_{j=0}^{[(n-1)/4]} c\left(F|_{F^{-1}(\mathbf{RP}^{4j+1} \times \Delta^1)}\right) \bigoplus_{k=1}^{[(n+1)/4]} \frac{\sigma}{8}\left(F|_{F^{-1}(\mathbf{RP}^{4k-1} \times \Delta^1)}\right), \end{aligned}$$

where $\sigma/8$ is the signature invariant.

Remark 5.2.3. For i odd our result agrees with [**LdM71**, Theorem IV.2.3], but for $i = 2\ell$ even it gives a more direct geometric description than "the splitting invariant along a singular $\mathbf{Z}_2$-manifold [**Sul96**] representing $\mathbf{RP}^{4\ell}$." It is strange that our observation was not made earlier, in light of Wall's computation, see Remark 5.2.4 and [**LdM71**, §III.2.2, Remarks IV.2].

PROOF. Recall the commutative diagram of abelian groups:

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n) \arrow[r, equal] \arrow[d, "\mathrm{restr}"] & {[\mathbf{RP}^n, \mathrm{G/TOP}]} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathrm{incl}^*"] & {[\mathbf{RP}^n, K]} \arrow[r, equal] \arrow[d, "\mathrm{incl}^*"] & \bigoplus_{i=1}^{[n/2]} H^{2i}(\mathbf{RP}^n; \pi_{2i}(\mathrm{G/TOP})) \arrow[d, "\mathrm{incl}^*"] \\
\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{n-1}) \arrow[r, equal] & {[\mathbf{RP}^{n-1}, \mathrm{G/TOP}]} \arrow[r, leftrightarrow, "\cong"] & {[\mathbf{RP}^{n-1}, K]} \arrow[r, equal] & \bigoplus_{i=1}^{[(n-1)/2]} H^{2i}(\mathbf{RP}^{n-1}; \pi_{2i}(\mathrm{G/TOP})).
\end{tikzcd}$$

Here K is the product H-space

$$K := \prod_{i>0} K(\pi_{2i}(\mathrm{G/TOP}), 2i),$$

and the right-most vertical map is a split epimorphism with zero kernel if n odd, and with kernel $\mathbf{Z}_2$ if $n = 2k$ even. In the latter case, this "top" $\mathbf{Z}_2$ summand is detected by the Kervaire-Arf invariant c, and so it is flipped by connecting sum with the Kervaire manifold K^{2k} (cf. Rem. 5.2.4), which fixed the "lower" summands. Each middle horizontal map is a *homomorphism,* since it is induced by maps of infinite loop space hence of H-spaces. They are themselves induced by maps of Ω-spectra[1]

$$\mathbf{L}. \xrightarrow{\mathrm{Loc}_{(2)}} (\mathbf{L}.)_{(2)} \xrightarrow{k\times\ell} \underline{K}(\pi_*(\mathbf{L}.)_{(2)})$$

$$\underline{K}(\pi_*(\mathbf{L}.)) \longrightarrow \underline{K}(\pi_*(\mathbf{L}.)_{(2)}).$$

Moreover, each of these induced middle maps is a bijection by [**LdM71**, Theorem IV.2.2]. Therefore the result for $\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n)$ follows by backwards induction on $n > 3$.

A similar diagram, with the H-spaces $\Omega(\mathrm{G/TOP})$ and ΩK replacing G/TOP and K, produces a homomorphism

$$\mathrm{restr} : \mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n \times (\Delta^1, \partial\Delta^1)) \longrightarrow \mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{n-1} \times (\Delta^1, \partial\Delta^1)).$$

This is a split epimorphism with zero kernel if n even, and with kernel $\mathbf{Z}_2$ (resp. $\mathbf{Z}$) if $n = 4\ell+1$ (resp. $n = 4\ell-1$). In the latter case, it is detected by the Kervaire-Arf c (resp. signature $\sigma/8$) invariant and is altered by connecting sum with the Kervaire manifold $K^{4\ell+2}$ (resp. Milnor manifold $M^{4\ell}$, cf. Rem. 5.2.4). Therefore the result for the relative case (2) follows by backwards induction on $n > 3$. $\square$

5.2.2. Surgery obstructions.

REMARK 5.2.4. Recall for all $k, \ell > 0$ the existence of the Kervaire manifold K^{2k} (resp. **Milnor manifold** $M^{4\ell}$). Its construction is plumbing k-disc bundles over k-spheres along certain graphs and then closing the result. It is an almost parallelized, simply connected, closed TOP $2k$- (resp. 4ℓ)-manifold such that the associated degree one TOP normal map $f : K \to S^{2k}$ (resp. $f : M \to S^{4\ell}$) has nontrivial Kervaire-Arf invariant $c(f) = 1 \in \mathbf{Z}_2$ (resp. signature $\sigma(f)/8 = 1 \in \mathbf{Z}$).

[1]Here $\underline{K}$ denotes the product of Eilenberg-MacLane spectra. The latter map $k \times \ell$ is the Sullivan characteristic class [**MM79**] for $\mathbf{L}_0 = \mathrm{G/TOP}$; see [**TW79**] for the proof that it is a homotopy equivalence.

Also recall [**Wal99**, Thm. 13A.1] the following table:

$n \mod 4$	0	invariant	1	invariant	2	invariant	3	invariant
$L_n(1,+1)$	$\mathbf{Z}$	$\sigma/8$	0		$\mathbf{Z}_2$	c	0	
$L_n(\mathbf{C}_2,+1)$	$\mathbf{Z}\oplus\mathbf{Z}$	$\sigma/8\oplus\widetilde{\sigma}/8$	0		$\mathbf{Z}_2$	c	$\mathbf{Z}_2$	d
$L_n(\mathbf{C}_2,-1)$	$\mathbf{Z}_2$	c	0		$\mathbf{Z}_2$	c	0.	

Proposition 5.2.5. *The surgery obstruction maps*

$$\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n) \xrightarrow{\theta_0} L_n(\mathbf{C}_2,(-1)^{n+1})$$

$$\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^n\times(\Delta^1,\mathbf{RP}^n\times\partial\Delta^1)) \xrightarrow{\theta_1} L_{n+1}(\mathbf{C}_2,(-1)^{n+1})$$

satisfy

$$\mathrm{Ker}(\theta_0) \;=\; \bigoplus_{i=1}^{[n/2]-1}\mathbf{Z}_2\oplus\begin{cases}\mathbf{Z}_2 & \text{if } n\equiv +1 \pmod 4\\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{Cok}(\theta_1) \;=\; \begin{cases}\mathbf{Z} & \text{if } n\equiv -1 \pmod 4\\ 0 & \text{otherwise.}\end{cases}$$

They are detected by Kervaire-Arf invariants restricted transversally to $\mathbf{RP}^{2i}$ and by signature $\widetilde{\sigma}/8$ on universal covers.

Proof. The maps θ_0,θ_1 are homomorphisms of groups by algebraic surgery sequence. By Remark 5.2.4,

$$\operatorname{index}\mathrm{Ker}(\theta_0)=\begin{cases}1 & \text{if } n\equiv 1 \pmod 4)\\ 2 & \text{if } n\equiv 0,2 \pmod 4.\end{cases}$$

The latter follows by taking connected sum of a degree one TOP normal map f to $\mathbf{RP}^{2k}$ with the Kervaire manifold map $K^{2k}\to S^{2k}$. This flips its Kervaire-Arf invariant $c(f)\in\mathbf{Z}_2$ and fixes the transverse inverse image of $\mathbf{RP}^{2i}$ for all $1\le i<k$. Then

$$\operatorname{index}\mathrm{Ker}(\theta_0)=2\quad\text{if } n\equiv -1 \pmod 4,$$

since the invariant d makes commutative the diagram

$$\begin{tikzcd} \mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{4\ell+3}) \arrow[r, "\mathrm{restr}"] \arrow[d, "\theta_0"] & \mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{4\ell+2}) \arrow[d, "c"] \\ L_{4\ell+3}(\mathbf{C}_2, +1) \arrow[r, "d", "\cong"'] & \mathbf{Z}_2. \end{tikzcd}$$

A similar argument with the Kervaire manifold shows that

$$\mathrm{Ker}(\theta_0) = \mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{n-1}) \quad \text{if } n \equiv -1 \pmod 4.$$

Also by Remark 5.2.4,

$$\operatorname{index} \mathrm{Im}(\theta_1) = \begin{cases} 1 & \text{if } n \equiv 0, 2 \pmod 4 \\ 1 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

The latter follows by taking interior connected sum of F with the Kervaire manifold map $K^{4\ell+2} \to S^{4\ell+2}$. Here F is a degree one TOP normal map of manifold triads that restricts to a homeomorphism on the boundary:

$$F : (W^{4\ell+2}, \partial W) \longrightarrow (\mathbf{RP}^{4\ell+1} \times (\Delta^1, \partial\Delta^1)).$$

Finally, we show that

$$\mathrm{Im}(\theta_1) = \mathbf{Z} \oplus 0 \quad \text{if } n \equiv -1 \pmod 4.$$

The containment $\supseteq$ holds, by taking interior connected sum of the identity map on $\mathbf{RP}^{4\ell-1} \times \Delta^1$ with the Milnor manifold map $M^{4\ell} \to S^{4\ell}$. Conversely, the containment $\subseteq$ holds, since θ_1 vanishes on the "lower" summands of $\mathcal{N}_{\mathrm{TOP}}(\mathbf{RP}^{4\ell-1} \times (\Delta^1, \partial\Delta^1))$, by Proposition 5.2.2. □

5.2.3. Structure group.

Remark 5.2.6. For any closed TOP n-manifold X with fundamental group π and orientation character ω, the Quinn-Ranicki algebraic surgery sequence [**Ran92a**, Dfn. 14.6] is an exact sequence of abelian groups equivalent to the **classical surgery sequence** of based sets ($n > 3$):

$$\cdots \xrightarrow{N} \mathcal{N}_{\mathrm{TOP}}(X \times (\Delta^1, \partial\Delta^1)) \xrightarrow{\theta_1} L^s_{n+1}(\pi, \omega)$$
$$\xrightarrow{\partial} \mathcal{S}^s_{\mathrm{TOP}}(X) \xrightarrow{N} \mathcal{N}_{\mathrm{TOP}}(X) \xrightarrow{\theta_0} L^s_n(\pi, \omega).$$

Here if $n = 4$ then we assume π is "good" in the sense of [**FQ90**], such as our main concern $\pi = \mathbf{C}_2$.

REMARK 5.2.7. The **simple structure set**

$$\mathcal{S}^s_{\mathrm{TOP}}(X)$$

is the set of s-bordism classes of simple homotopy equivalences $g : M \to X$ for some closed TOP n-manifold M. By the s-cobordism theorem (cf. [**FQ90**]), this equivalence relation is the same as pre-composition with any homeomorphism making the appropriate triangle homotopy-commute. As stated above, the set $\mathcal{S}^s_{\mathrm{TOP}}(X)$ can be given the structure of an abelian group such that the surgery exact sequence consists of homomorphisms.

PROPOSITION 5.2.8. *The structure group fits into a short exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{Cok}(\theta_1) \xrightarrow{\ \partial\ } \mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^n) \xrightarrow{\ N\ } \mathrm{Ker}(\theta_0) \longrightarrow 0.$$

PROOF. Immediate from the existence of the algebraic surgery sequence. $\square$

PROPOSITION 5.2.9. *The above sequence splits for all $n > 3$.*

PROOF. By Proposition 5.2.5 we may assume $n = 4\ell + 3$ for some $\ell > 0$, hence $\mathrm{Cok}(\theta_1) = \mathbf{Z}$ is detected by $\widetilde{\sigma}/8$. Define a set map

$$BL : \mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^{4\ell+3}) \longrightarrow \mathbf{Z},$$

the **Browder-Livesay desuspension invariant**, as the composite of the following three maps:

(1) the obstruction to splitting a homotopy equivalence g along the one-sided submanifold $\mathbf{RP}^{4\ell+2}$:

$$\mathrm{split} : \mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^{4\ell+3}) \longrightarrow LN_{4\ell+2}(1 \to \mathbf{C}_2^+),$$

(2) the antiquadratic kernel isomorphism

$$\mathrm{aqk} : LN_{4\ell+2}(1 \to \mathbf{C}_2^+) \longrightarrow L_{4\ell+4}(\mathbf{Z}[1]),$$

obtained from consideration of the lift of the transverse inverse image of $\mathbf{RP}^{4\ell+2}$ to a $\mathbf{C}_2$-equivariant degree one normal map with target the universal cover $S^{4\ell+2}$, and

(3) the signature isomorphism

$$\sigma/8 : L_{4\ell+4}(\mathbf{Z}[1]) \longrightarrow \mathbf{Z}.$$

The set map BL is in fact a homomorphism, since

$$\text{split} : \mathcal{S}^s_{\text{TOP}}(X^n) = \mathbf{S}_{n+1}(X) \longrightarrow LS_{n-q}(\Phi)$$

is a homomorphism for all codimensions q by [**Ran92a**, Prop. 23.2, 23.3].

It remains to show that

$$BL \circ \partial = \mathbf{1} : \mathbf{Z} \longrightarrow \mathbf{Z}.$$

By Proposition 5.2.8 and the TOP version of [**LdM71**, Theorem V.2.1][2], the composite $BL \circ \partial$ is surjective[3], hence equals $\pm\mathbf{1}$. But the sign is in fact $+$ by the TOP version of [**LdM71**, Theorem IV.4.1], which relates BL to $\widetilde{\sigma}/8$. □

THEOREM 5.2.10. *Let $n > 3$ and write $k := [n/2]$. As abelian groups,*

$$\mathcal{S}_{\text{TOP}}(\mathbf{RP}^n) \cong \begin{cases} (k-1)\mathbf{Z}_2 & \text{if } n \equiv 0,2 \pmod 4 \\ k\mathbf{Z}_2 & \text{if } n \equiv +1 \pmod 4 \\ \mathbf{Z} \oplus (k-1)\mathbf{Z}_2 & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

For each homotopy equivalence $g : M \to \mathbf{RP}^n$, it is detected in lower dimensions by the Kervaire-Arf invariant $c\left(g|_{g^{-1}(\mathbf{RP}^{2i})}\right)$ for all $1 \le i < k$. The structure group is detected near the top dimension by

$$\begin{cases} \text{the Kervaire-Arf invariant } c\left(g\big|_{g^{-1}(\mathbf{RP}^{n-1})}\right) & \text{if } n \equiv +1 \pmod 4 \\ \text{the Browder-Livesay invariant } BL(g) & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

PROOF. Immediate from Propositions 5.2.8, 5.2.9, 5.2.5. □

[2]Regard this as the one-sided precursor to Cappell's nilpotent normal cobordism construction.

[3]See [**Wal99**, Lemma 12.10] for realization of the one-sided splitting invariants.

5.2.4. Homotopy automorphisms. The group

$$\mathrm{hAut}(X)$$

of **homotopy automorphisms** of a topological space X is defined as the set of homotopy classes of all homotopy equivalences $X \to X$ under composition of maps.

PROPOSITION 5.2.11. *The group* $\mathrm{hAut}(\mathbf{RP}^n)$ *is isomorphic to* $\mathbf{Z}^\times$*, generated by reflection about* $\mathbf{RP}^{n-1}$*.*

PROOF. Since the reflection ρ of S^n about S^{n-1} commutes with the antipodal map of S^n, it induces a self-homeomorphism $\overline{\rho}$ of $\mathbf{RP}^n$ such that the following diagram commutes:

$$\begin{tikzcd} S^n \arrow[r, "\rho"] \arrow[d, "\pi"] & S^n \arrow[d, "\pi"] \\ \mathbf{RP}^n \arrow[r, "\overline{\rho}"] & \mathbf{RP}^n \end{tikzcd}$$

where $\pi : S^n \to \mathbf{RP}^n$ is the quotient map. Since $\rho \circ \rho = \mathbf{1}_{S^n}$, we have $\overline{\rho} \circ \overline{\rho} = \mathbf{1}_{\mathbf{RP}^n}$. Then define a homomorphism

$$\varphi : \mathbf{Z}^\times \longrightarrow \mathrm{hAut}(\mathbf{RP}^n); \quad -1 \longmapsto \overline{\rho}.$$

Let $\omega : \pi_1(\mathbf{RP}^n) \to \mathbf{C}_2$ be the orientation character of $\mathbf{RP}^n$: if n is even, then ω is an isomorphism, and if n is odd, then ω is the trivial map. By computation with the $\mathbf{C}_2$-equivariant cellular chain complex of S^n, for all n, observe that

$$H_n(\mathbf{RP}^n; \mathbf{Z}^\omega) \cong \mathbf{Z}.$$

Here, twice the generator $[\mathbf{RP}^n]$ is the fundamental class $\pi_*[S^n]$. Then define a homomorphism

$$\psi : \mathrm{hAut}(\mathbf{RP}^n) \longrightarrow \mathbf{Z}^\times; \quad [h] \longmapsto h_*[\mathbf{RP}^n].$$

Since reflection through a hyperplane reverses orientation of S^n for all n, we have $\rho_*[S^n] = -[S^n]$. Hence $\overline{\rho}_*[\mathbf{RP}^n] = -[\mathbf{RP}^n]$. Thus $\psi \circ \varphi = \mathbf{1}_{\mathbf{Z}^\times}$ and ψ is surjective.

Suppose $[h] \in \mathrm{hAut}(\mathbf{RP}^n)$ satisfies $h_*[\mathbf{RP}^n] = [\mathbf{RP}^n]$. We may assume that h is a cellular map. Then the only two obstructions to constructing a homotopy between

h and $\mathbf{1}_{\mathbf{RP}^n}$ are:

$$\begin{aligned} h_\# - \mathbf{1}_\# &\in \operatorname{End}(\pi_1(\mathbf{RP}^n)) \\ h_* - \mathbf{1}_* &\in \operatorname{End}(H_n(\mathbf{RP}^n; \mathbf{Z}^\omega)). \end{aligned}$$

But these difference classes vanish by hypothesis. So $[h] = [\mathbf{1}_{\mathbf{RP}^n}] \in \mathrm{hAut}(\mathbf{RP}^n)$. Thus ψ is injective and $\varphi \circ \psi = \mathbf{1}_{\mathrm{hAut}(\mathbf{RP}^n)}$. Therefore φ and ψ are inverses. $\square$

THEOREM 5.2.12. *Let $n > 3$. Consider the set $\mathcal{H}$ of homeomorphism classes of closed* TOP *n-manifolds in the homotopy type of $\mathbf{RP}^n$. There exist a bijection*

$$\mathcal{H} \longleftrightarrow \begin{cases} \mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^n) & \text{if } n \not\equiv -1 \pmod 4 \\ \mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^{n-1}) \times \mathbf{Z}_{\geq 0} & \text{if } n \equiv -1 \pmod 4. \end{cases}$$

The former is a finite abelian group, computed in Theorem 5.2.10. The latter is a finite abelian monoid, where the codimension one Browder-Livesay invariant modulo orientation takes values in $\mathbf{Z}_{\geq 0} = \mathbf{Z}/\mathbf{Z}^\times$.

PROOF. By selection of a comparison homotopy equivalence, the desired set is in bijection to the quotient of the finite abelian group $\mathcal{S}_{\mathrm{TOP}}(\mathbf{RP}^n)$ by the group $\mathrm{hAut}(\mathbf{RP}^n)$. The latter group is $\mathbf{Z}^\times$, generated by reflection through $\mathbf{RP}^{n-1}$, by Proposition 5.2.11. Since $\mathbf{RP}^{n-1}$ is fixed, all invariants in Theorem 5.2.10 are fixed except possibly the top one. Under the flip of orientation in S^n induced by the reflection, observe that the Arf invariant is fixed but the codimension one Browder-Livesay invariant is flipped. $\square$

REMARK 5.2.13. Every fake $\mathbf{RP}^n$ is a homotopy $\mathbf{RP}^n$ for all $n > 1$, by [**Wal99**, Theorem 14E.1].

