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On fibering and splitting of 5-manifolds over the circle

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ABSTRACT

Our main result is a generalization of Cappell's 5-dimensional splitting theorem. As an application, we analyze, up to internal s-cobordism, the smoothable splitting and fibering problems for certain 5-manifolds mapping to the circle. For example, these maps may have homotopy fibers which are in the class of finite connected sums of certain geometric 4-manifolds. Most of these homotopy fibers have non-vanishing second mod 2 homology and have fundamental groups of exponential growth, which are not known to be tractable by Freedman–Quinn topological surgery. Indeed, our key technique is topological cobordism, which may not be the trace of surgeries.

1. Introduction

The problem of whether or not a continuous mapping $f: M \to S^1$ to the circle from a closed manifold M of dimension > 5 is homotopic to a fiber bundle projection was solved originally in the thesis of F. Thomas Farrell (cf. [10]). The sole obstruction lies in the Whitehead group of the fundamental group $\pi_1 M$ and has been reformulated in several ways [34,9,19]. Precious little is known about the 5-dimensional fibering problem. The purpose of this paper is to provide more information using recent advances in rigidity. Our approach here blends together the systematic viewpoint of high-dimensional surgery theory and the more ad-hoc vanishing results known for certain geometric 4-manifolds.

First, we extend some surgery theory. The central theorem of this paper is a generalization of the Cappell–Weinberger theorem [3,39] for splitting compact 5-manifolds along certain incompressible, two-sided 4-submanifolds (Theorem 4.1). Indeed, the development of additional tools for our main splitting theorem motivated the author's initial investigation of 4-manifolds [21].

Then, we attack the fibering problem. A first application is a version of the Farrell fibering theorem for smooth s-block bundles (Definition 5.1, Theorem 5.8) over the circle S^1 with homotopy fiber $\mathbb{RP}^4$ (1.1); compare [8,13,18]. The more central geometric applications are to topological s-block bundles (Theorem 5.6). Namely, we allow the fibers to be compact, orientable 4-manifolds whose interiors admit a complete, finite volume metrics of euclidean, real hyperbolic, or complex hyperbolic type (1.2). Moreover, we allow the fiber to be a finite connected sum of orientable surface bundles over surfaces of positive genus, and of H-bundles over the circle S^1 such that the compact irreducible 3-manifold H either is S^3 or D^3, or is orientable with non-zero first Betti number (hence Haken), or has complete, finite volume hyperbolic interior (1.4). The hypotheses require smoothness of the total space and the conclusions assert smoothability of the fiber.

1.1. Examples of fibers

Our examples are chosen so that Farrell's fibering obstruction in K-theory and Cappell's splitting obstruction in L-theory vanish.

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First family of examples

These fibers are certain non-orientable, smooth 4-manifolds with fundamental group cyclic of order two [18]. Assume:

Hypothesis 1.1. Suppose Q is a non-orientable DIFF 4-manifold of the form

$$Q = Q_0 \# Q_1$$

where:

(1) $Q_0 = \# r(S^2 \times S^2)$ for some $r \geqslant 0$, and
(2) $Q_1 = S^2 \times \mathbb{RP}^2$ or $Q_1 = S^2 \rtimes \mathbb{RP}^2$ or $Q_1 = \#_{S^1} n(\mathbb{RP}^4)$ for some $1 \leqslant n \leqslant 4$.

Second family of examples

These irreducible, possibly non-orientable fibers have torsion-free fundamental groups of exponential growth and have non-vanishing second homotopy group. Assume:

Hypothesis 1.2. Suppose S is a compact, connected DIFF 4-manifold such that:

(1) S is the total space of a DIFF fiber bundle $S^2 \to S \to \Sigma$, for some compact, connected, possibly non-orientable 2-manifold Σ of positive genus, or
(2) S is the total space of a DIFF fiber bundle $H \to S \to S^1$, for some closed, connected, hyperbolic 3-manifold H, or
(3) the interior $S - \partial S$ admits a complete, finite volume metric of euclidean, real hyperbolic, or complex hyperbolic type.

Moreover, assume $H_1(S;\mathbb{Z})$ is 2-torsionfree if S is non-orientable. Furthermore, in the fiber bundle $S \rtimes_h S^1$ (resp. $S \rtimes_\alpha S^1$) considered for types (2) and (3) in this section, assume $h : S \to S$ (resp. α) is homotopic rel ∂S to an isometry of $S - \partial S$.[1]

Remark 1.3. According to [17, Lemma 5.9], the isomorphism classes of fiber bundles $S^2 \to S \to \Sigma$ in type (1) are in bijective correspondence with the product $H^1(\Sigma;\mathbb{Z}_2) \times H^2(\Sigma;\mathbb{Z}_2)$. The orientable S^2-bundles over Σ are classified by the second factor. The isomorphism classes of fiber bundles $H \to S \to S^1$ in type (2) are in bijective correspondence with $\pi_0(\operatorname{Isom} H)$.

Third family of examples

These reducible, orientable fibers have torsion-free fundamental groups of exponential growth and have vanishing second homotopy groups. A simple example of such a fiber is $F = \# n(S^3 \times S^1)$, whose fundamental group $\pi_1(F)$ is the free group of rank n. Assume:

Hypothesis 1.4. Suppose F is an orientable DIFF 4-manifold of the form

$$F = F_1 \# \cdots \# F_n$$

for some $n > 0$, under the following conditions on the compact, connected, orientable 4-manifolds F_i. Assume:

(1) F_i is the total space of a DIFF fiber bundle $H_i \to F_i \to S^1$, for some compact, connected, orientable 3-manifold H_i such that:
   (a) H_i is S^3 or D^3, or
   (b) H_i is irreducible with non-zero first Betti number, or
(2) F_i is the total space of a DIFF fiber bundle $\Sigma_i^f \to F_i \to \Sigma_i^b$, for some compact, connected, orientable 2-manifolds Σ_i^f and Σ_i^b of positive genus.

1.2. Main results

The first splitting theorem is a specialization of the general splitting theorem (Theorem 4.1) to the mapping torus $X \rtimes_h S^1$ of a homotopy self-equivalence $h : X \to X$ for certain classes of smooth 4-manifolds X.

Theorem (5.4). *Let X be any of the* 4*-manifolds Q, S, F defined in* (1.1, 1.2, 1.4)*. Let $h : X \to X$ be a homotopy equivalence which restricts to a diffeomorphism $\partial h : \partial X \to \partial X$. Suppose M is a compact* DIFF 5*-manifold and $g : M \to X \rtimes_h S^1$ is a homotopy equivalence which restricts to a diffeomorphism $\partial g : \partial M \to \partial X \rtimes_{\partial h} S^1$.*

Then g is homotopic to a map g' which restricts to a simple homotopy equivalence $g' : X' \to X$ such that the TOP *inverse image $X' := (g')^{-1}(X)$ is homeomorphic to X and the exterior M' of X' in M is a smoothable* TOP *self s-cobordism of X.*

[1] This hypothesis is required since Mostow rigidity fails for product geometries: the $\mathbb{E}^4$-manifold $T^2 \times T^2$ has monodromies made from non-conformal elements of $\pi_0 \operatorname{Homeo}(T^2) = PSL_2(\mathbb{Z})$.

In other words, cutting M along the bicollared, smoothable TOP 4-submanifold $X' := (g')^{-1}(X) \approx X$ yields a smoothable TOP s-cobordism $(M'; X, X)$ and a simple homotopy equivalence $(g'_\infty; g'_0, g'_1) : (M'; X, X) \to X \times (\Delta^1; 0, 1)$ of manifold triads such that $g'_1 = \alpha \circ g'_0$. Be aware that the existence of a smooth structure on $X' \approx X$ does not imply that X' is a DIFF submanifold of M.

The second splitting theorem connects homotopy structures on mapping tori to smoothable s-cobordisms, homotopy self-equivalences, and the smoothing invariant.