5.3. The connected sum of real projective spaces, $\mathbf{RP}^n \# \mathbf{RP}^n$

Consider the quotient map f in the definition of real projective space:

$$\mathbf{C}_2 \longrightarrow S^{n-1} \xrightarrow{f} \mathbf{RP}^{n-1}.$$

Then observe that the connected sum

$$X := \mathbf{RP}^n \# \mathbf{RP}^n$$

of real projective spaces admits a decomposition

$$X = S^n \times I \bigcup_{f \times \mathbf{1}_{\partial I}} \mathbf{RP}^{n-1} \times \partial I.$$

REMARK 5.3.1. The analogous result to Theorem 5.2.12 is the TOP classification of closed manifolds in the simple homotopy type of X. This is the main result of the paper of Brookman-Davis-Khan [**BDK**] (compare with Jahren-Kwasik [**JK06**] for $n = 4$). The analysis involves the study of the complicated action of $\mathrm{hAut}(X)$ on the structure set $\mathcal{S}_{\mathrm{TOP}}(X)$, in particular the action of the switch automorphism, interchanging the two factors of $\mathbf{C}_2 * \mathbf{C}_2$, on the surgery group $L_*(\mathbf{Z}[\mathbf{C}_2 * \mathbf{C}_2])$.

The following list is a classification of covers of the manifold $X = \mathbf{RP}^n \# \mathbf{RP}^n$; see Figure 5.3.1.

THEOREM 5.3.2.

(1) *The* ***universal cover***

$$\mathbf{D}_\infty \longrightarrow S^{n-1} \times \mathbf{R} \longrightarrow X$$

is defined by the proper action

$$(p, r) \cdot t^i a^\epsilon := ((-1)^\epsilon p, (-1)^\epsilon (r + i)) \quad \textit{for all } i \in \mathbf{Z} \textit{ and } \epsilon \in \{0, 1\}.$$

(2) *For $1 \leq j$, the regular covers*

$$\mathbf{D}_j \longrightarrow S^{n-1} \times S^1 \longrightarrow X$$

are given by the induced free action

$$(p, q) \cdot t^i a^\epsilon := \left((-1)^\epsilon p, e^{-2\pi\sqrt{-1}\, i/j} \cdot \begin{cases} q & \textit{if } \epsilon = 0 \\ \overline{q} & \textit{if } \epsilon = 1 \end{cases} \right).$$

In particular, if n is even then the ***orientation cover***

$$\mathbf{C}_2 \longrightarrow S^{n-1} \times S^1 \longrightarrow X$$

is given by the free involution

$$(p, q) \cdot a := (-p, \overline{q}),$$

which reverses orientation. If n is odd, then X is already orientable.

(3) *The two regular self-covers*

$$\mathbf{C}_2 \longrightarrow X \longrightarrow X$$

are obtained as the orbit spaces of $S^{n-1} \times \mathbf{R}$ *by the subgroups*

$$\left\langle a, t^2 \right\rangle \quad \textit{and} \quad \left\langle b, t^2 \right\rangle .$$

(4) *For* $2 < j$ *odd, the irregular self-cover (with trivial covering group)*

$$j \longrightarrow X \longrightarrow X$$

is obtained as the orbit space of $S^{n-1} \times \mathbf{R}$ *by the subgroup*

$$\left\langle a, t^j \right\rangle .$$

(5) *For* $2 < j$ *even, the two irregular self-covers (with covering group* $\mathbf{C}_2$*)*

$$j \longrightarrow X \longrightarrow X$$

are obtained as the orbit spaces of $S^{n-1} \times \mathbf{R}$ *by the subgroups*

$$\left\langle a, t^j \right\rangle \quad \textit{and} \quad \left\langle ta, t^j \right\rangle .$$

(6) *The two irregular, one-ended covers (with trivial covering group)*

$$\infty \longrightarrow X \setminus \mathrm{pt} \longrightarrow X$$

are obtained as the orbit spaces of $S^{n-1} \times \mathbf{R}$ *by the subgroups*

$$\langle a \rangle \quad \textit{and} \quad \langle b \rangle .$$

Proof of 5.3.2. Part (1) is immediate from the above description of X. Then Parts (2)–(6) follow from the conjugacy classification of subgroups of $\pi_1(X) = \mathbf{D}_\infty$ in Proposition 5.1.1. $\square$

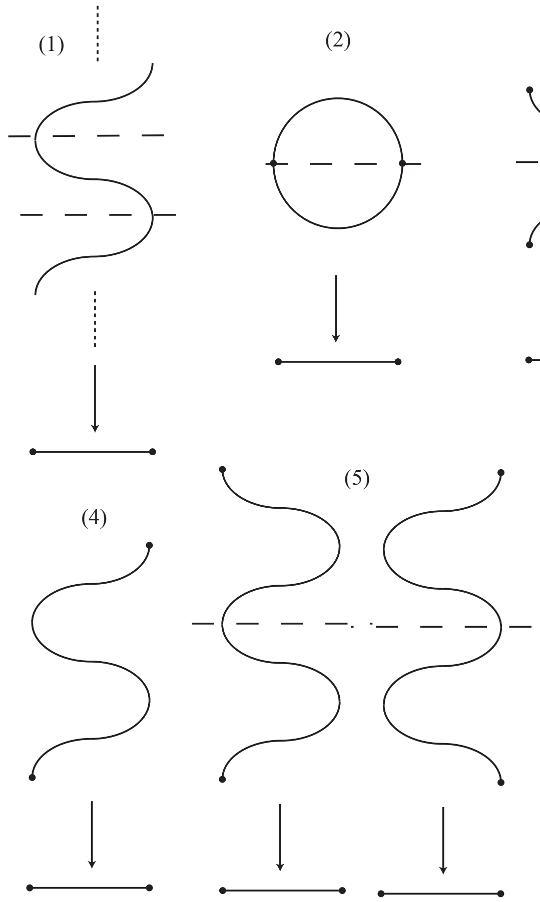

FIGURE 5.3.1. Covers of $X = \mathbf{RP}^n \# \mathbf{RP}^n$ by folding.

CHAPTER 6

Splitting homotopy equivalences in finite covers

Definition. A group G is **self-similar of index** k if there exists a subgroup of index k which is isomorphic to G.

We are motivated by Parts (4) and (5) of Proposition 5.1.1 and Theorem 5.3.2. For an earlier study of this concept, see [**Far79**, §4, Introduction].

Tom Farrell showed that

$$L_3^h(\mathbf{D}_\infty^{+,+}) = L_3^h(\mathbf{C}_2^+ * \mathbf{C}_2^+)$$

is either $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or not finitely generated [**Far79**, Thm. 4.1]. Also, he proved that each element, under a sufficiently large self-similar transfer [**Far79**, Lem. 4.2], has zero component in Cappell's summand

$$\mathrm{UNil}_3^h(\mathbf{Z}; \mathbf{Z}^+, \mathbf{Z}^+) \cong \mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-).$$

His method was a reduction to a topological result of Browder on splitting a proper homotopy equivalence to $M \times \mathbf{R}$, where M is a certain simply-connected closed $(4k+1)$-manifold.

Bjørn Jahren and Sławomir Kwasik later conclude [**JK06**, Thm. 1, Rmk. 1], using Farrell's result, that the infinitely many non-homeomorphic non-split topological 4-manifolds in the tangential homotopy type of $\mathbf{RP}^4 \# \mathbf{RP}^4$ are, in fact, finitely covered by $\mathbf{RP}^4 \# \mathbf{RP}^4$ itself. These manifolds are obtained from the Freedman-Quinn surgery sequence (compare §8.2).

Shmuel Weinberger suggested that there should be a proof in terms of controlled topology– I am grateful to Jim Davis for communicating and outlining this alternative method of proof to me. This method has the advantage of generalizing Farrell's result to $L_n^h(\mathbf{D}_\infty^\omega)$ for all n and orientation characters $\omega : \mathbf{D}_\infty \to \mathbf{C}_2$.

Consider the Bass-Heller-Swan fundamental theorem of algebraic K-theory (see [**Bas68**, §XII.7]). Let

$$\eta \subset \mathrm{Wh}(\pi \times \mathbf{C}_\infty)$$

be the subgroup consisting of all the elements that vanish under transfer to the self-similar subgroup $\pi \times k\mathbf{C}_\infty$ for some $k > 0$. It can be shown that η is exactly the internal direct sum of the two copies of the Bass group $\widetilde{\mathrm{Nil}}_0(\mathbf{Z}[\pi])$ and the torsion subgroup of the Whitehead group $\mathrm{Wh}(\pi)$.

Of course, there is an analogous proof of this vanishing theorem for $\mathrm{Wh}(\pi \times \mathbf{C}_\infty)$ using controlled K-theory. But there is also a purely algebraic approach involving only elementary row and column operations. In Theorem 6.2.3, we provide a similar elementary operation proof for the self-similar transfers in the L-theory of the infinite dihedral group $\mathbf{D}_\infty^\omega$. This theorem is a quantitative, algebraic alternative to Theorem 6.1.1 because it does not require stabilization in the Witt group, and the required degree is straightforward to compute from a matrix representative.

6.1. Existence of the required cover

Recall the notation of §1.2 and Proposition 1.3.8, the latter of which is stated in the language of controlled topology.

THEOREM 6.1.1 (Weinberger). *Let $n \in \mathbf{Z}$ and $\epsilon := (-1)^n$. Consider the infinite dihedral group*

$$G^\omega = \mathbf{C}_2^\epsilon * \mathbf{C}_2^\epsilon$$

with the given orientation character ω. Then for any element

$$\vartheta \in L_n^h(G^\omega),$$

there exists $K > 0$ such that for every $k \geq K$ the geometric self-similar transfer $\iota_k^!(\vartheta)$ lies in the summand

$$L_n^h([0,1];p).$$

That is, it gains infinitesimal control. Equivalently, for any

$$\vartheta \in \mathrm{UNil}_n^h(\Phi^\omega),$$

there exists $K > 0$ such that $k \geq K$ implies

$$\mathrm{split}_L(\iota_k^!(\vartheta)) = 0.$$

In particular, for all $m > 3$, any homotopy equivalence of manifolds

$$h : W^{m+1} \longrightarrow \mathbf{RP}^{m+1} \#_\Sigma \mathbf{RP}^{m+1}$$

is splittable along the homotopy sphere Σ^m in sufficiently large self-similar covers.

REMARK 6.1.2. Consider the lower dimensional cases. If $m = 4$, then Cappell's 5-dimensional splitting theorem is applicable (see Chapter 7). If $m = 3$ and $\Sigma = S^3$ and h is tangential, then Jahren and Kwasik [**JK06**, Thm. 1, Rmk. 1] have a stronger conclusion (compare Chapter 8).

REMARK 6.1.3. The above controlled L-groups have been calculated as follows. Observe that

$$L_n([0,1]; p) \cong H_n([0,1]; \mathbf{L}.(p))$$

fits into a Mayer-Vietoris sequence (§1.3), consisting of the classical L-groups of the trivial group 1 and the order two group $\mathbf{C}_2^\epsilon$ (see Table 5.2.4). Implicitly using this result, Farrell showed [**Far79**, §4] for $\epsilon = +1$ that

$$L_3([0,1]; p) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and is detected by codimension one Arf invariants at both endpoints $\{0, 1\}$, since the group 1 is a retract of $\mathbf{C}_2^+$ implies that

$$L_2(1) \longrightarrow L_2(\mathbf{C}_2^+) \oplus L_2(\mathbf{C}_2^+)$$

is a split monomorphism (see Remark 5.2.4).

More generally, for $\epsilon = +1$ note that $L_n([0,1]; p)$ is isomorphic to the canonical complementary summand of $L_n(1)$ in $L_n(\mathbf{C}_2^+) \oplus L_n(\mathbf{C}_2^+)$. Hence

$$L_2([0,1]; p) \cong \mathbf{Z}_2$$

and is detected by the codimension zero Arf invariant at either endpoint 0 or 1, and

$$L_1([0,1]; p) = 0.$$

Also

$$L_0([0,1];p) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$$

and is detected by the signature on the universal cover at both endpoints $\{0,1\}$, and by the signature at either endpoint 0 or 1.

On the other hand, for $\epsilon = -1$ note [**Wal99**, §13A] that

$$L_3([0,1];p) = 0,$$

and

$$L_2([0,1];p) \cong \mathbf{Z}_2$$

and is detected by codimension zero Arf invariant at either endpoint 0 or 1. Also

$$L_1([0,1];p) \cong \mathbf{Z}$$

and is detected by codimension one signature at the midpoint 1/2, and

$$L_0([0,1];p) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and is detected by the codimension zero Arf invariant at both endpoints $\{0,1\}$. $\square$

PROOF OF THEOREM 6.1.1. Recall the notation of §5.1.2, where for each $k > 0$ there is defined (cf. [**Far79**, §4, Intro.]) a group monomorphism

$$\iota_k : \mathbf{D}_\infty \longrightarrow \mathbf{D}_\infty; \qquad t, a \longmapsto t^k, a.$$

It is an extension of the self-covering of the circle, mentioned above for K-theory. In correspondence with passage to the k-fold irregular self-cover $\iota_k : E \to E$, the geometric transfer map

$$\iota_k^! : L_n^B(G^\omega) \longrightarrow L_n^B(G^\omega)$$

induces a homomorphism

$$\mathrm{split}_L \circ \iota_k^! \circ \mathrm{incl} : \mathrm{UNil}_n^h(\Phi^\omega) \longrightarrow \mathrm{UNil}_n^h(\Phi^\omega).$$

This transfer map leaves the summand $L_n([0,1];p)$ invariant, since infinitesimal control is preserved.

Let (C, ψ) be the geometric n-dimensional Poincaré quadratic complex[1] representing ϑ, as defined in [**Yam87**, Proof 3.5]. In particular, recall that C consists of geometric based $\mathbf{Z}$-modules in the space

$$E = EG \times_G \mathbf{R} \simeq BG \simeq \mathbf{RP}^\infty \# \mathbf{RP}^\infty,$$

and that each morphism, of the chain complex C and the quadratic structure ψ, is a matrix of loops which represent the elements of G.

Our goal is to gain infinitesimal control over the radius of $\iota_k^!(C, \psi)$ for sufficiently large k.

The Squeezing Lemma of Connolly-Davis [**CD**, Lems. 4, 6], which is a sharper version of the Shrinking Lemma of Yamasaki [**Yam87**, Lem. 3.10], gives the following conclusion. There exists $0 < \varepsilon \leq 1$ such that: if $\iota_k^!(C, \psi)$ has radius $< \varepsilon$ in $X = [0, 1]$ then the cobordism class

$$[\iota_k^!(C, \psi)]$$

lies in the image of the infinitesimally controlled classes:

$$\text{forgetcontrol} : L_n([0, 1]; p) \longrightarrow L_n(E).$$

Consider the index two subgroup generated by $t = ab$:

$$H = \mathbf{C}_\infty \subset G = \mathbf{D}_\infty.$$

Then there is a right coset decomposition

$$G = H \sqcup Ha.$$

Define loops

$$\tau, \alpha : [0, 1] \longrightarrow E$$

representing the generators t, a as follows. Their lifts to the universal cover

$$\widetilde{E} = EG = S^\infty \times \mathbf{R}$$

[1] The control space $X = [0, 1]$ technically should be $[0, 1/2]$, but for convenience we re-scale its metric by a factor of two.

are given by the paths

$$\widetilde{\tau}(\theta) := \begin{cases} \exp(4\theta) \times 0 & \text{if } \theta \in \left[0, \frac{1}{4}\right] \\ \exp(1) \times (2\theta - \frac{1}{2}) & \text{if } \theta \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \exp(3 - 4\theta) \times \frac{1}{2} & \text{if } \theta \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ \exp(0) \times (2\theta - 1) & \text{if } \theta \in \left[\frac{3}{4}, 1\right], \end{cases} \quad \text{and} \quad \widetilde{\alpha}(\theta) := \exp(\theta) \times 0.$$

where the exponential is defined by

$$\exp : [0,1] \longrightarrow S^1 \subset S^\infty; \qquad \theta \longmapsto e^{i\pi\theta} \in \mathbf{R}^2.$$

Consider the control map $p : E \to X$. It equals the quotient of the G-invariant composite map

$$\widetilde{E} \longrightarrow \mathbf{R} \longrightarrow S^1 \longrightarrow X,$$

where the last map folds the circle S^1 onto the upper semi-circle $\exp(\frac{1}{2}X)$.

Note, for all $\epsilon \in \{0, 1\}$ and $m \in \mathbf{Z}$, by measuring a path in X via post-composition with p, that the diameter of the image is

$$\operatorname{diam}_X(\tau^m \alpha^\epsilon) = \begin{cases} 1 & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases}$$

Select $K > 0$ such that $1/K < \varepsilon$. Also select $N > 0$ such that every matrix element $\tau^m \alpha^\epsilon$ of (C, ψ) satisfies $|m| \leq N$. Now let $k \geq KN$. Note that each path

$$\iota_k^!(\tau^m \alpha^\epsilon) : [0,1] \longrightarrow E$$

satisfies that its lift to the universal cover $\widetilde{E}$ has projection to $\mathbf{R}$ being the closed interval from 0 to m/k of diameter $|m|/k < \varepsilon$. Therefore the radius in X of the transfer $\iota_k^!(C, \psi)$ is $< \varepsilon$, so we are done. □

6.2. Degree of the required cover

As mentioned in the introduction, we compute an upper bound on the degree of the self-similar cover required for splitting a homotopy equivalence to $\mathbf{RP}^n \# \mathbf{RP}^n$ ($n > 4$). Only basic algebraic definitions are involved, but the number of the elementary row and column operations in the proof is substantial.

6.2.1. Notation for Laurent transfers (see [HRT87, §5] for definitions). Let R be a ring with involution, and let $\tau \in \mathbf{C}_2$. The ring of Laurent polynomials

$$R[t,t^{-1}]^{\tau} = \bigoplus_{i\in\mathbf{Z}} Rt^i$$

has an involution $^-$ defined over sums by

$$\overline{rt^i} := \tau^i \bar{r} t^i \quad \text{for all} \quad r \in R \text{ and } i \in \mathbf{Z}.$$

Topologically, $R[t,t^{-1}]^{\tau}$ corresponds to the group ring $R[\mathbf{C}_\infty]$ with the involution $(t \mapsto t^{-1})$, but with a certain twisting; see Theorem 6.2.1 and Corollary 6.2.2.

Let $k > 0$ be an integer, and write

$$\upsilon := \tau^k \in \mathbf{C}_2.$$

Define a morphism ι_k of rings with involution by

$$\iota_k : R[u,u^{-1}]^{\upsilon} \longrightarrow R[t,t^{-1}]^{\tau}; \qquad ru \longmapsto rt^k.$$

Define an ι_k-trace X by

$$X : R[t,t^{-1}]^{\tau} \longrightarrow R[u,u^{-1}]^{\upsilon}; \qquad rt^i \longmapsto \begin{cases} ru^{i/k} & \text{if } k|i \\ 0 & \text{else.} \end{cases}$$

Topologically, X corresponds to projection onto the trivial coset of the index k subgroup $k\mathbf{C}_\infty$ in the infinite cyclic group $\mathbf{C}_\infty$.

Let $\kappa = s, h, p$ be a decoration. Denote ι_k^X as the transfer homomorphism induced by (the coform induced by the form induced by) the ι_k-trace X:

$$\iota_k^X : L_n^\kappa(R[t,t^{-1}]^{\tau}) \longrightarrow L_n^\kappa(R[u,u^{-1}]^{\upsilon}).$$

Topologically, ι_k^X corresponds to lifting the (simple, free, projective) surgery obstruction of a degree one normal map to the k-fold self-similar cover.

By abuse of notation, we shall call by the same names the extension $(a \mapsto a)$ of the maps ι_k and X from $\mathbf{C}_\infty$ to the infinite dihedral group with orientation character:

$$\mathbf{D}_\infty = \left\langle t, a \mid ata^{-1} = t^{-1}, a^2 = 1 \right\rangle \quad \text{with} \quad \omega : \mathbf{D}_\infty \longrightarrow \mathbf{C}_2.$$

Suppose $\omega(t) = \tau$, and define

$$\nu(u) := \omega(t^k) = \upsilon.$$

Then there are extensions

$$\iota_k : R[\mathbf{D}_\infty^\nu] \longrightarrow R[\mathbf{D}_\infty^\omega] \quad \text{and} \quad X : R[\mathbf{D}_\infty^\omega] \longrightarrow R[\mathbf{D}_\infty^\nu].$$

Therefore we obtain an extension of the transfer homomorphism:

$$\iota_k^X : L_n^\kappa(R[\mathbf{D}_\infty^\omega]) \longrightarrow L_n^\kappa(R[\mathbf{D}_\infty^\nu]).$$

6.2.2. Statement of results. The following algebraic theorems and topological corollaries concern nilpotence of elements under the self-similar transfers ι_k^X.

THEOREM 6.2.1. *Let $\kappa = h, p$ be a decoration, and let $\tau \in \mathbf{C}_2$. Select an element*

$$\vartheta \in \operatorname{Im}\left(L_n^\kappa(R[t]^\tau) \oplus L_n^\kappa(R[t^{-1}]^\tau) \longrightarrow L_n^\kappa(R[t,t^{-1}]^\tau)\right).$$

Then there exists even $k > 0$ such that the self-similar transfer vanishes:

$$\iota_k^X(\vartheta) = 0$$

if and only if the element has null-augmentation:

$$\operatorname{eval}_0(\vartheta) = 0.$$

Moreover, on the level of matrix representatives of ϑ, the vanishing of transfers can be achieved without stabilization in the L-group.

COROLLARY 6.2.2. *Let $(Y, \partial Y)$ be a connected, finite Poincaré pair of dimension $m > 5$. Suppose that the fundamental group of Y is a product of the form*

$$\pi_1(Y) = \pi \times \mathbf{D}_\infty$$

and that its orientation character $\omega : \pi_1(Y) \to \mathbf{C}_2$ satisfies[2]

$$\omega(\pi \times 1) = 1 \quad \textit{and} \quad \omega(1 \times t) = 1.$$

Write

$$R = \mathbf{Z}[\pi] \quad \textit{and} \quad \epsilon = \omega(a).$$

Let $(X, \partial X)$ be a properly embedded, connected, one-sided Poincaré subpair of $(Y, \partial Y)$. Suppose W is a compact m-manifold and

$$g : (W, \partial W) \longrightarrow (Y, \partial Y)$$

[2]These orientation hypotheses are only assumed for simplicity of the conclusion.

is a homotopy equivalence that restricts to a homeomorphism on the boundary.

Then g is splittable along X in some self-similar $(\pi \times k\mathbf{D}_\infty)$-cover of X, if and only if the anti-quadratic signature[3]

$$BL(g) \in LN^h_{m-1}(R[\mathbf{C}_\infty] \to R[\mathbf{D}_\infty]) \xrightarrow{\mathrm{aqk}\cong} L^h_{m+\epsilon}(R[t,t^{-1}])$$

has zero component in $L^p_{m+\epsilon}(R)$ and has null-augmentation in $L^h_{m+\epsilon}(R)$. □

THEOREM 6.2.3. *Let $\kappa = h, p$ be a decoration, and let $v \in \mathbf{C}_2$. Select an element*

$$\vartheta \in \mathrm{Im}\left(\mathrm{UNil}^\kappa_n(R; R^v, R^v) \longrightarrow L^\kappa_n(R[\mathbf{D}^{v,v}_\infty])\right).$$

Then there exists odd $k > 0$ such that the self-similar transfer vanishes:

$$\iota^X_k(\vartheta) = 0.$$

Moreover, on the level of matrix representatives of ϑ, the vanishing of transfers can be achieved without stabilization in the L-group.

COROLLARY 6.2.4. *Let $(Y, \partial Y)$ be a connected, finitely dominated Poincaré pair of dimension $m > 5$. Let $(Z, \partial Z)$ be a properly embedded, connected, separating Poincaré subpair, and denote*

$$\pi := \pi_1(Z).$$

Suppose that the fundamental group of Y is a product of the form

$$\pi_1(Y) = \pi \times \mathbf{D}_\infty$$

and that its orientation character $\omega : \pi_1(Y) \to \mathbf{C}_2$ satisfies

$$\omega(1 \times t) = 1.$$

Suppose W is a compact m-manifold and

$$g : (W, \partial W) \longrightarrow (Y, \partial Y)$$

is a homotopy equivalence that restricts to a homeomorphism on the boundary.

Then g is splittable along Z in some odd self-similar $(\pi \times k\mathbf{D}_\infty)$-cover of Y. □

[3] See Proof 5.2.9 for the Browder-Livesay invariant in the simply-connected case.

6.2.3. Proofs.

PROOF OF THEOREM 6.2.1. It suffices, by the Ranicki-Shaneson exact sequence in L-theory, to prove the theorem for all n even and decorations $\kappa = s, h, p$.

Let $\epsilon := (-1)^{n/2}$. Abbreviate

$$\mathcal{A} := R[t, t^{-1}].$$

Represent $\vartheta \in L_n^\kappa(\mathcal{A}^\tau)$ by a (simple) nonsingular ϵ-quadratic form (C, Θ). By Higman linearization (2.1.10) and hypothesis on ϑ, we may assume

$$\Theta = \theta_0 + (t-1)\theta_1 \quad \text{and} \quad C = P \oplus P^*,$$

where each θ_i is induced from R and $P = \mathcal{A} \otimes_R P_0$ for some finitely generated projective R-module P_0. Define R-module morphisms

$$\theta := \theta_0 - \theta_1 \quad \text{and} \quad \lambda_i := \theta_i + \epsilon\theta_i^* \quad \text{and} \quad \lambda := \theta + \epsilon\theta^* = \lambda_0 - \lambda_1.$$

Since the following map Λ is an isomorphism of A-modules:

$$\Lambda := \Theta + \epsilon\Theta^* : C \longrightarrow \mathrm{Hom}_A(C, A),$$

by Lemma 2.1.13, the R-module morphism

$$\nu := \lambda_0^{-1}\lambda_1$$

is nilpotent, say, of degree $r > 0$. Then, since $\mathrm{eval}_0(\Lambda) = \lambda$ is isomorphism of R-modules, we must have that the following map η is also nilpotent of degree r:

$$\eta := \lambda^{-1}\lambda_1 = (1-\nu)^{-1}\nu.$$

Now we shall define $k := 2r$. It suffices to show that

$$\iota_k^X(\vartheta) = \mathrm{eval}_0(\vartheta).$$

Note that our transfer ι_k^X lands in the subring

$$\mathcal{B} := R[u, u^{-1}],$$

which has trivial involution on u:

$$\upsilon(u) = \tau(t^{2r}) = 1.$$

We shall employ $k \times k$ blocks of morphisms and use $0 \leq i, j < k$ to index rows & columns. Unless otherwise indicated, blocks are 0.

First, we compute some basic transfers. For any Laurent polynomial $f \in \mathcal{A}$, define a left $\mathcal{A}$-module morphism

$$\mathrm{rmult}_f : \mathcal{A} \longrightarrow \mathcal{A}; \qquad a \longmapsto a \cdot f.$$

The transfer of the free module of rank one is indexed by right cosets:

$$\iota_k^!(\mathcal{A}) = \bigoplus_{0 \leq h < k} \mathcal{B} t^h.$$

Then, for $f = 1$ and $f = t$, we calculate left $\mathcal{B}$-module morphisms $\iota_k^!(\mathcal{A}) \to \iota_k^!(\mathcal{A})^*$:

$$\iota_k^X(\mathrm{rmult}_1) = \begin{cases} 1 & \text{if } i + j = 0 \\ u & \text{if } i + j = k \end{cases} \quad \text{and} \quad \iota_k^X(\mathrm{rmult}_t) = \begin{cases} u & \text{if } i + j = k - 1. \end{cases}$$

We remind the reader that $0 \leq i, j < k$ are the indices for these $(k \times k)$-matrices. Therefore, since $\Theta = \theta + t\theta_1$ is a linear combination of left R-module morphisms, we obtain

$$\iota_k^X(\Theta) = \begin{cases} \theta & \text{if } i + j = 0 \\ u\theta & \text{if } i + j = k \\ u\theta_1 & \text{if } i + j = k - 1 \end{cases} \quad \text{and} \quad \iota_k^X(\mathrm{eval}_0(\Theta)) = \begin{cases} \theta & \text{if } i + j = 0 \\ u\theta & \text{if } i + j = k. \end{cases}$$

Next, we set up some short-hand notation, to be used in our formal algebraic manipulations. Recursively define left R-module morphisms

$$A_1 := u(\eta^*\theta - \lambda_1)\eta \quad \text{and} \quad A_{h+1} := \eta^* A_h \eta \quad \text{for all } h \in \mathbf{Z}_{\geq 0}.$$

Note $A_r = 0$.

Let $0 \leq x \neq y < k$ be indices. For any left R-module morphism $\Gamma : C \to C^*$, define an elementary-block-operation isomorphism

$$\delta_{x,y}^{\Gamma} := \begin{cases} 1 & \text{if } i = j \\ \Gamma & \text{if } (i, j) = (x, y) \end{cases}$$

Define $(-\epsilon)$-even morphisms

$$\varepsilon^{\Gamma}_{x,y} := \begin{cases} \Gamma & \text{if } (i,j) = (x,y) \\ -\epsilon\Gamma^* & \text{if } (i,j) = (y,x) \end{cases} \quad \text{and} \quad \varepsilon^{\Gamma}_{x,x} := \begin{cases} \Gamma - \epsilon\Gamma^* & \text{if } (i,j) = (x,x) \end{cases} \cdot$$

Finally, as ϵ-quadratic forms $\iota^!_k(C) \to \iota^!_k(C)^*$ over $\mathcal{B}$, it suffices to show that the transfer $\iota^X_k(\Theta)$ is (simple) homotopy equivalent [**Ran80a**] to the ϵ-quadratic form

$$\Delta_h := \begin{cases} \theta & \text{if } i+j = 0 \\ u\theta & \text{if } i+j = k \\ u\theta_1 & \text{if } i+j = k-1, h \leq i,j \\ uA_h & \text{if } i = k-h = j \end{cases}$$

for each $1 \leq h \leq r$. We shall proceed by induction on h. Observe for $h = r$ that

$$A_r = 0 \quad \text{and} \quad \{\, (i,j) \mid i+j = k-1 \text{ and } h \leq i,j \,\} = \varnothing.$$

Hence, we can conclude that $\iota^X_k(\Theta)$ is (simple) homotopy equivalent to

$$\Delta_r = \iota^X_k(\mathrm{eval}_0(\Theta)).$$

Therefore

$$\iota^X_k(\vartheta) = \iota^X_k(\mathrm{eval}_0(\vartheta))$$

as Witt classes over $\mathcal{B}$, as desired.

Basic Step: Suppose $h = 1$.

Note

$$\left(\delta^{-u\eta}_{0,k-1}\right)^{\%}\left(\iota^X_k\Theta\right) + \left(\varepsilon^{u\eta^*\theta - u\theta_1}_{k-1,0} + \varepsilon^{u^2\eta^*\theta}_{k-1,k-1}\right) = \Delta_1.$$

So the transfer $\iota^X_k(\Theta)$ is (simple) homotopy equivalent to the ϵ-quadratic form Δ_1.

Inductive Step: Suppose that the transfer $\iota^X_k(\Theta)$ is (simple) homotopy equivalent to the ϵ-quadratic form Δ_h for some $1 \leq h < r$.

First, note

$$\left(\delta_{h,k-h}^{-\lambda^{-1*}A_h^*}\right)^{\%}(\Delta_h) + \left(\varepsilon_{k-h,k-h}^{u\theta\lambda^{-1*}A_h^*} + \varepsilon_{k-(h+1),k-h}^{u\theta_1\lambda^{-1*}A_h^*}\right)$$
$$= \Delta_h' := \begin{cases} \theta & \text{if } i+j=0 \\ u\theta & \text{if } i+j=k \\ u\theta_1 & \text{if } i+j=k-1, h \leq i,j \\ -uA_h\eta & \text{if } i=k-h, i-j=1 \end{cases}.$$

So Δ_h is (simple) homotopy equivalent to Δ_h'.

Next, note

$$\left(\delta_{h,k-(h+1)}^{\lambda^{-1}A_n\eta}\right)^{\%}(\Delta_h') + \left(\varepsilon_{k-(h+1),k-h}^{-u\eta^*A_h^*\lambda^{-1*}\theta} + \varepsilon_{k-(h+1),k-(h+1)}^{-u\eta^*A_n^*\lambda^{-1*}\theta_1}\right)$$
$$= \Delta_h'' := \begin{cases} \theta & \text{if } i+j=0 \\ u\theta & \text{if } i+j=k \\ u\theta_1 & \text{if } i+j=k-1, h \leq i,j \\ uA_{h+1}\eta & \text{if } i=k-(h+1)=j \end{cases}.$$

So Δ_h' is (simple) homotopy equivalent to Δ_h''.

Finally, note

$$\left(\delta_{k-h,k-(h+1)}^{-\eta}\right)^{\%}(\Delta_h'') + \left(\varepsilon_{k-(h+1),h}^{u\eta^*\theta-u\theta_1}\right) = \Delta_{h+1}.$$

Therefore Δ_h'', hence the transfer $\iota_k^X(\Theta)$, is (simple) homotopy equivalent to the ϵ-quadratic form Δ_{h+1}. This completes the induction on h. □

Proof of Theorem 6.2.3. It suffices, by Proposition 1.1.9 and the Ranicki-Shaneson exact sequence in L-theory, to prove the theorem for all n even and decorations $\kappa = s, h, p$. Let $\epsilon := (-1)^{n/2}$.

First, we set up some short-hand notation. Define an R-algebra and (R, R)-bimodule[4] with involution

$$\mathcal{C} := R[\mathbf{D}_\infty^{v,v}] \quad \text{and} \quad \mathscr{B} := R[\mathbf{C}_2^v \setminus 1] = a_\pm R^v.$$

[4]For symmetry of notation, we present $\mathbf{D}_\infty = \mathbf{C}_2 * \mathbf{C}_2 = \langle a_-, a_+ \mid (a_-)^2 = 1 = (a_+)^2 \rangle$.

Represent $\vartheta \in L_n^\kappa(\mathcal{C})$ by a (simple) nonsingular ϵ-quadratic unilform over $(R; \mathscr{B}, \mathscr{B})$:

$$(P_\pm, \widehat{\theta}_\pm : P_\pm \to \mathscr{B} \otimes_R P_\mp).$$

Its image is the (simple) nonsingular (± 1)-quadratic form over $\mathcal{C}$:

$$\vartheta = \left(\overline{P}_- \oplus \overline{P}_+, \Theta := \begin{pmatrix} a_-\theta_- & \mathbf{1} \\ 0 & a_+\theta_+ \end{pmatrix} \right).$$

Here, the finitely generated projective R-modules and R-module morphisms

$$P_\pm \quad \text{and} \quad \theta_\pm : P_\pm \longrightarrow P_\mp$$

are extended over $\mathcal{C}$ and $\mathscr{B}$ as

$$\overline{P}_\pm := \mathcal{C} \otimes_R P_\pm \quad \text{and} \quad \widehat{\theta}_\pm(x)(y) := a_\pm \theta_\pm(x)(y) \in \mathscr{B}.$$

Recall that non-singularity of the unilform means there are canonical identifications $P_\mp = P_\pm^*$. The dual morphisms are defined by

$$\theta_\pm^* : P_\pm \to P_\mp; \qquad \theta_\pm^*(x)(y) := \overline{\theta_\pm(y)(x)}$$

$$\widehat{\theta}_\pm^* : P_\pm \to \mathscr{B} \otimes_R P_\mp; \qquad \widehat{\theta}_\pm(x)(y) := \upsilon a_\pm \theta_\pm^*(x)(y).$$

Their ϵ-symmetrizations are defined by

$$\lambda_\pm := \theta_\pm + \epsilon\upsilon\theta_\pm^* : P_\pm \longrightarrow P_\mp$$

$$\widehat{\lambda}_\pm := \widehat{\theta}_\pm + \epsilon\widehat{\theta}_\pm^* : P_\pm \longrightarrow \mathscr{B} \otimes_R P_\mp.$$

Then $\widehat{\lambda}_\pm = a_\pm \lambda_\pm$. Their composites are defined by

$$\lambda := \lambda_+ \circ \lambda_- : P_- \longrightarrow P_-$$

$$\widehat{\lambda} := \widehat{\lambda}_+ \circ \widehat{\lambda}_- : P_- \longrightarrow \mathscr{B} \otimes_R \mathscr{B} \otimes_R P_- \subset \overline{P}_-.$$

Then $\widehat{\lambda} = t^{-1}\lambda$, where

$$t := a_+ a_- \in \mathbf{C}_\infty \subset \mathbf{D}_\infty.$$

By the definition of unilform, there exists $r > 0$ such that $\widehat{\lambda}^{r-1} = 0$ hence $\lambda^{r-1} = 0$.

Now we shall define $k := 2r + 1$. Consider the element

$$u := t^k \in \mathbf{C}_\infty \subset \mathbf{D}_\infty$$

and the subring[5] with involution

$$\mathcal{D} := R[\langle u, a \mid aua^{-1} = u^{-1}, a^2 = 1\rangle]$$

of the ring with involution

$$\mathcal{C} = R[\langle t, a \mid aua^{-1} = t^{-1}, a^2 = 1\rangle].$$

We shall employ $2k \times 2k$ blocks of morphisms and use $0 \leq i^{\pm}, j^{\pm} < k$ to index rows & columns. Unless otherwise indicated, blocks are 0. Furthermore, we re-use the notation for transfers ($\iota_k^!, \iota_k^X$), right multiplication (rmult), and elementary operations (δ, ϵ) from Proof 6.2.3.

First, we compute some basic transfers. The transfer of the free module of rank one is indexed by right cosets:

$$\iota_k^!(\mathcal{C}) = \bigoplus_{0 \leq h < k} \mathcal{D}t^h.$$

Then we calculate left $\mathcal{D}$-module morphisms $\iota_k^!(\mathcal{C}) \to \iota_k^!(\mathcal{C})^*$:

$$\iota_k^X(\mathrm{rmult}_a) = \begin{cases} a & \text{if } i+j=0 \\ ua & \text{if } i+j=k \end{cases} \quad \text{and} \quad \iota_k^X(\mathrm{rmult}_b) = \begin{cases} a & \text{if } i+j=1 \\ ua & \text{if } i+j=k+1. \end{cases}$$

Therefore, since $\Theta = \begin{pmatrix} a\theta_- & \mathbf{1} \\ 0 & b\theta_+ \end{pmatrix}$ is a block matrix of left R-module morphisms, we obtain

$$\iota_k^X(\Theta) = \begin{cases} \mathbf{1} & \text{if } i^+ = j^- \\ a\theta_+ & \text{if } i^+ + j^+ = 0 \\ ua\theta_+ & \text{if } i^+ + j^+ = k \\ a\theta_- & \text{if } i^- + j^- = 1 \\ ua\theta_- & \text{if } i^- + j^- = k+1. \end{cases}$$

Next, we set up some more short-hand notation, to be used in our formal algebraic manipulations. For any left R-module morphisms $\Gamma : \overline{P}_+ \to \overline{P}_-$ and $0 \leq h \leq r$,

[5]For readability of the manipulations, we revert our notation: $a := a_-$ and $b := a_+ = ta$.

we define a nonsingular ϵ-quadratic form Δ_h^Γ over $\mathcal{D}$ by

$$\Delta_h^\Gamma := \begin{cases} \mathbf{1} & \text{if } i^+ = j^- \\ a\theta_- & \text{if } i^- + j^- = 1 \\ ua\theta_- & \text{if } i^- + j^- = k+1 \\ -\upsilon\theta_+^*\lambda_- & \text{if } j^- = i^+ + 1, i = 0 \\ -\epsilon\lambda & \text{if } j^- = i^+ + 1, 0 < i \leq r \\ \Gamma & \text{if } i^+ = r - h = j^+. \end{cases}$$

On the $\mathcal{D}$-module $\iota_k^!(\overline{P}_+ \oplus \overline{P}_-)$, it suffices to show that the transfer $\iota_k^X(\Theta)$ is (simple) homotopy equivalent [**Ran80a**] to the ϵ-quadratic form

$$\Delta_h^{\upsilon ua\lambda^h\lambda_+^*\,\theta_-\,\lambda_+(\lambda^*)^h}$$

for each $0 \leq h < r$. We shall proceed by induction on h. Observe for $h = r-1$ that

$$\Delta_{r-1}^0$$

is a ϵ-hyperbolic form with (simple) lagrangian $\iota_k^!(\overline{P}_+)$. Here $\Gamma = 0$, since $\lambda^{r-1} = 0$ by nilpotence of ϑ. Hence, by pullback of this lagrangian, we can conclude that the transfer $\iota_k^X(\Theta)$ is a (simple) ϵ-hyperbolic form. Therefore

$$\iota_k^X(\vartheta) = 0$$

as a Witt class over $\mathcal{D}$, as desired.

Basic Step: Suppose $h = 0$.