Theorem (5.5). *Let X be any of the* 4*-manifolds Q, S, F defined in* (1.1, 1.2, 1.4)*. Let $\alpha : X \to X$ be a diffeomorphism. Then there is an exact sequence of based sets*:

$$\pi_1^\alpha\big(\widetilde{\mathbb{S}}^s_{\mathrm{TOP}+}(X), \widetilde{\mathrm{G}}^s(X)\big) \xrightarrow{\cup} \mathcal{S}^h_{\mathrm{TOP}}\big(X \rtimes_\alpha S^1\big) \xrightarrow{\mathrm{ks}} \mathbb{F}_2 \oplus H_1(X; \mathbb{F}_2)_\alpha.$$

Our fibering theorem is proven using a key strategy of Tom Farrell [9]. If the smooth 4-manifold X is closed and simply-connected, the analogous theorem was proven by J. Shaneson [33, Thm. 5.1]. We do not assume ∂X is connected.

Theorem (5.6). *Let X be any of the* 4*-manifolds Q, S, F defined in* (1.1, 1.2, 1.4)*. Let M be a* DIFF 5*-manifold, and let $f : M \to S^1$ be a continuous map. Suppose $\partial X \to \partial M \xrightarrow{\partial f} S^1$ is a* DIFF *fiber bundle and the homotopy equivalence $\partial X \to \mathrm{hofiber}(\partial f)$ extends to a homotopy equivalence $X \to \mathrm{hofiber}(f)$. Then $f : M \to S^1$ is homotopic* rel ∂M *to the projection of a smoothable* TOP *s-block bundle with fiber X.*

Remark. Let X be an aspherical, compact, orientable DIFF 4-manifold with fundamental group π. Suppose the non-connective L-theory assembly map $H_n(\pi; \mathbb{L}.^h) \to L^h_n(\mathbb{Z}[\pi])$ is an isomorphism for $n = 4, 5$. Then the general splitting and fibering theorems (4.1, 5.6) hold for X, with the inclusion of the standard high-dimensional algebraic K- and L-theory obstructions.

1.3. Techniques

Our methods employ geometric topology: topological transversality in all dimensions (Freedman and Quinn [12]) and the prototype of a nilpotent normal cobordism construction for smooth 5-manifolds (Cappell [7,3]). Our hypotheses are algebraic-topological in nature and come from the surgery characteristic class formulas of Sullivan–Wall [38] and from the assembly map components of Taylor–Williams [35]. For the main application, the difficulty is showing that vanishing of algebraic K- and L-theory obstructions is *sufficient* for a solution to the topological fibering problem as an s-block bundle over the circle.

The reader should be aware that the topological transversality used in Section 2 produces 5-dimensional TOP normal bordisms $W \to X \times \Delta^1$ which may not be smoothable, although $\partial W = \partial_- W \cup \partial_+ W$ is smoothable. In particular, W may not admit a TOP handlebody structure relative to $\partial_- W$. Hence W may not be the trace of surgeries on topologically embedded 2-spheres in X. Therefore, W may not be produced by Freedman–Quinn surgery theory, which is developed only for fundamental groups $\pi_1(X)$ of class SA, containing subexponential growth [22].

2. Five-dimensional assembly on 4-manifolds

Let $(X, \partial X)$ be a based, compact, connected, TOP 4-manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(X) : \pi \to \mathbb{Z}^\times$. Recall, for any $\alpha \in \pi$ and $\beta \in \pi_2(X)$, that there is a Whitehead product $[\alpha, \beta] \in \pi_2(X)$ which vanishes if and only if the loop α acts trivially on β. The π-**coinvariants** are the abelian group quotient $\pi_2(X)_\pi := \pi_2(X)/\langle [\alpha, \beta] \mid \alpha \in \pi,\ \beta \in \pi_2(X)\rangle$.