Note

$$\left(\delta_{0^-,0^+}^{-a\theta_+} \circ \prod_{0<j\leq r} \delta_{k-j^-,j^+}^{-ua\lambda_+}\right)^{\%} (\iota_k^X\Theta) + \left(\sum_{0<i\leq r} \varepsilon_{i^+,k-i^+}^{-ua\theta_+} + \varepsilon_{1^-,0^+}^{\theta_-\theta_+} + \sum_{0<j\leq r} \varepsilon_{1+j^-,j^+}^{\theta_-\lambda_+}\right) = \Delta_0^{\upsilon ua\lambda_+^*\theta_-\lambda_+}.$$

So the transfer $\iota_k^X(\Theta)$ is (simple) homotopy equivalent to the ϵ-quadratic form

$$\Delta_0^{\upsilon ua\lambda_+^*\theta_-\lambda_+}.$$

Inductive Step: Suppose that the transfer ι_k^X is (simple) homotopy equivalent to the ϵ-quadratic form

$$\Delta_h^{\upsilon ua\lambda^h\lambda_+^*\theta_-\lambda_+(\lambda^*)^h}$$

for some $0 \leq h < r-1$.

Abbreviate

$$\Gamma := \upsilon ua\lambda^h\lambda_+^*\theta_-\lambda_+\lambda^{*h}.$$

Note the pullback

$$\begin{aligned}&\left(\delta_{r+2+h^+,r-1-h^+}^{-\upsilon ua\lambda_-^*\Gamma^*\lambda^*}\right)^{\%}\\
&\qquad\qquad\Bigg(\left(\delta_{r+2+h^+,r-h^+}^{\upsilon ua\lambda_-^*\Gamma}\circ\delta_{r-h^-,r-1-h^+}^{\Gamma^*\lambda^*}\right)^{\%}\\
&\qquad\left(\left(\delta_{r-h^-,r-h^+}^{-\Gamma}\right)^{\%}(\Delta_h^{\Gamma})+\left(\varepsilon_{r-1-h^+,r-h^+}^{\epsilon\lambda\Gamma}+\varepsilon_{r+2+h^-,r-h^+}^{ua\theta_-\Gamma}\right)\right)\\
&\qquad\qquad+\left(\varepsilon_{r+2+h^-,r-1-h^+}^{-ua\theta_-\Gamma^*\lambda^*}+\varepsilon_{r-1-h^+,r-1-h^+}^{\upsilon ua\lambda^{(h+1)}\lambda_+^*\lambda_-\lambda_+\lambda^{*(h+1)}}\right)\Bigg)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad= \Delta_{h+1}^{\lambda\Gamma\lambda^*}.\end{aligned}$$

So the transfer $\iota_k^X(\Theta)$ is (simple) homotopy equivalent to the ϵ-quadratic form

$$\Delta_{(h+1)}^{\upsilon ua\lambda^{(h+1)}\lambda_+^*\theta_-\lambda_+\lambda^{*(h+1)}}.$$

This completes the induction on h. □

6.3. Kernel and image of self-similar transfers

Our purpose here is to negatively resolve a question of Shmuel Weinberger: Are the kernels in quadratic L-theory of the self-similar transfers of the infinite dihedral group $\mathbf{D}_\infty$ finitely generated $\mathbf{Z}$-modules? There is some overlap with concurrent work of Frank Connolly and Bjørn Jahren.

Let $k > 0$, and recall the notation of §6.2.1. There is a unique extension ι_k of the index k inclusion of maximal infinite cyclic subgroups:

$$\iota_k : \mathbf{D}_\infty \longrightarrow \mathbf{D}_\infty; \qquad u, a \longmapsto t^k, a.$$

For simplicity, we write the corresponding geometric transfer as $\iota_k^! = \iota_k^X$.

In the following lemma, there exists an extension $\iota_k^!$ to the Browder-Livesay LN-groups of the index two subgroup $\mathbf{C}_\infty \subset \mathbf{D}_\infty$. Refer to Proof 5.2.9 for the antiquadratic kernel isomorphism aqk.

LEMMA 6.3.1. *For all $k > 0$ and $n \in \mathbf{Z}$ the following diagram commutes:*

$$\begin{array}{ccccc}
L_n(\mathbf{Z}[t,t^{-1}]) & \xrightarrow{\mathrm{aqk}^{-1}} & LN_n(\mathbf{Z}[\mathbf{C}_\infty^+] \to \mathbf{Z}[\mathbf{D}_\infty^{-,-}]) & \xrightarrow{\partial} & L_n(\mathbf{Z}[\mathbf{D}_\infty^{+,+}]) \\
\downarrow{\scriptstyle W_k} & & \downarrow{\scriptstyle \iota_k^!} & & \downarrow{\scriptstyle \iota_k^!} \\
L_n(\mathbf{Z}[u,u^{-1}]) & \xrightarrow{\mathrm{aqk}^{-1}} & LN_n(\mathbf{Z}[\mathbf{C}_\infty^+] \to \mathbf{Z}[\mathbf{D}_\infty^{-,-}]) & \xrightarrow{\partial} & L_n(\mathbf{Z}[\mathbf{D}_\infty^{+,+}]).
\end{array}$$

□

Recall from §4 the Verschiebung algebra

$$\mathcal{V} := \mathbf{Z}[V_n \mid n > 0] = \mathbf{Z}[V_p \mid p \text{ prime}]$$

of n-th power operators

$$V_n := (x \longmapsto x^n)$$

on the polynomial ring $\mathbf{Z}[x]$.

THEOREM 6.3.2. *Consider* $\mathrm{UNil}_2(\mathbf{Z})$ *and the self-similar transfer homomorphism*

$$\iota_k^! : L_2(\mathbf{Z}[\mathbf{D}_\infty]) \longrightarrow L_2(\mathbf{Z}[\mathbf{D}_\infty]).$$

(1) *Suppose k is even. Then there is an inclusion*

$$\mathrm{UNil}_2(\mathbf{Z}) \subseteq \mathrm{Ker}(\iota_k^!).$$

(2) *Suppose $k > 1$ is odd. Then*

$$\mathrm{Ker}\left(\iota_k^! | \mathrm{UNil}_2(\mathbf{Z})\right)$$

is not a finitely generated $\mathbf{Z}$-module but is a finitely generated $\mathbf{Z}[V_k]$-submodule. Moreover, there is an inclusion

$$\mathrm{UNil}_2(\mathbf{Z}) \subseteq \mathrm{Im}\left(\iota_k^! | \mathrm{UNil}_2(\mathbf{Z})\right).$$

PROOF. The following statement[6] is a consequence of Lemma 6.3.1 and the Connolly-Ranicki isomorphism (see Thm. 2.1.2)

$$r : \mathrm{UNil}_2(\mathbf{Z}) \longrightarrow NL_2(\mathbf{Z}).$$

The monomorphic image of $NL_2(\mathbf{Z})$ *in* $L_2(\mathbf{Z}[t,t^{-1}])$ *maps isomorphically to the monomorphic image of* $\mathrm{UNil}_2(\mathbf{Z})$ *in* $L_2(\mathbf{Z}[\mathbf{D}_\infty])$. So it suffices to compute the image of $L_2(\mathbf{Z}[t])$ under the transfer homomorphism for Laurent polynomials:

$$W_k : L_2(\mathbf{Z}[t,t^{-1}]) \longrightarrow L_2(\mathbf{Z}[u,u^{-1}]).$$

By [**CD04**, Theorem 4.6(2)], the Witt group $L_2(\mathbf{Z}[t])$ is isomorphic to the idempotent quotient of the Tate homology group $\widehat{H}_1(\mathbf{C}_2;\mathbf{Z}[t])$:

$$\mathbf{F}_2[t]/\left\{ p^2 - p \mid p \in \mathbf{F}_2[t] \right\}.$$

The map from $L_2(\mathbf{Z}[t])$ to this quotient is given by restriction of the Arf invariant of the characteristic 2 field $\mathbf{F}_2(t)$. The map from the quotient to $L_2(\mathbf{Z}[t])$ is defined [**CD04**, 4.4(1)] for all representatives $p \in \mathbf{F}_2[t]$ by

$$[p] \longmapsto P_{p,1} = \left[\mathbf{Z}[t]e_1 \oplus \mathbf{Z}[t]e_2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix} \right].$$

Denote F_p as the image of $P_{p,1}$ in $L_2(\mathbf{Z}[t,t^{-1}])$. Write the polynomial

$$p = p_0 + \cdots + p_r t^r \in \mathbf{F}_2[x].$$

Observe

$$W_k(F_p) = \Bigg[\bigoplus_{0\le i<k} \mathbf{Z}[u,u^{-1}]t^i e_1 \bigoplus_{0\le i<k} \mathbf{Z}[u,u^{-1}]t^i e_2, \\ \begin{pmatrix} 0 & A_k \\ -A_k & 0 \end{pmatrix}, \begin{pmatrix} p_0 E_k^0 + \cdots + p_r E_k^r \\ E_k^0 \end{pmatrix} \Bigg].$$

[6]This observation was the starting point for the computations of the switch map in [**BDK**].

Here, using indices $0 \leq i, j < k$, we define a matrix A_k and vector E_k^r by

$$(A_k)_{i,j} := \begin{cases} 1 & \text{if } i+j=0 \\ u & \text{if } i+j=k \\ 0 & \text{else} \end{cases} \quad \text{and} \quad (E_k^r)_i := \begin{cases} u^\ell & \text{if } r+2i=\ell k \\ 0 & \text{else.} \end{cases}$$

Consider the homomorphism

$$\mathrm{Arf} : L_2(\mathbf{Z}[u,u^{-1}]) \longrightarrow \mathbf{F}_2[u,u^{-1}]/\left\{ f^2 - f \mid f \in \mathbf{F}_2[u,u^{-1}] \right\}.$$

Using the fundamental theorem of algebraic L-theory ([**Ran81**, §5, p. 430], [**Ran74**, §4]) and [**CD04**, Theorem 4.6(2)], Arf is an epimorphism. Its kernel is the image of the right-hand term $L_2(\mathbf{Z}) \cong \mathbf{F}_2$. This image is generated by the Witt class

$$\left[\bigoplus_2 \mathbf{Z}[t,t^{-1}], \begin{pmatrix} t & 0 \\ 0 & -1 \end{pmatrix}\right] \otimes \left[\bigoplus_2 \mathbf{Z}[t,t^{-1}], \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$
$$= \left[\bigoplus_4 \mathbf{Z}[t,t^{-1}], \begin{pmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} t \\ t \\ -1 \\ -1 \end{pmatrix}\right].$$

This particular example comes from the L-theory product (see [**Ran81**, §1.9]):

$$L^m(R) \otimes L_n(S) \longrightarrow L_{m+n}(R \otimes_{\mathbf{Z}} S).$$

A symplectic basis for the nonsingular (-1)-quadratic form $W_k(F_p)$ is

$$\left\{ t^0 e_1, t^0 e_2;\ t^1 e_1, u^{-1} \cdot t^{k-1} e_2;\ t^2 e_1, u^{-1} \cdot t^{k-2} e_2;\ \ldots;\ t^{k-1} e_1, u^{-1} \cdot t^1 e_2 \right\}.$$

Therefore a direct computation shows that

$$\mathrm{Arf}(W_k(F_p)) = \begin{cases} p_0 + p_k u + \cdots + p_{rk} u^r & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even.} \end{cases}$$

The desired conclusions now follow. $\square$

CHAPTER 7

Codimension one splitting for DIFF 5-manifolds

The goal of this chapter is to relax the Cappell/Wall fundamental group conditions (see [**Cap76b**, Ch. V] and [**Wal99**, §16]) for splitting homotopy equivalences between DIFF = PL 5-dimensional manifolds along a two-sided 4-submanifold. Instead, they are replaced with the vanishing of a surgery characteristic class (7.1.1).

7.1. On exactness of the surgery sequence for PL 4-manifolds

Recall that the PL normal invariant set and structure set of a manifold pair $(X, \partial X)$ satisfy (see [**Wal99**, p. 106]) that the restriction of each representative $f : (M, \partial M) \to (X, \partial X)$ to the boundary is a PL homeomorphism, and that the base point of each set is the identity map $\mathbf{1}_{(X,\partial X)}$. Throughout we denote $\mathbf{L}. := \mathbf{L}.^h\langle 1\rangle(\mathbf{Z})$ as the 1-connective cover $\mathbf{G/TOP}$ of the $\mathbf{Z}$-graded 4-periodic Ω-spectrum $\mathbf{L}.^h(\mathbf{Z})$. Its relevant homotopy groups $\pi_2(\mathbf{L}.)$ and $\pi_4(\mathbf{L}.)$ are given by Arf invariant and signature.

7.1.1. Statement of results.

Theorem 7.1.1. *Let $(X, \partial X)$ be a compact connected oriented* PL *4-manifold with fundamental group π.*

(1) *The surgery sequence*

$$\mathcal{S}^s_{\mathrm{PL}}(X, \partial X) \xrightarrow{\ N\ } \mathcal{N}_{\mathrm{PL}}(X, \partial X) \xrightarrow{\ \sigma\ } L^s_4(\mathbf{Z}[\pi])$$

of based sets is exact if

$$\text{(7.1.1.1)} \qquad \mathrm{Ker}\,(I_0 + \kappa_2) \ \cap \ [u_0(A \cap [X]) \oplus u_2(\mathrm{Ker}\, v_2)] \ = \ 0.$$

(2) *The subgroup A is uniquely determined by the index formula*

$$[H^4(X, \partial X; \mathbf{Z}) : A] + [H_2(X; \mathbf{Z}_2) : \mathrm{Ker}\, v_2] = 3.$$

We comment on the terms in Condition (7.1.1.1) of surgery characteristic classes.

(i) The element
$$[X] \in H_4(X, \partial X; \mathbf{Z})$$
is a choice of orientation in the orientation cover.

(ii) The subgroup A is defined by
$$A := \left\{ f^*(2\iota_4) \in H^4(X, \partial X; \mathbf{Z}) \;\mid\; \exists\, f : X/\partial X \to \mathrm{G/PL} \right\}.$$

(iii) The following maps are induced by the classifying map $u : X \to B\pi$ of the universal cover:
$$u_0 : H_0(X; \mathbf{Z}) \longrightarrow H_0(\pi; \mathbf{Z}) \quad \text{and} \quad u_2 : H_2(X; \mathbf{Z}_2) \longrightarrow H_2(\pi; \mathbf{Z}_2).$$

(iv) The element
$$v_2 = w_2 \in H^2(X; \mathbf{Z}_2)$$
is the second Wu class.

(v) The following maps are the 0- and 2-dimensional components of the assembly map asmb:
$$I_0 : H_0(\pi; \mathbf{Z}) \to L_4^h(\mathbf{Z}[\pi]) \quad \text{and} \quad \kappa_2 : H_2(\pi; \mathbf{Z}_2) \to L_4^h(\mathbf{Z}[\pi]).$$

Taylor and Williams [**TW79**] show that the spectrum $\mathbf{L}.(\mathbf{Z}[\pi])_{(2)}$ is a coproduct of Eilenberg-MacLane spectra, such as $\underline{K}(\mathbf{Z}_{(2)}, 0)$ and $\underline{K}(\mathbf{Z}_2, 2)$. The maps I_0 and κ_2 are induced from the inclusion of these factors by the smash product with $B\pi$.

Remark 7.1.2. Observe that u_0 is an isomorphism, and that u_2 is an epimorphism with kernel a homomorphic image of $\pi_2(X) \otimes \mathbf{Z}_2$, by the Hopf sequence of pullback fibration
$$\widetilde{X} \longrightarrow X \xrightarrow{\;u\;} B\pi.$$
Hence if the compact 4-manifold X is aspherical, then u_2 is an isomorphism.

Remark 7.1.3. Let $f : X/\partial X \to \mathrm{G/PL}$ be a normal invariant, and recall [**Dav05**, Proposition 3.6] as follows (its proof adapts to ∂X nonempty, see Remark

7.1.9). The surgery characteristic class $f^*(\iota_2)$ equals the codimension two Kervaire class

$$\mathrm{kerv}(f) \in H^2(X, \partial X; \mathbf{Z}_2).$$

The surgery characteristic class $f^*(2\iota_4)$ equals the signature class

$$\frac{1}{8}\mathrm{sign}(f) \cdot [X]^* \in H^4(X, \partial X; \mathbf{Z}).$$

Its surgery obstruction in $L_4^s(\mathbf{Z}[\pi])$ equals

$$\sigma(f) = (I_0 \circ u_0)(f^*(2\iota_4) \cap [X]) + (\kappa_2 \circ u_2)(f^*(\iota_2) \cap [X]).$$

COROLLARY 7.1.4. *The assembly map*

$$\mathrm{asmb} = I_0 + \kappa_2 + \cdots : H_*(\pi; \mathbf{L}.) \longrightarrow L_*^s(\mathbf{Z}[\pi]) \longrightarrow L_*^h(\mathbf{Z}[\pi])$$

is known to be injective (that is, the Borel/Novikov conjecture is true [**FRR95**]*) for a large class of groups* π*. Then any compact connected oriented* PL *4-manifold* X *with such a fundamental group* π *has its surgery sequence exact.* □

COROLLARY 7.1.5 ([**Wal99**, Theorem 16.6]). *Let* X *be a compact connected oriented* DIFF *or* PL *4-manifold. Suppose that*

$$H_2(\pi_1(X); \mathbf{Z}_2) = 0.$$

Then the following surgery sequence of based sets is exact:

$$\mathcal{S}^s_{\mathrm{PL}}(X, \partial X) \xrightarrow{\ N\ } \mathcal{N}_{\mathrm{PL}}(X, \partial X) \xrightarrow{\ \sigma\ } L_4^s(\mathbf{Z}[\pi_1(X)]).$$

PROOF. Observe that $\mathrm{Ker}(I_0) = 0$ since X is oriented and that $\mathrm{Ker}(\kappa_2) = 0$ by assumption. □

COROLLARY 7.1.6. *The surgery sequence is exact at* $\mathcal{N}_{\mathrm{PL}}(T^4)$*. This example is excluded in the previous corollary, see* [**Wal99**, Thm. 16.6, Remark].

PROOF. This follows from Corollary 7.1.4, using Novikov's proof that the assembly map $\mathrm{asmb} : H_4(\pi; \mathbf{L}.) \to L_4(\mathbf{Z}[\pi])$ is an isomorphism for all poly-$\mathbf{Z}$ groups π, see [**Wal99**, §15AB, §17H]. □

Remark 7.1.7. Since $\mathbf{RP}^4$ is non-orientable, we cannot apply Theorem 7.1.1(1) to obtain exactness of the surgery sequence at $\mathcal{N}_{\mathrm{PL}}(\mathbf{RP}^4)$. In fact, this case remains unknown, see [**Wal99**, Thm. 16.6, Remark] [**CS76**] [**FS81**]. I am grateful to Julius Shaneson for emphasizing this issue.

Remark 7.1.8. The closed simply-connected case [**Wal99**, Thm. 16.5] (whose proof is erroneous) of Corollary 7.1.5 is proven correctly as a corollary of [**CH90**, Theorem 5.2]. The proof below of Theorem 7.1.1 uses the Cochran-Habegger formula [**CH90**, Thm. 5.1] for normal invariants, and so their formula cannot be returned as a corollary for $\pi_1(X) = 1$ and $\partial X = \varnothing$.

7.1.2. Proofs.

Remark 7.1.9. We comment on extension to nonempty ∂X in our references.

(1) The Davis surgery characteristic class formula [**Dav05**, Prop. 3.6] for $\partial X = \varnothing$ relies on the Sullivan-Wall factorization of the surgery obstruction map σ through reduced bordism:
$$\widetilde{\Omega}_4^{\mathrm{STOP}}(B\pi_+ \wedge G/TOP) \longrightarrow L_4^s(\mathbf{Z}[\pi]).$$
However this ingredient remains the same if we assume a homeomorphism on ∂X (as opposed to a homotopy equivalence on ∂X), which is required for transverse inverse images in the codimension two Kervaire invariant (see [**Dav05**, Dfn. 3.5]).

(2) The Kirby-Taylor generalization [**KT01**, Thm. 18, Remarks] of the Cochran-Habegger formula [**CH90**, Thm. 5.1] already includes arbitrary ∂X. The original formula of [**CH90**] is stated for simply-connected compact topological 4-manifolds X with $\partial X = \varnothing$.