Hypothesis 2.1. Suppose that the following homomorphism is surjective:

$$(\, I_1 \quad \kappa_3 \,) : H_1\big(\pi; \mathbb{Z}^\omega\big) \oplus H_3(\pi; \mathbb{Z}_2) \longrightarrow L^h_5\big(\mathbb{Z}[\pi]^\omega\big)$$

and that the following induced homomorphism is injective:

$$\mathrm{Hurewicz} : \big(\pi_2(X) \otimes \mathbb{Z}_2\big)_\pi \longrightarrow H_2(X; \mathbb{Z}_2).$$

Theorem 2.2. *Assume Hypothesis* 2.1*. Then the following surgery obstruction map is surjective*:

$$\sigma_* : \mathcal{N}_{\mathrm{TOP}}\big(X \times \Delta^1\big) \longrightarrow L^h_5\big(\mathbb{Z}[\pi]^\omega\big). \qquad (2.2.1)$$

Following Sylvain Cappell's work on the Novikov conjecture, Jonathan Hillman obtained the same conclusion under different, group-theoretic hypotheses for a square-root closed graph of certain class of groups [17, Lem. 6.9].

Proof. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}_{\mathrm{TOP}}(X \times \Delta^1) & \xrightarrow{\sigma_*} & L^h_5(\mathbb{Z}[\pi]^\omega) \\ \cap[X]_{\mathbb{L}\cdot} \downarrow \cong & & \uparrow A_\pi\langle 1\rangle \\ H_5(X; \mathrm{G/TOP}^\omega) & \xrightarrow{u_*} & H_5(\pi; \mathrm{G/TOP}^\omega). \end{array}$$

Since $A_\pi\langle 1\rangle = I_1 + \kappa_3$ is surjective and $u_1 : H_1(X; \mathbb{Z}^\omega) \to H_1(\pi; \mathbb{Z}^\omega)$ is an isomorphism, it suffices to show that $u_3 : H_3(X; \mathbb{Z}_2) \to H_3(\pi; \mathbb{Z}_2)$ is surjective.

Consider the Leray–Serre spectral sequence, with π-twisted coefficients, of the fibration $\widetilde{X} \to X \xrightarrow{u} B\pi$, where $\widetilde{X}$ is the universal cover of X. Then the map u_3 is an edge homomorphism with image subgroup $E^\infty_{3,0}$. Note

$$E^\infty_{3,0} = \mathrm{Ker}\big(d^3_{3,0} : H_3(B\pi; \mathbb{Z}_2) \longrightarrow \big(\pi_2(X) \otimes \mathbb{Z}_2\big)_\pi\big).$$

There is an exact sequence involving the associated graded groups $E^\infty_{0,2}$ and $E^\infty_{2,0}$ and inducing the classical Hopf sequence:

$$0 \to \mathrm{Cok}\big(d^3_{3,0}\big) \xrightarrow{\mathrm{Hurewicz}_*} H_2(X; \mathbb{Z}_2) \xrightarrow{u_*} H_2(B\pi; \mathbb{Z}_2) \to 0.$$

It follows from the second part of the hypothesis that the transgression $\partial = d^3_{3,0}$ is zero. Therefore $\mathrm{Im}(u_3) = E^\infty_{3,0} = H_3(\pi; \mathbb{Z}_2)$, hence σ_* is surjective. □

Some families of reducible examples X of the theorem are obtained as finite connected sums of certain compact, aspherical 4-manifolds X_i which are constructed from non-positively curved manifolds. Recall that the interior of any compact surface Σ of positive genus has the structure of a complete, finite volume, euclidean or hyperbolic 2-manifold; hence Σ is aspherical. The following corollary gives a rich source of examples, including $X = \#n(S^3 \times S^1)$, whose fundamental group is free.