(3) The Siebenmann-Morita formulas [**KS77**, Thm. C.15.1] in the Puppe sequence, for the identification of the image of $[X, \mathrm{TOP}/\mathrm{PL}]$ with the cokernel of reduction modulo 2 on $H^3(X)$, and for the Kirby-Siebenmann invariant ks in terms of the Postnikov tower for G/TOP, only rely on X being a

countable finite-dimensional simplicial complex. So it is already true for arbitrary ∂X.

Proof of Theorem 7.1.1. Since any simple homotopy equivalence has vanishing surgery obstruction, the image of N is contained in the kernel of σ. For the remainder of the proof, we show that the kernel of σ is contained in the image of N.

Recall the commutative diagram:

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{PL}}(X,\partial X) \arrow[rr, "\sigma"] \arrow[d, equal] & & L_4^s(\mathbf{Z}[\pi^\omega]) \\
{[X/\partial X, \mathrm{G/PL}]} \arrow[r] \arrow[d, "\mathrm{red}_{\mathrm{TOP}}"] & \widetilde{\Omega}_4^{\mathrm{SPL}}(B\pi_+ \wedge \mathrm{G/PL}, \omega) \arrow[ur] \arrow[d, "\mathrm{red}_{\mathrm{TOP}}"] & H_4(\pi; \mathbf{L}.^\omega) \arrow[u, "\mathrm{asmb}"'] \arrow[d, equal] \\
{[X/\partial X, \mathrm{G/TOP}]} \arrow[r] \arrow[d, equal] & \widetilde{H}_4(B\pi_+ \wedge \mathrm{G/TOP}; \mathbf{MSTOP}^\omega) \arrow[dr, "\mathrm{act}_*"] & H_0(\pi; \mathbf{Z}^\omega) \oplus H_2(\pi; \mathbf{Z}_2) \arrow[d, equal] \\
H^0(X, \partial X; \mathbf{G/TOP}) \arrow[r, "\cap[X]_{\mathbf{L}.}"] & H_4(X; \mathbf{G/TOP}^\omega) \arrow[r, "u_*"] & H_4(B\pi; \mathbf{G/TOP}^\omega)
\end{tikzcd}$$

due to Sullivan-Wall [**Wal99**, Thm. 13B.3] and Quinn-Ranicki [**Ran92a**, Thm. 18.5]. It follows that the image $\hat{\sigma}(f)$, through the scalar product act[1], in $H_4(\pi; \mathbf{L}.^{\omega})$ of a normal invariant $f : X/\partial X \to \mathrm{G/PL}$ consists of two characteristic classes:

$$\hat{\sigma}(f) = u_0(f^*(2\iota_4) \cap [X]) \oplus u_2(f^*(\iota_2) \cap [X]),$$

which are determined by the TOP manifold-theoretic invariants of Remark 7.1.3.

Now let $f : X/\partial X \to \mathrm{G/PL}$ be a normal invariant with vanishing surgery obstruction. Then

$$0 = \sigma(f) = (I_0 + \kappa_2)\, \hat{\sigma}(f).$$

Observe by definition and Lemma 7.1.10 that

$$f^*(2\iota_4) \in A \quad \text{and} \quad f^*(\iota_2) \cap [X] \in \mathrm{Ker}(v_2).$$

Then the components of $\hat{\sigma}(f)$ vanish:

$$u_0(f^*(2\iota_4) \cap [X]) = 0 \quad \text{and} \quad u_2(f^*(\iota_2) \cap [X]) = 0,$$

by hypothesis (7.1.1.1). So

$$f^*(2\iota_4) = 0 \quad \text{and} \quad f^*(\iota_2) \cap [X] = (\mathrm{red}_2 \circ \mathrm{Hurewicz})(\alpha)$$

for some spherical class $\alpha \in \pi_2(X)$, by Remark 7.1.2 and Poincaré duality. Therefore the Kirby-Taylor generalization [**KT01**, Thm. 18, Remarks] of the Cochran-Habegger formula [**CH90**, Thm. 5.1] to arbitrary (π, ω) is applicable. We obtain that the Novikov pinch map $h_\alpha : (X, \partial X) \to (X, \partial X)$, which is a simple homotopy equivalence of X restricting to a PL homeomorphism on ∂X, has normal invariant

$$\mathrm{red}_{\mathrm{TOP}}\,(N(h_\alpha)) = (1 + \langle v_2, f^*(\iota_2) \cap [X] \rangle) \cdot f^*(\iota_2) = f^*(\iota_2)$$

in

$$H^2(X, \partial X; \mathbf{Z}_2) = \mathrm{Ker}(\mathcal{N}_{\mathrm{TOP}}(X, \partial X) \xrightarrow{\mathrm{retr}} H^4(X, \partial X; \mathbf{Z})).$$

Both the lifts of $f^*(\iota_2)$:

$$f, N(h_\alpha) \in \mathcal{N}_{\mathrm{PL}}(X, \partial X) = [X/\partial X, \mathrm{G/PL}]$$

[1]Recall that $\mathbf{L}. = \mathbf{G/TOP}$ is a module spectrum over the ring spectrum $\mathbf{L}^{\cdot} = \mathbf{MSTOP}$ via Brown representation; see [**Ran92a**, Rmk. B9] [**Wal99**, Thm. 9.8] on homotopy groups and Sullivan's method of proof in his thesis/notes.

have vanishing pullback of $2\iota_4$, as detected by signature in Remark 7.1.3. The class

$$2\iota_4 \in H^4(\mathrm{G/PL}; \mathbf{Z})$$

is the pullback of the class

$$\iota_4 \in H^4(K(\mathbf{Z}, 4); \mathbf{Z})$$

under the composite

$$\mathrm{G/PL} \xrightarrow{\mathrm{red}_{\mathrm{TOP}}} \mathrm{G/TOP} \xrightarrow{\mathrm{retr}} K(\mathbf{Z}, 4).$$

Therefore there exists a TOP normal bordism F, which restricts to a homeomorphism on $\partial X \times \Delta^1$, from the PL normal map f with $\sigma(f) = 0$ to the simple homotopy self-equivalence h_α of X. Consider the exact sequence of triangulation obstructions in Proof 7.1.10. By Poincaré duality with twisted coefficients, the difference class (i.e. the existence of a PL normal bordism) between f and h_α in $\mathcal{N}_{\mathrm{PL}}(X, \partial X)$ is identified with an element of the cokernel of the reduction modulo 2 map

$$\mathrm{Ker}(\omega)^{\mathrm{ab}} = H_1(X; \mathbf{Z}^\omega) \xrightarrow{\quad \mathrm{red}_2 \quad} H_1(X; \mathbf{Z}_2) = \pi^{\mathrm{ab}} \otimes \mathbf{Z}_2.$$

Here $\omega : \pi \to \mathbf{C}_2$ is the orientation character. But X is oriented, so this cokernel vanishes. Therefore f and h_α are PL normally bordant, thus we are done.

Part (2) of the theorem is relegated to Lemma 7.1.11. □

For the purpose of comparison between the following arguments and the literature, we refer the reader to [**CH90**, p. 435, par. 1] [**Wal99**, Proofs 16.5, 16.6].

Lemma 7.1.10. $\mathrm{Ker}(v_2) = B \cap [X]$ *where the subgroup* B *is defined as*

$$B := \big\{ f^*(\iota_2) \in H^2(X, \partial X; \mathbf{Z}_2) \ \ | \\ \exists f : X/\partial X \to \mathrm{G/PL} \textit{ satisfying } f^*(2\iota_4) \equiv 0 \pmod 2 \big\}.$$

Proof. Recall the Siebenmann-Morita exact sequence [**KS77**, Thm. C.15.1]:

$$0 \longrightarrow \mathrm{Cok}\big(\mathrm{red}_2 : H^3(X, \partial X; \mathbf{Z}) \to H^3(X, \partial X; \mathbf{Z}_2)\big) \longrightarrow [X/\partial X, \mathrm{G/PL}] \\ \xrightarrow{\mathrm{red}_{\mathrm{TOP}}} [X/\partial X, \mathrm{G/TOP}] \xrightarrow{\mathrm{ks}} [X/\partial X, B(\mathrm{TOP/PL})].$$

Therein is stated that for all

$$(a,b) \in H^4(X,\partial X;\mathbf{Z}) \times H^2(X,\partial X;\mathbf{Z}_2)$$
$$= [X/\partial X, K(\mathbf{Z},4) \times K(\mathbf{Z}_2,2)] = [X/\partial X, \mathrm{G/TOP}^{[5]}],$$

the Kirby-Siebenmann invariant is given by

$$\mathrm{ks}(a,b) = \mathrm{red}_2(a) + \mathrm{Sq}^2(b) \in H^4(X,\partial X;\mathbf{Z}_2) = [X/\partial X, K(\mathbf{Z}_2,4)].$$

Let $b \in H^2(X,\partial X;\mathbf{Z}_2)$. There exists $a \in H^4(X,\partial X;\mathbf{Z})$ such that

$$\mathrm{red}_2(a) = \mathrm{Sq}^2(b),$$

since $(X,\partial X)$ is a 4-dimensional Poincaré pair implies that

$$\mathrm{red}_2 : H^4(X,\partial X;\mathbf{Z}) \to H^4(X,\partial X;\mathbf{Z}_2)$$

is surjective. Define a map

$$t := (a,b) : X/\partial X \to \mathrm{G/TOP}.$$

Hence $b = t^*(\iota_2)$ is satisfied. Then there exists a lift $f : X/\partial X \to \mathrm{G/PL}$ of $t \in \mathrm{Ker}(\mathrm{ks})$. Note

$$f^*(2\iota_4) = f^*(\mathrm{red}^*_{\mathrm{TOP}}(\iota_4)) = t^*(\iota_4) = a.$$

Therefore $b \in B$ if and only if $\mathrm{Sq}^2(b) = 0$, if and only if $b \cap [X] \in \mathrm{Ker}(v_2)$. Thus

$$B \cap [X] = \mathrm{Ker}(v_2).$$

$\square$

LEMMA 7.1.11. $[H^4(X,\partial X;\mathbf{Z}) : A] + [H_2(X;\mathbf{Z}_2) : \mathrm{Ker}\, v_2] = 3.$

PROOF (AN ANALOGOUS ARGUMENT). Let $a \in H^4(X,\partial X;\mathbf{Z})$. Then

$$a = f^*(2\iota_4) \quad \text{for some} \quad f : X/\partial X \longrightarrow \mathrm{G/PL}$$

if and only if

$$\mathrm{red}_2(a) = \mathrm{Sq}^2(b) = v_2 \cup b \quad \text{for some} \quad b \in H^2(X,\partial X;\mathbf{Z}_2).$$

Thus if $v_2 = 0$ then

$$A = \mathrm{Ker}(\mathrm{red}_2).$$

Hence the subgroup A has index 2, independent of the orientability of X. If $v_2 \neq 0$ then, since $(X, \partial X)$ is a 4-dimensional Poincaré pair, there exists b such that

$$\langle v_2 \cup b, \mathrm{red}_2[X] \rangle = 1,$$

so the reduction modulo two map is surjective:

$$\mathrm{red}_2 : A \to H^4(X, \partial X; \mathbf{Z}_2).$$

Hence the subgroup A has index 1, independent of the orientability of X, and the subgroup $\mathrm{Ker}(v_2)$ has index 2. $\square$

7.2. Stable splitting obstructions between DIFF 5-manifolds

The main theorem (7.2.1) determines the existence and uniqueness of smooth split solutions in dimension five. Throughout this section, we use the notation of §1.2 for splitting homotopy equivalences, with $\dim(Y) = 5$ and $\dim(X) = 4$ substituted. Recall that the structure set $\mathcal{S}^h_{\mathrm{PL}}(X, \partial X)$ denotes the set of PL h-bordism classes rel ∂ of homotopy equivalences from compact PL 4-manifolds to X which are PL homeomorphisms on ∂X.

Let $r \in \mathbf{Z}_{\geq 0}$ be a stabilization parameter. By trivial handle-exchanges in Y, denote the **r-stabilized submanifold**

$$X_r := X \# r(S^2 \times S^2).$$

The stable surgery exact sequence of Cappell-Shaneson [**CS71**] for smooth 4-manifolds is the motivation to define (see [**KT01**, p. 393]) the **r-stable structure set**

$$\begin{aligned} {}^r\widetilde{\mathcal{S}}^h_{\mathrm{PL}}(X, \partial X) := \big\{ f \in \mathcal{S}^h_{\mathrm{PL}}(X_r, \partial X) \mid \\ \text{the normal invariant of } f \text{ is stabilized from } \mathcal{N}_{\mathrm{PL}}(X, \partial X) \big\}. \end{aligned}$$

In particular for $r = 0$, we recover $\mathcal{S}^h_{\mathrm{PL}}(X, \partial X)$. We remark that the TOP version of the main theorem for $\partial X = \varnothing$ and $H, G_{\pm}$ small is achieved without stabilization by S. Weinberger in [**Wei87**, Theorem 1].

7.2.1. Statement of results. Recall the topological notation of §1.2.1.

THEOREM 7.2.1. *Consider a two-sided incompressible* PL *4-submanifold* $(X, \partial X)$ *of a compact connected* DIFF *5-manifold* $(Y, \partial Y)$. *Suppose that* $(X, \partial X)$ *satisfies Condition (7.1.1.1) and that there exists* $r \in \mathbf{Z}_{\geq 0}$ *such that Wall realization restricts as follows:*

$$L_5^h(\mathbf{Z}[H]) \times \mathcal{S}_{\mathrm{PL}}^h(X, \partial X) \longrightarrow {}^r\widetilde{\mathcal{S}}^h_{\mathrm{PL}}(X, \partial X). \tag{7.2.1.1}$$

Let $(W, \partial W)$ *be a compact* DIFF *5-manifold, and let* $g : W \to Y$ *be any homotopy equivalence that restricts to a diffeomorphism on* ∂Y. *Then, relative to the boundary, the map* $g : (W, \partial W) \to (Y, \partial Y)$ *is*

(1) *normally bordant to a split map along* X_{2r} *if the following* K*-theory class vanishes:*

$$[\partial_K(\tau(g))] \in \widehat{H}^5(I^\omega),$$

(2) *h-bordant to a split map along* X_{2r} *if and only if in addition the following L-theory element vanishes:*

$$\mathrm{split}_L^h(g) \in \mathrm{UNil}_6^h(\Phi^\omega),$$

(3) *homotopic to a split map along* X_{2r} *if and only if the following* K*- and L-theory elements vanish:*

$$(\partial_K \oplus \mathrm{split}_K)(\tau(g)) \in I \oplus \widetilde{\mathrm{Nil}}_0(\Phi) \quad \textit{and then} \quad \mathrm{split}_L^s(g) \in \mathrm{UNil}_6^s(\Phi^\omega).$$

(4) *Furthermore, the set of split homotopy classes of split solutions stands in bijection with the controlled* $\mathbf{L}.$*-homology group*

$$H_6^{h\to h}(T/G; \mathbf{L}.(p)).$$

COROLLARY 7.2.2 ([**Cap76b**, Theorem 5, Remark]). *Consider a connected incompressible, oriented two-sided 4-submanifold* X *of a closed* DIFF *5-manifold* Y. *Write* $H = \pi_1(X)$ *and* $G = \pi_1(Y)$. *Suppose the following conditions are satisfied:*

(1) *(Cappell) the fundamental group* H *is square-root closed in* G,
(2) *(Wall) the group homology* $H_2(H; \mathbf{Z}_2)$ *vanishes, and*

(3) *the surgery obstruction map*

$$\mathcal{N}_{\mathrm{PL}}(X \times (\Delta^1, \partial\Delta^1)) \xrightarrow{\ \sigma\ } L_5^h(\mathbf{Z}[H]) \quad \textit{is surjective.} \tag{7.2.2.1}$$

Then any homotopy equivalence $g : W \to Y$, for W a closed DIFF *5-manifold, is*

(1) *h-bordant to a split map along $X = X_0$ if and only if the following K-theory class vanishes:*

$$[\partial_K(\tau(g))] \ \in \ \widehat{H}^5(I^\omega),$$

(2) *homotopic to a split map along $X = X_0$ if and only if the following K-theory elements vanish:*

$$(\partial_K \oplus \mathrm{split}_K)(\tau(g)) \in I \oplus \widetilde{\mathrm{Nil}}_0(\Phi).$$

Proof. Note $\partial X = \partial Y = \varnothing$. Since H is square-root closed in G, by [**Cap74b**, Corollary 4], we have

$$\mathrm{UNil}_6^h(\Phi^\omega) = 0.$$

Also, by Corollary 7.1.5, since

$$H_2(H; \mathbf{Z}_2) = 0,$$

we have that Condition (7.1.1.1) is satisfied. Finally, observe that Condition (7.2.2.1) implies Condition (7.2.1.1) for $r = 0$, since the L_5-action must be trivial.[2] $\square$

Remark 7.2.3. Cappell observed [**Cap76b**, Ch. V] that the proofs of [**CS71**, Theorems 4.1, 5.1] can be modified as to not depend on the Cappell-Shaneson stable surgery sequence [**CS71**, Thms. 2.1, 3.1]. Observe our theorem generalizes the conditions of [**Cap76b**, Theorem 5, Remark] and below generalizes its proof, which is essentially contained in [**CS71**, Theorems 4.1, 5.1] when UNil vanishes.

[2]This occurs if $L_5^h(\mathbf{Z}[H^\omega]) = 0$, e.g. if H is an odd torsion group [**Bak75**].

7.2.2. Proof of main theorem. For simplicity, we shall abbreviate

$$\begin{aligned} L_n(\pi) &:= L_n(\mathbf{Z}[\pi^\omega]) \\ \widetilde{\mathrm{Nil}}_0 &:= \widetilde{\mathrm{Nil}}_0(\Phi) \\ \mathrm{UNil}_n &:= \mathrm{UNil}_n(\Phi^\omega). \end{aligned}$$

Throughout the proofs of the four parts, we write DIFF instead of PL in order to deal with rounding corners. This notational convenience is justified by the fact that any PL 4- or 5-manifold admits a unique DIFF structure compatible with the triangulation [**FQ90**, Ch. 8].

For the reader's convenience, we recall a version of Cappell's Mayer-Vietoris sequence (1.3.5):

$$\cdots \xrightarrow{\partial_L} L_n^h(H) \xrightarrow{i_- - i_+} L_n^h(J) \xrightarrow{j} \frac{L_n^B(G)}{\mathrm{UNil}_n^s} \xrightarrow{\partial_L} L_{n-1}^h(H) \longrightarrow \cdots .$$

REMARK 7.2.4. Consider

$$Z := \mathbf{CP}^4 \# 2(S^3 \times S^5),$$

which is a simply-connected closed PL 8-manifold whose Euler characteristic and signature are equal to one (see [**Wei87**, Thm. 1]). Let $n \in \mathbf{Z}_{\geq 0}$. Suppose h is a homotopy equivalence between n-manifolds with fundamental group π and orientation character ω. Now let κ be a $*$-invariant subgroup of $\mathrm{Wh}(\pi)$. Observe that the surgery obstruction $\sigma(f)$ maps to $\sigma(f \times \mathbf{1}_Z)$ under the decorated periodicity isomorphism[3]

$$\otimes \sigma^*(Z) : L_n^\kappa(\pi) \longrightarrow L_{n+8}^\kappa(\pi).$$

This follows from Kwun-Szczarba's torsion product formula [**Lüc02**, Thm. 2.1] and from Ranicki's surgery product formula [**Ran80b**, Prop. 8.1(ii)]. □

Throughout the proofs, the reader should keep in mind the following figure.

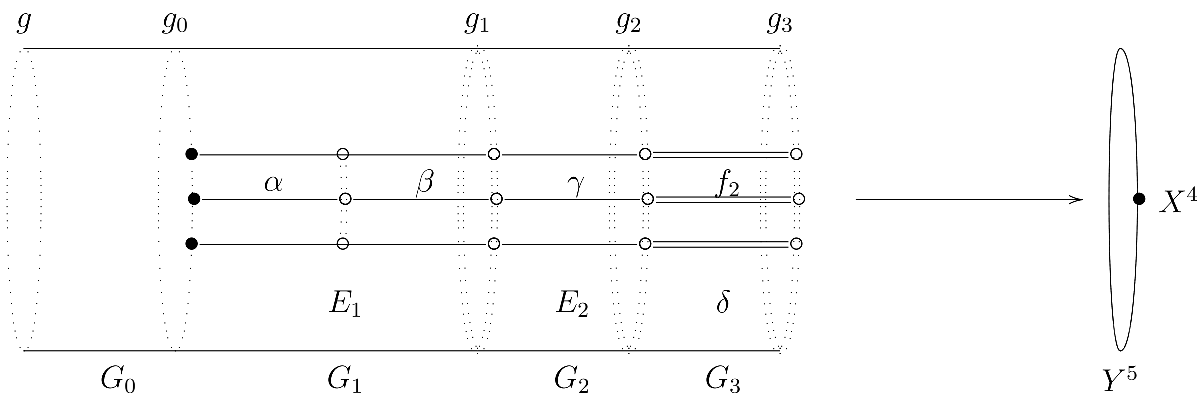

FIGURE 7.2.1. Our extended Cappell-Shaneson replacement G for the high-dimensional Cappell nncc.