Corollary 2.3. *Suppose X is a* TOP 4-*manifold of the form*

$$X = X_1 \# \cdots \# X_n$$

for some $n > 0$ and some compact, connected 4-*manifolds X_i with* (*torsion-free*) *fundamental groups Λ_i such that*:

(1) *the interior $X_i - \partial X_i$ admits a complete, finite volume metric of real or complex hyperbolic type, or*
(2) *X_i is the total space of a fiber bundle $\Sigma^f_i \to X_i \to \Sigma^b_i$, for some compact, connected* 2-*manifolds Σ^f_i and Σ^b_i of positive genus,*[2] *or*
(3) *X_i is the total space of a fiber bundle $H_i \to X_i \to S^1$, for some compact, connected, irreducible* 3-*manifold H_i such that*:
   (a) *H_i is S^3 or D^3,*
   (b) *H_i is orientable with non-zero first Betti number* (*i.e. $H^1(H_i; \mathbb{Z}) \neq 0$, e.g. the boundary ∂H_i is non-empty*)*, or*
   (c) *the interior $H_i - \partial H_i$ admits a complete, finite volume metric of hyperbolic type.*

Then the topological 5-*dimensional surgery obstruction map* (2.2.1) *is surjective. Moreover, X_i only needs to have type* (1), (2), *or* (3) *up to homotopy equivalences which respect orientation characters.*

Proof. Let $\Lambda_i := \pi_1(X_i)$ be the fundamental group of X_i with orientation character $\omega_i : \Lambda_i \to \mathbb{Z}^\times$. Consider the connective assembly map

$$A_{\Lambda_i}\langle 1\rangle = (\, I_1 \quad \kappa_3 \,) : H_5\big(B\Lambda_i; \mathrm{G/TOP}^{\omega_i}\big) \longrightarrow L^h_5\big(\mathbb{Z}[\Lambda_i]^{\omega_i}\big).$$

In order to verify Hypothesis 2.1 and apply Theorem 2.2, it suffices to show that:

(i) $\pi_2(X_i) = 0$,
(ii) $H_d(B\Lambda_i; \mathbb{Z}) = 0$ for all $d > 4$, and
(iii) the non-connective assembly map is an isomorphism:

$$A_{\Lambda_i} : H_5\big(B\Lambda_i; \mathbb{L}.^{\omega_i}\big) \longrightarrow L^h_5\big(\mathbb{Z}[\Lambda_i]^{\omega_i}\big).$$

Then $A_{\Lambda_i}\langle 1\rangle$ is an isomorphism. So, since the trivial group 1 is square-root closed in the torsion-free groups Λ_i, the UNil-groups associated to the free product $\pi = \bigstar^n_{i=1} \Lambda_i$ vanish, by [2, Corollary 4], which was proven in [3, Lemmas II.7, 8, 9].

[2] **Positive genus**: implies torsion-free; each surface Σ^f_i and Σ^b_i is a finite connected sum of at least either one torus T^2 or two real projective planes $\mathbb{RP}^2$, with arbitrary punctures. The first non-orientable example is the Klein bottle $Kl = \mathbb{RP}^2 \# \mathbb{RP}^2$.

Therefore, by the Mayer–Vietoris sequence in L-theory [2, Thm. 5(ii)], Proposition 5.9 for h-decorations, the five-lemma, and induction on n, we obtain that $A_\pi\langle 1\rangle$ is an isomorphism. Moreover, by the Mayer–Vietoris sequence in singular homology and the Hurewicz theorem applied to the universal cover $\widetilde{X}$, we obtain that $\pi_2(X)=0$.

There are three types of connected summands X_i.

Type 1. Since $X_i-\partial X_i$ is covered by $\mathbb{H}^4$ or $\mathbb{CH}^2$, we obtain X_i is aspherical. That is, the compact 4-manifold X_i is model for $B\Lambda_i$. Then (i) and (ii) are satisfied. Since $X-\partial X_i$ is complete, homogeneous, and has non-positive sectional curvatures, by [11, Proposition 0.10], condition (iii) is satisfied.

Type 2. Since the surfaces Σ_i^f and Σ_i^b are aspherical, by the homotopy fibration sequence, we obtain that the compact 4-manifold X_i is aspherical. Then (i) and (ii) are satisfied. By a result of J. Hillman [16, Lemma 6] for closed, aspherical surface bundles over surfaces, condition (iii) is satisfied. Indeed, the Mayer–Vietoris argument extends to compact, aspherical surfaces with boundary: each circle C_j in the connected-decomposition of the aspherical surface $\Sigma_i^b=F_1\#\cdots\#F_r$ generates an indivisible element in the free fundamental group of the many-punctured torus or Klein bottle F_k, hence each inclusion $\pi_1(C_j)\to\pi_1(F_k)$ of fundamental groups is square-root closed (see [5, Thm. 2.4] for detail).