[3] If R is a ring with involution then $\mathbf{L}.(R)$ is a module spectrum over the ring spectrum $\mathbf{L}^{\cdot}(\mathbf{Z})$; see [**Ran92a**, p. 318, line 4] on the level of spectra and [**Ran80a**, Prop. 8.1] on the level of homotopy groups.

PROOF OF THEOREM 7.2.1(1). Let $g : W \to Y$ be a homotopy equivalence which restricts to a PL homeomorphism on ∂Y, where W is a compact PL 5-manifold. Since

$$[\partial_K(\tau(g))] = 0 \in \widehat{H}^5(I^\omega),$$

by [**Cap76b**, Lemma II.4] and Waldhausen's Mayer-Vietoris sequence (1.3.4) and Milnor's torsion duality, there exists an h-bordism

$$G_0 : W \times [0,1] \longrightarrow Y \operatorname{rel} \partial Y$$

from g to a homotopy equivalence $g_0 : W \to Y$ such that

$$(\partial_K \oplus \mathrm{split}_K)(\tau(g_0)) = 0 \in I \oplus \widetilde{\mathrm{Nil}}_0.$$

Hence $\tau(g_0) \in B$. By general position, we may assume that g_0 is transversal to X. Consider the degree one normal map

$$f := g_0|_M : M \longrightarrow X,$$

where $M := g_0^{-1}(X)$ is the PL transverse inverse image.

By [**Cap76b**, Lemma I.1], there exists a homotopy from $g_0 \times \mathbf{1}_Z$ to a certain homotopy equivalence $\overline{g}$. The homotopy itself is obtained via handle-exchanging along "nilpotent" relative homotopy elements. The map $\overline{g}$ is transversal to $\overline{X} := X \times Z$ and restricts to a 4-connected degree one normal map $\overline{f} : \overline{M} \to \overline{X}$. Then by [**Cap76b**, Lemma II.1], there exists a f.g. projective lagrangian P for the nonsingular $(+1)$-quadratic form $\sigma(\overline{f})$ over $\mathbf{Z}[H^\omega]$. The lagrangian P is obtained from a certain null-bordism in the cover of $\overline{Y} := Y \times Z$ corresponding to the subgroup H. Moreover by [**Cap76b**, Lemma II.2], its projective class is

$$[P] = \partial_K(\tau(\overline{g})) \in I$$

and satisfies $[P^*] = -[P]$.

We now replace the use of Condition (d) occurring in [**Cap76b**, Lemmas II.2,3,4] *at the cost of losing control of the resultant bordism. However, in a roundabout fashion, we will obtain a replacement for Cappell's nilpotent normal cobordism construction, which was observed in neither* [**Cap76b**] *nor* [**CS71**].

Consider the Ranicki-Rothenberg exact sequence [**Ran80a**, Prop. 9.1] over the ring $\mathbf{Z}[H^\omega]$ with involution:

$$L^p_{13} \longrightarrow \widehat{H}^{13}(\widetilde{K}_0) \longrightarrow L^h_{12} \longrightarrow L^p_{12}.$$

The existence of P implies that $\sigma(\overline{f})$ has zero image in $L^p_{12}(H)$. Hence it is the image of the Tate cohomology class

$$[[P]] \in \widehat{H}^{13}(\widetilde{K}_0(\mathbf{Z}[H])^\omega).$$

Namely, $\sigma(\overline{f}) \in L^h_{12}(H)$ is the cobordism class of the hyperbolic construction $\mathscr{H}_+(P)$. Note that it vanishes in a stronger sense:

$$\begin{aligned} [P] &= \partial_K(\tau(\overline{g})) \\ &= \partial_K(\tau(g_0)) \cdot \chi(Z) \\ &= \partial_K(\tau(g_0)) \\ &= 0 \in I. \end{aligned}$$

Then $\sigma(\overline{f}) = 0$, so $\sigma(f) = 0$ by normal bordism invariance and periodicity of the surgery obstruction map σ. Therefore, by Exactness at the Normal Invariants (7.1.1), there exists a PL normal bordism α from f to a homotopy equivalence f_0.

Now consider the degree one normal map of triads, relative to the PL homemorphism $\partial W \to \partial Y$:

$$W \setminus M \times \overset{\circ}{D^1} \bigcup \mathrm{Domain}(\alpha) \times \partial D^1 \xrightarrow{\quad e_0 \quad} Y \setminus X \times \overset{\circ}{D^1}.$$

It is defined by cutting g_0 along f, and then pasting in two copies of α (see Figure 7.2.1). In particular, the degree one normal map e_0 restricts to a homotopy equivalence on the "seams" $X \times \partial D^1$. Observe that

$$G_0 \bigcup \alpha \times \mathbf{1}_{D^1}$$

is a normal bordism over Y between $e_0 \cup \partial_+(\alpha) \times \mathbf{1}_{D^1}$ and the original homotopy equivalence g. Then its surgery obstruction $\sigma(e_0)$ has zero image in $L^B_5(G)$. So by Cappell's Mayer-Vietoris sequence (1.3.5), the surgery obstruction

$$\sigma(e_0) \in L^h_5(J)$$

is the image of some element $y \in L_5^h(H)$.

By Condition (7.2.1.1),[4] there exists a PL normal bordism β over X_r from

$$\overline{f_0} := f_0 \# \mathbf{1}_{r(S^2 \times S^2)}$$

to another homotopy equivalence f_1 with surgery obstruction $\sigma(\beta) = -y$. Then the degree one normal map

$$e_0' := e_0 \bigcup_{\overline{f_0} \times \mathbf{1}_{D^1}} \beta \times \mathbf{1}_{\partial D^1}$$

over $Y \setminus X_r \times \overset{\circ}{D}{}^1$ restricts to the homotopy equivalence $f_1 \times \mathbf{1}_{\partial D^1}$ on the seam $X_r \times \partial D^1$. Note that its surgery obstruction is

$$\begin{aligned} \sigma(e_0') &= \sigma(e_0) + i_-\sigma(\beta \times -1) - i_+\sigma(\beta \times 1) \\ &= \sigma(e_0) - (i_- - i_+)(y) \\ &= 0. \end{aligned}$$

So by the 5-dimensional surgery exact sequence [**Wal99**, Thm. 10.8], there exists a PL normal bordism E_1 of triads over $Y \setminus X_r \times \overset{\circ}{D}{}^1$ from e_0' to a homotopy equivalence e_1, which restricts to the fixed homotopy equivalence $f_1 \times \mathbf{1}_{\partial D^1}$ on the seam. Therefore we obtain a normal bordism (see Figure 7.2.1)

$$G_1 := \left(g_0 \times \mathbf{1}_{[0,1]} \bigcup_{f \times \mathbf{1}_{D^1}} (\alpha \bigcup_{f_0} \beta) \times \mathbf{1}_{D^1} \right) \cup E_1.$$

Note $G_0 \cup G_1$ connects the original g to a B-torsion homotopy equivalence

$$g_1 := \bigcup_{f_1} e_1$$

split along X_r.

In a sense, the splitting problem is now solved, except that we would like more precise control on the path to the split solution. Namely, its surgery obstruction should equal the splitting obstruction in UNil_6^h.

[4]We perform r trivial 2-handle exchanges along X in Y in order to internalize the desired stabilization.

Since its boundary has torsion $\tau(\partial G_1) \in B$, we can define the intermediate surgery obstruction

$$x := \sigma(G_1) \in L_6^B(G).$$

By Condition (7.2.1.1), there exists a PL normal bordism γ over X_{2r} from

$$\overline{f_1} := f_1 \# \mathbf{1}_{r(S^2 \times S^2)}$$

to a homotopy equivalence f_2 with

$$\sigma(\gamma) = -\partial_L(x) \in L_5^h(H).$$

Note that the Mayer-Vietoris sequence (1.3.5) implies that

$$(i_- - i_+)\sigma(\gamma) = 0.$$

Then by the 5-dimensional surgery exact sequence, there exists a PL normal bordism E_2 over $Y \setminus X_{2r} \times \overset{\circ}{D}{}^1$ from the degree one normal map

$$e_1' := e_1 \bigcup_{\overline{f_1} \times \mathbf{1}_{\partial D^1}} \gamma \times \mathbf{1}_{\partial D^1}$$

to a homotopy equivalence e_2, which restricts to the homotopy equivalence $f_2 \times \mathbf{1}_{\partial D^1}$ on the seam. Therefore we obtain a normal bordism (see Figure 7.2.1)

$$G_2 := \gamma \times \mathbf{1}_{D^1} \bigcup E_2$$

over Y from g_1 to another B-torsion homotopy equivalence g_2 split along X_{2r}.

Observe from the definition of ∂_L, using periodicity and the action of the group $L_{14}^B(G)$ on the split homotopy structure $g_1 \times \mathbf{1}_Z$ (see Remark 1.3.7), that

$$\begin{aligned} \partial_L(\sigma(G_2)) &= \partial_L(\sigma(G_2 \times \mathbf{1}_Z)) \\ &= \sigma(\gamma \times \mathbf{1}_Z) \\ &= \sigma(\gamma) \\ &= -\partial_L(x - \mathrm{split}_L(x)). \end{aligned}$$

Then the surgery obstruction of G_2 must satisfy

$$\sigma(G_2) \equiv -(x - \mathrm{split}_L(x)) \mod \mathrm{Ker}(\partial_L).$$

So by the Mayer-Vietoris sequence (1.3.5), there exists $z \in L_6^h(J)$ such that

$$\sigma(G_2) + (x - \mathrm{split}_L(x)) \equiv j(z) \mod \mathrm{UNil}_6^s.$$

By the 5-dimensional surgery exact sequence, there exists a PL normal bordism δ over $Y \setminus X_{2r} \times \mathring{D}^1$, which restricts to the homotopy equivalence $f_2 \times \mathbf{1}_{\partial D^1}$ on the seam, whose surgery obstruction is

$$\sigma(\delta) = -z.$$

Lastly, define a PL normal bordism (see Figure 7.2.1) over Y:

$$G_3 := (f_2 \times \mathbf{1}_{[0,1]}) \times \mathbf{1}_{D^1} \bigcup_{(f_2 \times \mathbf{1}_1) \times \mathbf{1}_{\partial D^1}} \delta$$

from g_2 to another B-torsion homotopy equivalence g_3 split along X_{2r}. Then

$$\sigma(G_3) = j\sigma(\delta).$$

The advantage of uniting G_1 with G_2 and G_3 is that (see Figure 7.2.1) the normal bordism over Y:

$$G := G_1 \bigcup G_2 \bigcup G_3$$

has surgery obstruction in the summand UNil_6^s of $L_6^B(G)$:

$$\begin{aligned}\sigma(G) &\equiv x + (j(z) - (x - \mathrm{split}_L(x))) + j(-z)\\ &= \mathrm{split}_L(x)\\ &\equiv 0 \mod \mathrm{UNil}_6^s.\end{aligned}$$

Consider the PL normal bordism $G \times \mathbf{1}_Z$ between the B-torsion homotopy equivalences $g_0 \times \mathbf{1}_Z$ and $g_3 \times \mathbf{1}_Z$. Recall Cappell's decomposition [**Cap74a**, Thm. 7] of the B-torsion structure set:

$$\mathrm{nncc} : \mathcal{S}_{\mathrm{PL}}^B((Y, \partial Y) \times Z) \xrightarrow{\cong} \mathcal{S}_{\mathrm{PL}}^{\mathrm{split}}((Y, \partial Y) \times Z; (X, \partial X) \times Z) \times \mathrm{UNil}_{14}^s.$$

It is a careful form of Wall realization, and so the connecting normal bordism enjoys the same uniqueness properties (see Remark 1.3.7 and [**Wal99**, Thm. 5.8(iii)]). Hence

$$\mathrm{nncc}([g_0 \times \mathbf{1}_Z]) = ([g_3 \times \mathbf{1}_Z], \sigma(G \times \mathbf{1}_Z)).$$

Therefore the L-theory splitting obstructions must be:

$$\begin{aligned}\sigma(G) &= \sigma(G \times \mathbf{1}_Z) \\ &= \mathrm{split}_L(g_0 \times \mathbf{1}_Z) \\ &= \mathrm{split}_L^s(g_0) \\ \sigma(G_0 \cup G) &= \mathrm{split}_L^h(g).\end{aligned}$$

□

PROOF OF THEOREM 7.2.1(2). Suppose the primary K-theory class and secondary L-theory element vanish:

$$\begin{aligned}[\partial_K(\tau(g))] &\in \widehat{H}^5(I^\omega) \\ \mathrm{split}_L^h(g) &\in \mathrm{UNil}_6^h.\end{aligned}$$

Then, by Part (1), since

$$\begin{aligned}\sigma(G_0 \cup G) &= \mathrm{split}_L^h(g) \\ &= 0,\end{aligned}$$

we may perform surgery on the interior of the normal bordism $G_0 \cup G$ to obtain an h-bordism between the homotopy equivalences g and g_3. The latter map is split along the stabilization X_{2r}.

Conversely, suppose that there exists an h-bordism G' from g to a split homotopy equivalence g'. Then, by decorated periodicity and Cappell's bijection nncc^h in high-dimensions (1.2.4), note that

$$\begin{aligned}\mathrm{split}_L^h(g) &= \mathrm{split}_L^h(g \times \mathbf{1}_Z) \\ &= \sigma(G' \times \mathbf{1}_Z) \\ &= 0 \in \mathrm{UNil}_{14}^h.\end{aligned}$$

□

PROOF OF THEOREM 7.2.1(3). Suppose the primary K-theory and secondary L-theory elements vanish:

$$\begin{aligned}(\partial_K \oplus \mathrm{split}_K)(\tau(g)) &\in I \oplus \widetilde{\mathrm{Nil}}_0(\Phi)\\ \mathrm{split}^s_L(g) &\in \mathrm{UNil}^s_6(\Phi^\omega).\end{aligned}$$

Then

$$\begin{aligned}\tau(G_0) &= 0\\ \sigma(G_1) &= \mathrm{split}^s_L(g)\\ &= 0.\end{aligned}$$

Therefore we may perform surgery on the interior of the normal bordism $G_0 \cup G_1$ so that it becomes an s-bordism between the homotopy equivalences g to g_3. The latter map is split along the stabilization X_{2r}. We are done by the s-cobordism for a smooth 5-manifold base.

Conversely, suppose that there exists a homotopy G' from g to a split homotopy equivalence g'. Hence

$$\tau(g) = \tau(g') \in B.$$

So, by Waldhausen's Mayer-Vietoris sequence (1.3.4), we must have

$$(\partial_K \oplus \mathrm{split}_K)(\tau(g)) = 0 \in I \oplus \widetilde{\mathrm{Nil}}_0(\Phi).$$

Then, by decorated periodicity and Cappell's bijection nncc^s in high-dimensions (1.2.4), note that

$$\begin{aligned}\mathrm{split}^s_L(g) &= \mathrm{split}^s_L(g \times \mathbf{1}_Z)\\ &= \sigma(G' \times \mathbf{1}_Z)\\ &= 0 \in \mathrm{UNil}^s_{14}.\end{aligned}$$

$\square$

PROOF OF THEOREM 7.2.1(4). Observe that any other candidate for G has surgery obstruction in the same coset of

$$H_6^{h\to h}(T/G; \mathbf{L}.(p)).$$

But by Proposition 1.3.8, this controlled homology group has zero intersection with the image in $L_6^B(G)$ of

$$\mathrm{UNil}_6^s(\Phi^\omega).$$

Therefore we have a well-defined decomposition nncc^s. $\square$

REMARK 7.2.5. In summary, by assuming a relaxation of the Borel/Novikov Conjecture (7.1.1.1) and Bounded Stable Realization (7.2.1.1) on the smooth four-submanifold $(X, \partial X)$, we have shown that there are decomposition of the decorated structure sets:

$$\begin{aligned}
\mathrm{nncc}^s \ &: \ \mathcal{S}^B_{\mathrm{DIFF}}(Y, \partial Y) \xrightarrow{\cong} \mathcal{S}^{\mathrm{split}}_{\mathrm{DIFF}}(Y, \partial Y; X_{2r}, \partial X) \times \mathrm{UNil}_6^s(\Phi^\omega) \\
\mathrm{nncc}^h \ &: \ \mathcal{S}^h_{\mathrm{DIFF}}(Y, \partial Y) \xrightarrow{\cong} \mathcal{S}^{\mathrm{split}}_{\mathrm{DIFF}}(Y, \partial Y; X_{2r}, \partial X) \times \widehat{H}^5(I^\omega) \times \mathrm{UNil}_6^h(\Phi^\omega).
\end{aligned}$$

They constitute the extensions to smooth five-manifolds $(Y, \partial Y)$ of Cappell's nilpotent normal cobordism construction. This is the main geometric ingredient of two-sided splitting obstruction theory for high-dimensional manifolds and for the Mayer-Vietoris sequence in the L-theory of generalized free products.

CHAPTER 8

Classification within a stable TOP type of 4-manifold

Suppose X is a compact connected smooth 4-manifold, with fundamental group π and orientation character $\omega : \pi \to \mathbf{C}_2$. Our motivation here involves that the Cappell-Shaneson stable surgery sequence [**CS71**, Thm. 3.1] produces certain stable diffeomorphisms. This leads to a modified Wall realization

$$L_5^s(\mathbf{Z}[\pi^\omega]) \times \mathcal{S}^s_{\mathrm{DIFF}}(X, \partial X) \longrightarrow \overline{\mathcal{S}}^s_{\mathrm{DIFF}}(X, \partial X),$$

where $\mathcal{S}$ is the simple smooth structure set and $\overline{\mathcal{S}}$ and is the stable structure set (compare §7.2). Recall that the equivalence relation on these structure sets is smooth s-bordism of smooth homotopy structures. The actual statement of [**CS71**, Theorem 3.1] is sharper in that the amount of stabilization, i.e. number of connected summands of $S^2 \times S^2$, depends only on the rank of a given representative of the L-group.

In the case X is *sufficiently large* in that it contains a two-sided incompressible smooth 3-submanifold Σ, an elementary periodicity argument using Cappell's decomposition (1.2.3) shows that the restriction of the above action on $\mathcal{S}^B_{\mathrm{DIFF}}(X, \partial X)$ to the summand[1] UNil_5^s of $L_5^B(\mathbf{Z}[\pi^\omega])$ is free[2]. Therefore for each nonzero element of this exotic UNil-group, there exists a distinct stable smooth homotopy structure on X, restricting to a diffeomorphism on ∂X, which is not $\mathbf{Z}[\pi_1(\Sigma)]$-homology splittable along Σ. For X a connected sum of two copies of $\mathbf{RP}^4$ and Σ a homotopy 3-sphere, see the case studies [**JK06**] and [**BDK**].

[1]We gave a general definition of $\mathrm{UNil}^s_{\mathrm{odd}}$ in (1.1.8), without assuming Waldhausen's vanishing condition on $\widetilde{\mathrm{Nil}}_0$.

[2]The same argument shows that the restricted action of UNil^s_5 on $\mathcal{S}^B_{\mathrm{TOP}}(X, \partial X)$ is free, if X is a compact connected topological 4-manifold with good fundamental group.

For any $r \geq 0$, denote (cf. §7.2) the r**-stabilization of** X by

$$X_r := X \# r(S^2 \times S^2).$$

8.1. On the TOP classification of 4-manifolds

The main theorem of this section is an upper bound on the number of $S^2 \times S^2$ connected-summands sufficient for a *stable homeomorphism,* where the fundamental group of X lies in a certain class of good groups. If X is sufficiently large, instead by using Freedman-Quinn surgery [**FQ90**, Ch. 11], each nonzero element ϑ of the UNil-group yields a certain TOP 4-manifold Y_ϑ simple homotopy equivalent to X, which stabilizes to a manifold TOP s-cobordant to the above DIFF manifold Y.