Type 3. There are three types of fibers H_i.

Type 3a. Conditions (i)–(iii) are immediately satisfied.

Type 3b. Since H_i is a compact, connected, irreducible, orientable 3-manifold and $\pi_1(H_i)$ is infinite, using the Sphere Theorem of Papakyriakopoulos and the Hurewicz theorem, it can be shown that H_i is aspherical. Then X_i is aspherical, so conditions (i) and (ii) are satisfied. Since H_i is irreducible and $H_i\neq D^3$, no connected component of ∂H_i is a 2-sphere. If ∂H_i is non-empty, then it can be shown that H_i is Haken [15, Lem. 6.8]. So, by theorems of S. Roushon, the non-connective assembly map $A_{\pi_1(H_i)}$ is an isomorphism in dimensions 4 and 5: if ∂H is non-empty, this follows from [32, Theorem 1.1(1)], and if ∂H is empty, this follows from [31, Theorem 1.2]. Therefore, by the Ranicki–Shaneson sequence [27, Thm. 5.2], Proposition 5.9 for h-decorations, and the five-lemma, we obtain that condition (iii) is satisfied.

Type 3c. Since $\mathbb{H}^3$ is the universal cover of $H_i-\partial H_i$, the interior $H_i-\partial H_i$ is aspherical. But, since ∂H_i has a collar implies that $H_i-\partial H_i\hookrightarrow H_i$ is a homotopy equivalence, we obtain that H_i is also aspherical. Then X_i is aspherical, so conditions (i) and (ii) are satisfied. Since $\mathbb{H}^3$ isometrically covers $H_i-\partial H_i$, by a result of Farrell and Jones [11, Prop. 0.10], the non-connective assembly map $A_{\pi_1(H_i)}$ is an isomorphism in dimensions 4 and 5. Therefore, by the Ranicki–Shaneson sequence [27, Thm. 5.2], Proposition 5.9 for h-decorations, and the five-lemma, we obtain that condition (iii) is satisfied. □

Here is a family of non-aspherical examples X of the theorem.

Corollary 2.4. *Suppose X is a compact* TOP 4*-manifold which is homotopy equivalent to the total space of a fiber bundle $S^2\to E\to\Sigma$, for some compact, connected* 2*-manifold Σ of positive genus. Then the topological* 5*-dimensional surgery obstruction map* (2.2.1) *is surjective.*

Proof. By [17, Theorem 6.16], X is s-cobordant to E. Hence there is an induced (simple) homotopy equivalence $X\to E$ which respects orientation characters. The same methods of Corollary 2.3(2) show that $(I_1\quad\kappa_3)$ is surjective. Note that $\pi_1(E)=\pi_1(\Sigma)$ may not act trivially on $\pi_2(E)=\pi_2(S^2)=\mathbb{Z}$ but does acts trivially on $\pi_2(E)\otimes\mathbb{Z}_2=\mathbb{Z}_2$. An elementary argument with the Leray–Serre spectral sequence shows that $H_2(E;\mathbb{Z}_2)=H_2(S^2;\mathbb{Z}_2)\oplus H_2(\Sigma;\mathbb{Z}_2)$. Therefore Hurewicz is injective, and we are done by Theorem 2.2. □

The difference between DIFF and TOP for σ_* is displayed in [21, Prop. 2.1]. Later, we shall refer to a hypothesis introduced by Cappell [3, Thm. 5, Rmk.].