8.1.1. Statement of results. The following result and technique are similar to Hambleton-Kreck [**HK93**, Thm. B] for finite groups π.

THEOREM 8.1.1. *Suppose π is a good group (in the sense of* [**FQ90**]*) with orientation character $\omega : \pi \to \mathbf{C}_2$. Define the following rings with involution:*

$$\begin{aligned} A &:= \mathbf{Z}[\pi^\omega] \\ R &\subseteq \mathrm{Center}(A) \\ R_0 &:= \left\{ \sum_i x_i \bar{x}_i \;\middle|\; x_i \in R \right\}. \end{aligned}$$

Suppose A is a finitely generated R_0-module. Further suppose the commutative ring R_0 is noetherian and

$$d := \dim(\mathrm{maxspec}\, R_0) < \infty.$$

Now suppose that X is a compact connected TOP *4-manifold with*

$$(\pi_1(X), w_1) = (\pi, \omega)$$

and that it has the form

$$(X, \partial X) = (X_{-1}, \partial X) \# (S^2 \times S^2).$$

If X_r is homeomorphic to Y_r for some $r \geq 0$, then X_d is homeomorphic to Y_d.

Our primary examples are the virtually cyclic groups considered in Chapter 3.

PROPOSITION 8.1.2. *The product $F \times \mathbf{D}_\infty$ of any finite group F and the infinite dihedral group, with any choice of ω, satisfies the above hypotheses with $d = 2$.*

COROLLARY 8.1.3. *Let X be a compact connected* TOP *4-manifold with fundamental group $\pi_1(X) \cong F \times \mathbf{D}_\infty$ and F finite. If a* TOP *4-manifold Y is stably homeomorphic to X, then $Y \# 3(S^2 \times S^2)$ is homeomorphic to $X \# 3(S^2 \times S^2)$.* $\square$

8.1.2. Definitions and lemmas. An exposition of the following concept with applications is available in the book of Anthony Bak [**Bak81**].

DEFINITION 8.1.4 ([**Bas73**, I.4.1]). A **unitary ring** (A, λ, Λ) consists of a ring with involution A, an element

$$\lambda \in \mathrm{Center}(A) \quad \text{satisfying} \quad \lambda\overline{\lambda} = 1,$$

and a **form parameter** Λ. This is an abelian subgroup of A satisfying

$$\{\, a + \lambda\overline{a} \mid a \in A \,\} \subseteq \Lambda \subseteq \{\, a \in A \mid a - \lambda\overline{a} = 0 \,\}$$

and

$$r a \overline{r} \in \Lambda \quad \text{for all} \quad r \in A \text{ and } a \in \Lambda.$$

The following subgroup of unitary automorphisms can be realized by diffeomorphisms [**CS71**, Thm. 1.5].

DEFINITION 8.1.5 ([**Bas73**, I.5.1]). Let $(M, \langle \cdot, \cdot \rangle, \mu)$ be a quadratic module over a unitary ring (A, λ, Λ). A **transvection** $\sigma_{u,a,v}$ is an isometry of M defined by

$$\sigma_{u,a,v}(x) := x + \langle v, x \rangle u - \overline{\lambda} \langle u, x \rangle v - \overline{\lambda} \langle u, x \rangle a u$$

where $u, v \in M$ and $a \in A$ satisfy

$$\begin{aligned} \mu(u) &= 0 \in A/\Lambda \\ \mu(v) &= [a] \in A/\Lambda \\ \langle u, v \rangle &= 0 \in A. \end{aligned}$$

The following lemmas involve, for any finitely generated projective A-module $P = P^{**}$, a nonsingular $(+1)$-quadratic form over A called the **hyperbolic construction**

$$\begin{aligned} \mathscr{H}(P) &:= (P \oplus P^*, \langle \cdot, \cdot \rangle, \mu); \\ \langle x+f, y+g \rangle &:= f(y) + \overline{g(x)} \\ \mu(x+f) &:= [f(x)]. \end{aligned}$$

Topologically, $\mathscr{H}(A)$ is the equivariant intersection form of $S^2 \times S^2$ with coefficients in A.

LEMMA 8.1.6. *Consider a compact connected* TOP *4-manifold X with good fundamental group π and orientation character $\omega : \pi \to \mathbf{C}_2$. Define a ring with involution*

$$A := \mathbf{Z}[\pi^\omega].$$

Suppose that there is an orthogonal decomposition

$$K := \operatorname{Ker} w_2(X) = V_0 \perp V_1$$

as quadratic submodules of the intersection form of X over A, with a nonsingular restriction to V_0.

Define a homology class and a free A-module

$$\begin{aligned} p_+ &:= [S^2_+ \times \mathrm{pt}] \\ P_+ &:= A p_+. \end{aligned}$$

Consider the summand

$$\mathscr{H}(P_+) = H_2(S^2_+ \times S^2; A)$$

of

$$H_2(X \# (S^2_+ \times S^2) \# (S^2_- \times S^2); A).$$

Then for any transvection $\sigma_{p,a,v}$ on the quadratic module $K \perp \mathscr{H}(P_+)$ with $p \in V_0 \oplus P_+$ and $v \in K$, the stabilized isometry $\sigma_{p,a,v} \oplus \mathbf{1}_{H_2(2(S^2 \times S^2);A)}$ can be realized by a self-homeomorphism of $X \# 3(S^2 \times S^2)$ which restricts to the identity on ∂X.

REMARK 8.1.7. In the case that ∂X is empty and $\pi_1(X)$ is finite, then Lemma 8.1.6 is exactly [**HK93**, Corollary 2.3]. Although it turns out that their proof works in our generality, we include a full exposition, providing details absent from Hambleton-Kreck [**HK93**].

LEMMA 8.1.8. *Suppose X and p satisfy the hypotheses of Lemma 8.1.6. If p is unimodular in $V_0 \oplus P_+$, then the summand $X_1 = X\#(S^2_+ \times S^2)$ of X_2 can be topologically re-split so that $S^2 \times \mathrm{pt}$ represents p.*

PROOF. Since $V_0 \perp \mathscr{H}(P_+)$ is nonsingular, there exists an element $q \in V_0 \perp \mathscr{H}(P_+)$ such that (p, q) is a hyperbolic pair. Since $p, q \in \operatorname{Ker} w_2(X_1)$ and w_2 is the sole obstruction to framing the normal bundle in the universal cover, each homology class is represented by a canonical regular homotopy class of framed immersion

$$\alpha, \beta : S^2 \times \mathbf{R}^2 \longrightarrow X_1$$

with transverse double-points. Then, since the self-intersection number of α vanishes, all its double-points pair to yield framed immersed Whitney disks; we consider each disk separately:

$$W : D^2 \times \mathbf{R}^2 \longrightarrow X_1.$$

Upon performing finger-moves to regularly homotope W, we may assume that one component of

$$\alpha(S^2 \times 0) \setminus W(\partial D^2 \times \mathbf{R}^2)$$

is a framed embedded disk

$$V : D^2 \times \mathbf{R}^2 \longrightarrow X_1$$

and, by an arbitrarily small regular homotopy of β, that $\beta | S^2 \times 0$ is transverse to $W|\operatorname{int} D^2 \times 0$ with algebraic intersection number 1 in $\mathbf{Z}[\pi_1(X_1)]$. Hence W is a framed properly immersed disk in

$$\overline{X}_1 := X_1 \setminus \operatorname{Im} V.$$

So, since $\pi_1(\overline{X}_1) \cong \pi_1(X)$ is a good group, by Freedman's disk theorem [**FQ90**, Thm. 5.1A], there exists a framed properly TOP embedded disk

$$W' : D^2 \times \mathbf{R}^2 \longrightarrow \overline{X}_1$$

such that

$$W' = W \text{ on } \partial D^2 \times \mathbf{R}^2 \quad \text{and} \quad \operatorname{Im} W' \subset \operatorname{Im} W.$$

Therefore, by performing a Whitney move along W', we obtain that α is regularly homotopic to a framed immersion with one fewer pair of self-intersection points. Thus α is regularly homotopic to a framed TOP embedding α'. A similar argument, allowing an arbitrarily small regular homotopy of α', shows that β is regularly homotopic to a framed TOP embedding β' transverse to α', with a single intersection point

$$\alpha'(x_0 \times 0) = \beta'(y_0 \times 0)$$

such that the open disk

$$\Delta := \beta'(y_0 \times \mathbf{R}^2) \subset \alpha'(S^2 \times 0).$$

Define a closed disk

$$\Delta' := S^2 \setminus (\alpha')^{-1}(\Delta).$$

Surgery on X_1 along β' yields a compact connected TOP 4-manifold X'. Hence X_1 is recovered by surgery on X' along the framed embedded circle

$$\gamma : S^1 \times \mathbf{R}^3 \approx \operatorname{nbhd}_{S^2}(\partial \Delta') \times \mathbf{R}^2 \xrightarrow{\alpha'} X_1 \setminus \operatorname{Im} \beta' \subset X'.$$

But the circle γ is trivial in X', since it extends via α' to a framed embedding of the disk Δ' in X'. Therefore we obtain a TOP re-splitting of the connected sum

$$X_1 \approx X' \# (S^2 \times S^2)$$

so that $S^2 \times \mathrm{pt}$ of the right-hand side represents the image of p. $\square$

The next algebraic lemma decomposes certain transvections so that the pieces fit into the previous topological lemma.

LEMMA 8.1.9. *Suppose (A, λ, Λ) is a unitary ring such that: the additive monoid of A is generated by a subset S of the unit group $(A^\times, \cdot)$. Let $K = V_0 \perp V_1$ be a quadratic module over (A, λ, Λ) with a nonsingular restriction to V_0, and let $P_\pm$ be free left A-modules of rank one. Then any stabilized transvection*

$$\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)} \quad on \quad K \perp \mathscr{H}(P_+) \perp \mathscr{H}(P_-)$$

with $p \in V_0 \oplus P_+$ and $v \in K$ is a composite of transvections $\sigma_{p_i,0,v_j}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_j \in K \oplus \mathscr{H}(P_-)$.

PROOF. Using a symplectic basis $\{p_\pm, q_\pm\}$ of each hyperbolic plane $\mathscr{H}(P_\pm)$, define elements of $K \oplus \mathscr{H}(P_+ \oplus P_-)$:

$$\begin{aligned} v_0 &:= v + p_- - aq_- \\ v_1 &:= -p_- \\ v_2 &:= aq_-. \end{aligned}$$

Then

$$v = \sum_{i=0}^{2} v_i.$$

Observe that each $v_i \in K \oplus \mathscr{H}(P_-)$ is isotropic with $\langle v_i, p \rangle = 0$. So transvections $\sigma_{p,0,v_j}$ are defined. Note, by Definition 8.1.5, for all $x \in K \oplus \mathscr{H}(P_+ \oplus P_-)$, that

$$\begin{aligned} (\sigma_{p,0,v_2} \circ \sigma_{p,0,v_1} \circ \sigma_{p,0,v_0})(x) &= x + \sum_i \langle v_i, x \rangle p - \sum_i \overline{\lambda} \langle p, x \rangle v_i - \overline{\lambda} \langle p, x \rangle \sum_{i<j} \langle v_j, v_i \rangle p \\ &= x + \langle v, x \rangle p - \overline{\lambda} \langle p, x \rangle v - \overline{\lambda} \langle p, x \rangle ap \\ &= \sigma_{p,a,v}(x) \oplus \mathbf{1}_{\mathscr{H}(P_-)}. \end{aligned}$$

Therefore it suffices to consider the case that $v \in K \oplus \mathscr{H}(P_-)$ is isotropic. Write

$$p = p' \oplus p'' \in V_0 \oplus P_+.$$

Define a unimodular element

$$p_0 := p' \oplus 1p_+.$$

Note, since P_+ has rank one and by hypothesis, there exist $n \in \mathbf{Z}_{\geq 0}$ and unimodular elements $p_1, \ldots, p_n \in Sp_+ \subseteq P_+$ such that

$$p - p_0 = p'' - 1p_+ = \sum_{i=1}^{n} p_i.$$

For each $1 \leq i \leq n$, write

$$p_i := s_i p_+ \quad \text{for some} \quad s_i \in S.$$

Observe for all $1 \leq i, j \leq n$ that

$$\begin{aligned}
\langle v, p_i \rangle &= 0 \\
\mu(p_i) &= s_i \mu(p_+) \overline{s_i} \\
&= 0 \\
\langle p_i, p_j \rangle &= s_i \langle p_+, p_+ \rangle \overline{s_j} \\
&= 0.
\end{aligned}$$

Hence, we also have

$$\begin{aligned}
\langle v, p_0 \rangle &= 0 \\
\mu(p_0) &= 0.
\end{aligned}$$

Then transvections $\sigma_{p_i,0,v}$ are defined and commute, so note

$$\sigma_{p,0,v} = \prod_{i=0}^{n} \sigma_{p_i,0,v}.$$

□

Proof of Lemma 8.1.6. Define a homology class and a free A-module

$$\begin{aligned}
p_- &:= [S^2_- \times \mathrm{pt}] \\
P_- &:= Ap_-.
\end{aligned}$$

Consider the A-module decomposition

$$H_2(X_2; A) = H_2(X; A) \oplus \mathscr{H}(P_+) \oplus \mathscr{H}(P_-).$$

Observe that the unitary ring

$$(A, \lambda, \Lambda) = (\mathbf{Z}[\pi^\omega], +1, \{\, a - \overline{a} \mid a \in A \,\})$$

satisfies the hypothesis of Lemma 8.1.9 with the multiplicative subset

$$S = \pi \cup -\pi.$$

Therefore the stabilized transvection

$$\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)}$$

is a composite of transvections $\sigma_{p_i,0,v_j}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_j \in K \oplus \mathscr{H}(P_-)$. Then by Lemma 8.1.8, for each i, a TOP re-splitting

$$f_i : X_1 \approx X' \#(S^2 \times S^2)$$

of the connected sum can be chosen so that $S^2 \times \mathrm{pt}$ represents p_i. So by the Cappell-Shaneson realization theorem [**CS71**, Thm. 1.5][3], for each i and j, the pullback under $(f_i)_*$ of the stabilized transvection

$$\sigma_{p_i,0,v_j} \oplus \mathbf{1}_{H_2(S^2\times S^2;A)} = \sigma_{p_i\oplus 0,0,v_j\oplus 0}$$

is an isometry induced by a self-diffeomorphism of

$$(X'\#(S^2 \times S^2))\#(S^2 \times S^2).$$

Hence, by conjugation with the homeomorphism f_i, the above isometry is induced by a self-homeomorphism of

$$X_2 = X_1\#(S^2 \times S^2).$$

Thus the stabilized transvection

$$(\sigma_{p,a,v} \oplus \mathbf{1}_{\mathscr{H}(P_-)}) \oplus \mathbf{1}_{H_2(S^2\times S^2;A)}$$

is induced by the stabilized composite self-homeomorphism of

$$X_3 = X_2\#(S^2 \times S^2).$$

$\square$

[3]More precisely, their theorem realizes any transvection of the form $\sigma_{p_+,a,v}$ by a diffeomorphism of the 1-stabilization.

8.1.3. Proof of main theorem. Now we modify the induction of [**HK93**, Proof B]; our result will be one $S^2 \times S^2$ connected-summand less efficient than Hambleton-Kreck [**HK93**] in the case that π is finite. The main algebraic technique is a theorem of Bass [**Bas73**, Thm. IV.3.4] on the transitivity of a certain subgroup of isometries on the set of hyperbolic planes. We refer the reader to [**Bas73**, §IV.3] for the terminology used in our proof. The main topological technique is a certain clutching construction of an s-cobordism.

PROOF OF THEOREM 8.1.1. We may assume $r \geq d+1$. Let

$$f : X\#r(S^2 \times S^2) \longrightarrow Y\#r(S^2 \times S^2)$$

be a homeomorphism. We show that

$$\overline{X} := X\#(r-1)(S^2 \times S^2)$$

is homeomorphic to

$$\overline{Y} := Y\#(r-1)(S^2 \times S^2),$$

thus the result follows by backwards induction on r.

Consider Definition 8.1.13 and [**Bas73**, Hypotheses IV.3.1]. By hypothesis and Lemma 8.1.14, the minimal form parameter

$$\Lambda := \{\, a - \overline{a} \mid a \in A \,\}$$

makes (A, Λ) a quasi-finite unitary $(R, +1)$-algebra. Note, since

$$X = X_{-1}\#((S')^2 \times S^2)$$

by hypothesis, that the rank $r+1$ free A-module summand

$$P := H_2\left((S')^2 \times \mathrm{pt} \sqcup r(S^2 \times \mathrm{pt}); A\right)$$

of

$$\operatorname{Ker} w_2(\overline{X}\#(S^2 \times S^2))$$

satisfies [**Bas73**, Case IV.3.2.a]. Then, by [**Bas73**, Theorem IV.3.4], the subgroup G of the group $U(\mathscr{H}(P))$ of unitary automorphisms defined by

$$G := \langle\, \mathscr{H}(E(P)),\ EU(\mathscr{H}(P))\, \rangle$$

acts transitively on the set of hyperbolic pairs in $\mathscr{H}(P)$. So, by [**Bas73**, Corollary IV.3.5] applied to the quadratic module

$$V := \operatorname{Ker} w_2(X_{-1}),$$

the subgroup G_1 of $U(V \perp \mathscr{H}(P))$ defined by

$$G_1 := \left\langle\, \mathbf{1}_V \perp G,\ EU(\mathscr{H}(P), P; V),\ EU(\mathscr{H}(P), \overline{P}; V) \,\right\rangle$$

acts transitively on the set of hyperbolic pairs in $V \perp \mathscr{H}(P)$. Let

$$(p_0, q_0) \quad \text{and} \quad (p_0', q_0')$$

be the standard basis of the summand $H_2(S^2 \times S^2; A)$ of

$$H_2(\overline{X}\#(S^2 \times S^2); A) \quad \text{and} \quad H_2(\overline{Y}\#(S^2 \times S^2); A).$$

Therefore there exists an isometry $\varphi \in G_1$ of

$$V \perp \mathscr{H}(P) = \operatorname{Ker} w_2(\overline{X}\#(S^2 \times S^2))$$

such that

$$\varphi(p_0, q_0) = (f_*)^{-1}(p_0', q_0').$$

LEMMA 8.1.10. *The isometry*

$$\varphi \oplus \mathbf{1}_{H_2(3(S^2 \times S^2); A)}$$

is induced by a self-homeomorphism g of

$$\overline{X}\#4(S^2 \times S^2).$$

Then the homeomorphism

$$h := (f\#\mathbf{1}_{3(S^2 \times S^2)}) \circ g : \overline{X}\#4(S^2 \times S^2) \longrightarrow \overline{Y}\#4(S^2 \times S^2)$$

satisfies the equation

$$h_*(p_i, q_i) = (p_i', q_i') \quad \text{for all} \quad 0 \le i \le 3.$$

Here the hyperbolic pairs

$$\{\, (p_i, q_i) \,\}_{i=1}^3 \quad \text{and} \quad \{\, (p_i', q_i') \,\}_{i=1}^3$$

in the last three $S^2 \times S^2$ summands are defined similarly to (p_0, q_0) and (p_0', q_0').

LEMMA 8.1.11. *The following manifold triad* $(W; \overline{X}, \overline{Y})$ *is a compact* TOP *s-cobordism* rel ∂X*:*

$$W^5 := \overline{X} \times [0,1] \natural 4(S^2 \times D^3) \bigcup_h \overline{Y} \times [0,1] \natural 4(S^2 \times D^3).$$

Therefore, since $\pi_1(\overline{X}) \cong \pi_1(X)$ is a good group, by the TOP s-cobordism theorem [**FQ90**, Thm. 7.1A], we obtain that $\overline{X}$ is homeomorphic to $\overline{Y}$. This proves the theorem by induction on r. □

REMARK 8.1.12. The reason for restriction to the A-submodule

$$K = \operatorname{Ker} w_2(\overline{X}\#(S^2 \times S^2))$$

is two-fold. Geometrically [**CS71**, p. 504], a unique quadratic refinement of the intersection form exists on K, hence K is maximal. Also, the inverse image of (p_0', q_0') under the isometry f_* is guaranteed to be a hyperbolic pair in K, hence K is simultaneously minimal.

8.1.4. Remaining lemmas and proofs.

DEFINITION 8.1.13 ([**Bas73**, IV.1.3]). An R_0-algebra A is **quasi-finite** if, for each maximal ideal $\mathfrak{m} \in \operatorname{maxspec}(R_0)$, the following containment holds:

$$\mathfrak{m}A_{\mathfrak{m}} \subseteq \operatorname{rad} A_{\mathfrak{m}}$$

and that the following ring is left artinian:

$$A[\mathfrak{m}] := A_{\mathfrak{m}}/\operatorname{rad} A_{\mathfrak{m}}.$$

Here

$$A_{\mathfrak{m}} := (R_0)_{\mathfrak{m}} \otimes_{R_0} A$$

is the localization of A at $\mathfrak{m}$, and $\operatorname{rad} A_{\mathfrak{m}}$ is its Jacobson radical. The pair (A, Λ) is a **quasi-finite unitary** (R, λ)**-algebra** if (A, λ, Λ) is a unitary ring, A is an R-algebra with involution, and A is a quasi-finite R_0-algebra. Here R_0 is the commutative subring of R generated by **norms**:

$$R_0 = \left\{ \sum_i r_i \overline{r}_i \;\middle|\; r_i \in R \right\}.$$

LEMMA 8.1.14. *Suppose A is an algebra over a ring R_0 such that A is a finitely generated left R_0-module. Then A is a quasi-finite R_0-algebra.*

PROOF. Let $\mathfrak{m} \in \mathrm{maxspec}(R_0)$. By [**Bas68**, Corollary III.2.5] to Nakayama's lemma, we have

$$A_{\mathfrak{m}} \cdot \mathfrak{m} = A_{\mathfrak{m}} \cdot \mathrm{rad}\,(R_0)_{\mathfrak{m}} \subseteq \mathrm{rad}\, A_{\mathfrak{m}}.$$

Then

$$A[\mathfrak{m}] = (A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}})/\left((\mathrm{rad}\, A_{\mathfrak{m}})/\mathfrak{m}A_{\mathfrak{m}}\right)$$

and is a finitely generated module over the field

$$(R_0)_{\mathfrak{m}}/\mathfrak{m}(R_0)_{\mathfrak{m}},$$

by hypothesis. Therefore $A[\mathfrak{m}]$ is left artinian, hence A is quasi-finite. $\square$

The existence of the realization g is proven algebraically–we use the terminology of Hyman Bass [**Bas73**, §II.3].

PROOF OF LEMMA 8.1.10. Consider Lemma 8.1.6 applied to

$$\overline{X}\#(S^2 \times S^2) \qquad V_0 = \mathscr{H}(P) \qquad V_1 = V.$$

It suffices to show that the group G_1 is generated by a subset of the transvections $\sigma_{p,a,v}$ with $p \in \mathscr{H}(P)$ and $v \in V \oplus \mathscr{H}(P)$.

By [**Bas73**, Cases II.3.10.1–2], the group

$$EU(\mathscr{H}(P))$$

is generated by all transvections $\sigma_{u,a,v}$ with $u, v \in \overline{P}$ or $u, v \in P$. By [**Bas73**, Case II.3.10.3], the group

$$\mathscr{H}(E(P))$$

is generated by a subset of the transvections $\sigma_{u,a,v}$ with $u \in P, v \in \overline{P}$ or $u \in \overline{P}, v \in P$. By [**HK93**, Definition 1.4], the group

$$EU(\mathscr{H}(P), P; V)$$

is generated by all transvections $\sigma_{u,a,v}$ with $u \in P, v \in V$, and the group

$$EU(\mathscr{H}(P), \overline{P}; V)$$

is generated by all transvections $\sigma_{u,a,v}$ with $u \in \overline{P}, v \in V$. In any case, $p \in \mathscr{H}(P)$ and $v \in V \oplus \mathscr{H}(P)$. $\square$

The assertion is essentially that $(W; \overline{X}, \overline{Y})$ is an h-cobordism $\mathrm{rel}\, \partial X$ with zero Whitehead torsion.

Proof of Lemma 8.1.11. Observe the following diagram is a pushout, by the Seifert-VanKampen theorem:

$$\begin{tikzcd} \pi_1\left(\overline{X} \times 1 \# 4(S^2 \times S^2)\right) \arrow[r, "h_*", "\cong"'] \arrow[d, "\mathbf{1}"'] & \pi_1\left(\overline{Y} \times [0,1] \natural 4(S^2 \times D^3)\right) \arrow[d] \\ \pi_1\left(\overline{X} \times [0,1] \natural 4(S^2 \times D^3)\right) \arrow[r] & \pi_1(W). \end{tikzcd}$$

So the maps induced by the inclusion $\overline{X} \sqcup \overline{Y} \to W$ are isomorphisms:

$$\begin{aligned} i_* &: \pi_1(\overline{X} \times 0) \longrightarrow \pi_1(W) \\ j_* &: \pi_1(\overline{Y} \times 0) \longrightarrow \pi_1(W). \end{aligned}$$

Denote π as the common fundamental group using these identifications.

Observe that the nontrivial boundary map ∂_3 of the cellular chain complex

$$C_*(j; \mathbf{Z}[\pi]) \ : \ 0 \longrightarrow \bigoplus_{0 \le k < 4} \mathbf{Z}[\pi] \cdot (S^2 \times D^3) \xrightarrow{h_\# \circ \partial} \bigoplus_{0 \le l < 4} \mathbf{Z}[\pi] \cdot (D^2 \times S^2) \longrightarrow 0$$

is obtained as follows. First, attach thickened 2-cells to kill 4 copies of the trivial circle in $\overline{Y}$. Then, onto the resultant manifold

$$\overline{Y} \# 4(S^2 \times S^2),$$

attach thickened 3-cells to kill certain belt 2-spheres, which are the images under h of the normal 2-spheres to the 4 copies of the trivial circle in $\overline{X}$. Hence, as morphisms of based left $\mathbf{Z}[\pi]$-modules, the boundary map

$$\partial_3 = h_\# \circ \partial$$

is canonically identified with the morphism

$$h_* = \mathbf{1} : H_2(4(S^2 \times S^2); \mathbf{Z}[\pi]) \longrightarrow H_2(4(S^2 \times S^2); \mathbf{Z}[\pi])$$

on homology induced by the attaching map h. This last equality holds by the construction of h, since

$$h_*(p_i, q_i) = (p_i', q_i') \quad \text{for all} \quad 0 \leq i < 4.$$

So the inclusion $j : \overline{Y} \to W$ has torsion

$$\begin{aligned} \tau(C_*(j; \mathbf{Z}[\pi])) &= [h_\#] \\ &= [h_*] \\ &= [\mathbf{1}] \\ &= 0 \in \mathrm{Wh}(\pi). \end{aligned}$$

A similar argument using h^{-1} shows that the inclusion $i : \overline{X} \to W$ has zero torsion in $\mathrm{Wh}(\pi)$. Therefore $(W; \overline{X}, \overline{Y})$ is a compact TOP s-cobordism $\mathrm{rel}\, \partial X$. $\square$

The last proof uses the basic language of algebraic geometry: spec and maxspec of commutative rings.

Proof of Proposition 8.1.2. Recall from Proposition 5.1.2 that $\mathbf{D}_\infty$ is a good group. Define rings with involution

$$\begin{aligned} A &:= \mathbf{Z}[\mathbf{D}_\infty^\omega] \\ R &:= \mathbf{Z}[t + t^{-1}] \\ R_0 &:= \mathbf{Z}[u]. \end{aligned}$$

Here, the following elements are transcendental over $\mathbf{Z}$:

$$\begin{aligned} t &:= ab \in \mathbf{D}_\infty \\ u &:= \omega(t)(t + t^{-1})^2 \in R_0. \end{aligned}$$

Observe that R is the center of A with norm subring R_0. Then, by a theorem of Serre [**Bas68**, Thm. III.3.1], we obtain

$$\dim(\mathrm{spec}\, \mathbf{Z}[u]) = \dim(\mathrm{spec}\, \mathbf{Z}) + 1.$$

Note $\dim(\operatorname{spec} \mathbf{Z}) = 1$, since $\mathbf{Z}$ is a euclidean domain and not a field. It remains to show that the noetherian topological space $\mathfrak{P}$ and its subspace $\mathfrak{M}$ defined by

$$\begin{aligned} \mathfrak{P} &:= \operatorname{spec}(\mathbf{Z}[u]) \\ \mathfrak{M} &:= \operatorname{maxspec}(\mathbf{Z}[u]) \end{aligned}$$

have equal dimension.

More generally, if R_0 is a Jacobson ring, such as the above polynomial ring $\mathbf{Z}[u]$, then

$$\mathrm{ClosedSets}(\mathfrak{P}) \longrightarrow \mathrm{ClosedSets}(\mathfrak{M}); \qquad C \longmapsto C \cap \mathfrak{M}$$

is a lattice isomorphism, whose inverse is given by

$$D \longmapsto \mathrm{closure}_{\mathfrak{P}}(D).$$

This follows from [**Gro66**, §IV.10: Prop. 1.2.c'; Déf. 1.3, 3.1, 4.1; Cor. 4.6]. Hence $\dim(\mathfrak{M}) = \dim(\mathfrak{P})$ as desired. $\square$

8.2. TOP manifolds in the homotopy type of $\mathbf{RP}^4 \# \mathbf{RP}^4$

Given a tangential homotopy equivalence to a certain type of topological 4-manifold, the main goal of this section is to uniformly quantify the amount of topological stabilization sufficient for smoothing and for splitting along a two-sided 3-sphere. In particular, we sharpen a result of Jahren-Kwasik [**JK06**, Thm. 1(f)] on the connected sum of real projective 4-spaces (8.2.5).

Let X be a compact connected DIFF 4-manifold, and write

$$(\pi, \omega) := (\pi_1(X), w_1(X)).$$

Suppose π is good [**FQ90**]. Let $\vartheta \in L_5^s(\mathbf{Z}[\pi^\omega])$; represent it by a simple unitary automorphism of the orthogonal sum of r copies of the hyperbolic plane for some $r \geq 0$. Recall [**FQ90**, Ch. 11] that there exists a unique homeomorphism class

$$(X_\vartheta, h_\vartheta) \in \mathcal{S}^s_{\mathrm{TOP}}(X, \partial X)$$

as follows. It consists of a compact TOP 4-manifold X_ϑ and a simple homotopy equivalence $h_\vartheta : X_\vartheta \to X$ that restricts to a homeomorphism $h : \partial X_\vartheta \to \partial X$ on the boundary, such that there exists a normal bordism rel ∂X from h_ϑ to $\mathbf{1}_X$ with surgery obstruction ϑ. Such a homotopy equivalence is called **tangential**.

Theorem 8.2.1. *The following r-stabilization admits a* DIFF *structure:*

$$X_\vartheta \# r(S^2 \times S^2).$$

Furthermore, there exists a TOP *normal bordism between h_ϑ and $\mathbf{1}_X$ with surgery obstruction $\vartheta \in L_5^s(\mathbf{Z}[\pi^\omega])$, such that it consists of exactly $2r$ many 2-handles and $2r$ many 3-handles. In particular X_ϑ is $2r$-stably homeomorphic to X.*

Proof. The existence and uniqueness of $(X_\vartheta, h_\vartheta)$ follow from [**FQ90**, Theorems 11.3A, 11.1A, 7.1A]. But by [**CS71**, Theorem 3.1], there exists a DIFF s-bordism class of

$$(X_\alpha, h_\alpha)$$

uniquely determined as follows. Given a rank r representative α of the isometry class ϑ, this pair (X_α, h_α) consists of a compact DIFF 4-manifold X_α and a simple homotopy equivalence h_α that restricts to a diffeomorphism on the boundary:

$$\begin{aligned} h_\alpha \quad &: \quad (X_\alpha, \partial X_\alpha) \longrightarrow (X_r, \partial X) \\ X_r \quad &:= \quad X \# r(S^2 \times S^2). \end{aligned}$$

It is obtained from a DIFF normal bordism rel ∂X

$$(W_\alpha, H_\alpha)$$

from h_α to $\mathbf{1}_{X_r}$ with of surgery obstruction ϑ, constructed with exactly r many 2-handles and r many 3-handles, and clutched along a diffeomorphism which induces the simple unitary automorphism α on the surgery kernel

$$K_2(W_\alpha) = \mathscr{H}\left(\bigoplus_r \mathbf{Z}[\pi]\right).$$

This is rather the consequence, and not the construction[4] itself, of Wall realization [**Wal99**, Thm. 6.5] in high odd dimensions.

[4] In the DIFF 4-dimensional case, embeddings are chosen, via a self-diffeomorphism φ inducing α, within certain regular homotopy class of framed immersions of 2-spheres. Cappell and Shaneson [**CS71**, Thm. 1.5] cleverly construct φ using a circle isotopy theorem of Whitney.

By uniqueness in the simple TOP structure set, the following simple homotopy equivalences are s-bordant:

$$h_\vartheta \# \mathbf{1}_{r(S^2 \times S^2)} \quad \text{and} \quad h_\alpha.$$

Hence they differ by pre-composition with a homeomorphism, by the s-cobordism theorem [**FQ90**, Thm. 7.1A]. In particular, the domain

$$X_\vartheta \# r(S^2 \times S^2) \quad \text{is homeomorphic to} \quad X_\alpha,$$

inheriting its DIFF structure. Therefore, post-composition of H_α with the collapse map $X_r \to X$ yields the desired normal bordism between the simple homotopy equivalences h_ϑ and $\mathbf{1}_X$, obtained by attaching $r + r$ many 2- and 3-handles. $\square$

Next, we recall Hambleton-Kreck-Teichner classification of the homemomorphism types and simple homotopy types of closed 4-manifolds with fundamental group C_2^-. Then, we shall give a partial classification of the simple homotopy types and stable homeomorphism types of their connected sums, which have fundamental group $\mathsf{D}_\infty^{-,-} = \mathsf{C}_2^- * \mathsf{C}_2^-$. In the sequel, the $*$ **star operation** [**FQ90**, §10.4] flips the Kirby-Siebenmann invariant of certain topological 4-manifolds.

THEOREM 8.2.2 ([**HKT94**, Thm. 3]). *Every closed non-orientable topological 4-manifold with fundamental group order two is homeomorphic to exactly one manifold in the following list of w_2-**types**.*

(I) *The connected sum of $*\mathsf{CP}^2$ with RP^4 or its star. The connected sum of $k \geq 1$ copies of CP^2 with RP^4 or $\mathsf{RP}^2 \times S^2$ or their stars.*

(II) *The connected sum of $k \geq 0$ copies of $S^2 \times S^2$ with $\mathsf{RP}^2 \times S^2$ or its star.*

(III) *The connected sum of $k \geq 0$ copies of $S^2 \times S^2$ with $S(\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1)$ or $\#_{S^1} r\mathsf{RP}^4$ or their stars, for unique $1 \leq r \leq 4$.*

We explain the terms in the above theorem. Firstly,

$$\mathsf{R} \longrightarrow \gamma^1 \longrightarrow \mathsf{RP}^2$$

denotes the canonical line bundle, and

$$\varepsilon^1 := \mathsf{R} \times \mathsf{RP}^2$$

denotes the trivial line bundle. Secondly,

$$S^2 \longrightarrow S(\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1) \longrightarrow \mathbf{RP}^2$$

is the sphere bundle of the Whitney sum. Finally, the **circle-connected sum**

$$M \#_{S^1} N := M \setminus E \bigcup_{\partial E} N \setminus E$$

is defined by codimension zero embeddings of E in M and N that are not null-homotopic, where E is the nontrivial bundle:

$$D^3 \longrightarrow E \longrightarrow S^1.$$

COROLLARY 8.2.3 ([**HKT94**, Cor. 1]). *Let M and M' be closed non-orientable topological 4-manifolds with fundamental group of order two. Then M and M' are (simple) homotopy equivalent if and only if*

(1) *M and M' have the same w_2-type,*
(2) *M and M' have the same Euler characteristic, and*
(3) *M and M' have the same Stiefel-Whitney number: $w_1^4[M] = w_1^4[M']$ mod 2;*
(4) *furthermore in w_2-type III, that M and M' have $\pm$ the same Brown-Arf invariant mod 8.*

The following theorem is the main focus of this section. The pieces M and M' are classified by Hambleton-Kreck-Teichner [**HKT94**], and the UNil-group is computed by Connolly-Davis [**CD04**]. Since $\mathbf{Z}$ is a regular coherent ring, by Waldhausen's vanishing theorem (1.3.2), we have $\widetilde{\mathrm{Nil}}_0(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-) = 0$. Hence $\mathrm{UNil}_5^s = \mathrm{UNil}_5^h$ by Definition 1.1.8.

THEOREM 8.2.4. *Let M and M' be closed non-orientable topological 4-manifolds with fundamental group of order two. Write $X = M \# M'$, and denote S as the 3-sphere defining the connected sum. Let $\vartheta \in \mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-)$.*

(1) *There exists a unique homeomorphism class $(X_\vartheta, h_\vartheta)$, consisting of a closed* TOP *4-manifold X_ϑ and a tangential homotopy equivalence $h_\vartheta : X_\vartheta \to X$, such that it has splitting obstruction*

$$\mathrm{split}_L(h_\vartheta; S) = \vartheta.$$

The function which assigns ϑ to such a $(X_\vartheta, h_\vartheta)$ is a bijection.

(2) *Furthermore,*

$$X_\vartheta\#3(S^2\times S^2) \quad \textit{is homeomorphic to} \quad X\#3(S^2\times S^2).$$

It admits a DIFF *structure if and only if X does. There exists a* TOP *normal bordism between h_ϑ and $\mathbf{1}_X$, with surgery obstruction $\vartheta \in L_5^h(\mathbf{D}_\infty^{-,-})$, such that it is composed of exactly six 2-handles and six 3-handles.*

Proof. Recall that the forgetful map

$$L_5^s(\mathbf{D}_\infty^{-,-}) \longrightarrow L_5^h(\mathbf{D}_\infty^{-,-})$$

is an isomorphism by Proposition 5.1.2. Then the existence and uniqueness of $(X_\vartheta, h_\vartheta)$ and its handle description follow from Theorem 8.2.1, using $r = d + 1 = 3$ from Proposition 8.1.2 and Proof 8.1.1. By definition (see Remarks 1.2.4 and 1.3.7), note that the following composite function is the identity on $\mathrm{UNil}_5^h(\mathbf{Z};\mathbf{Z}^-,\mathbf{Z}^-)$:

$$\vartheta \longmapsto (X_\vartheta, h_\vartheta) \longmapsto \mathrm{split}_L(h_\vartheta; S).$$

In order to show that the other composite is the identity, note that two tangential homotopy equivalences $(X_\vartheta, h_\vartheta)$ and $(X'_\vartheta, h'_\vartheta)$ with the same splitting obstruction ϑ must be homeomorphic, by freeness[2] of the UNil_5^h action on the structure set $\mathcal{S}^h_{\mathrm{TOP}}(X)$. Finally, since the 4-manifolds X_ϑ and X are 6-stably homeomorphic via the TOP normal bordism between h_ϑ and $\mathbf{1}_X$, we conclude that they are in fact 3-stably homeomorphic by Corollary 8.1.3. □

Corollary 8.2.5. *The above theorem is true for $X = \mathbf{RP}^4\#\mathbf{RP}^4$, where the 4-manifold $\mathbf{RP}^4$ has w_2-type III.* □

Remark 8.2.6. We comment on a specific aspect of the topology of X. Every homotopy automorphism of $\mathbf{RP}^4\#\mathbf{RP}^4$ is homotopic to a homeomorphism [**JK06**, Lem. 1]. Then any automorphism of the group $\mathbf{D}_\infty$ can be realized [**JK06**, Claim]. The homeomorphism classes of closed topological 4-manifolds X' in the (not necessarily tangential) homotopy type of X has been computed in [**BDK**, Theorem 2].

The classification involves the study [**BDK**, Thm. 1] of the effect of transposition of the bimodules $\mathbf{Z}^-$ and $\mathbf{Z}^-$ in the countably infinite group $\mathrm{UNil}_5^h(\mathbf{Z}; \mathbf{Z}^-, \mathbf{Z}^-)$. As promised in the introduction, Corollary 8.2.5 provides a *uniform upper bound* on the number of $S^2 \times S^2$ connected-summands sufficient for [**JK06**, Theorem 1(f)], and on the number of 2- and 3-handles sufficient for [**JK06**, Proof 1(f)].

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