Hypothesis 2.5. Suppose that the following map is surjective:

$$\sigma_*:\mathcal{N}_{\mathrm{DIFF}}\big(X\times\Delta^1\big)\longrightarrow L_5^h\big(\mathbb{Z}[\pi]^\omega\big).$$

Remark 2.6. Suppose X is a DIFF 4-manifold and $\pi^\omega=(C_2)^-$. By [38, Theorem 13A.1], the following surgery obstruction map is automatically surjective:

$$\sigma_*:\mathcal{N}_{\mathrm{DIFF}}\big(X\times\Delta^1\big)\longrightarrow L_5^h\big(\mathbb{Z}[C_2]^-\big)=0.$$

Topological surjectivity fails for a connected sum $X\#X'$ of such manifolds: the Mayer–Vietoris sequence [2] shows that the cokernel is $\mathrm{UNil}_5^h(\mathbb{Z};\mathbb{Z}^-,\mathbb{Z}^-)\cong\mathrm{UNil}_3^h(\mathbb{Z};\mathbb{Z},\mathbb{Z})$, and this abelian group was shown to be infinitely generated [6].

3. Exactness at 4-dimensional normal invariants

For the convenience of the reader, we first recall the relevant hypotheses from the precursor [21, §3]. Let $(X,\partial X)$ be a based, compact, connected TOP 4-manifold with fundamental group π and orientation character $\omega:\pi\to\mathbb{Z}^\times$. Let $u:X\to B\pi$ be a based, continuous map that induces an isomorphism on fundamental groups. Denote the induced homomorphism

$$u_2:H_2(X;\mathbb{Z}_2)\longrightarrow H_2(B\pi;\mathbb{Z}_2).$$

Recall that X satisfies Poincaré duality with a unique mod 2 orientation class $[X] \in H_4(X, \partial X; \mathbb{Z}_2)$. The second Wu class $v_2(X) \in H^2(X; \mathbb{Z}_2)$ is a homomorphism

$$v_2(X) : H_2(X; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

uniquely determined, for all cohomology classes $a \in H^2(X, \partial X; \mathbb{Z}_2)$, by the formula

$$\langle v_2(X), a \cap [X] \rangle = \langle a \cup a, [X] \rangle.$$

Consider three cases for the orientation character ω below. The homomorphism

$$\kappa_2 : H_2(B\pi; \mathbb{Z}_2) \longrightarrow L_4^h(\mathbb{Z}[\pi]^\omega)$$

is the 2-dimensional component of the L-theory assembly map.

Hypothesis 3.1. Let X be orientable. Suppose that κ_2 is injective on the subgroup $u_2(\operatorname{Ker} v_2(X))$.

Hypothesis 3.2. Let X be non-orientable such that π contains an orientation-reversing element of finite order, and if CAT = DIFF, then suppose that orientation-reversing element has order two. Suppose that κ_2 is injective on all $H_2(B\pi; \mathbb{Z}_2)$, and suppose that $\operatorname{Ker}(u_2) \subseteq \operatorname{Ker}(v_2)$.

Hypothesis 3.3. Let X be non-orientable such that there exists an epimorphism $\pi^\omega \to \mathbb{Z}^-$. Suppose that κ_2 is injective on the subgroup $u_2(\operatorname{Ker} v_2(X))$.

Next, we recall the relevant results from [21, §4] used frequently in the later proofs in this paper. The subcategory TOP0 ⊂ TOP consists of those maps $f : M \to X$ with Kirby–Siebenmann stable smoothing invariant $\operatorname{ks}(f) := \operatorname{ks}(M) - \operatorname{ks}(X) = 0 \in \mathbb{Z}_2$. All structure sets and normal invariants below are relative to a diffeomorphism on ∂X.

Theorem 3.4. *Let $(X, \partial X)$ be a based, compact, connected,* CAT 4*-manifold with fundamental group $\pi = \pi_1(X)$ and orientation character $\omega = w_1(X) : \pi \to \mathbb{Z}^\times$.*

(1) *Suppose Hypothesis* 3.1 *or* 3.2. *Then the surgery sequence of based sets is exact at the smooth normal invariants*:

$$\mathcal{S}^s_{\mathrm{DIFF}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{DIFF}}(X) \xrightarrow{\sigma_*} L_4^h(\mathbb{Z}[\pi]^\omega). \tag{3.4.1}$$

(2) *Suppose Hypothesis* 3.1 *or* 3.2 *or* 3.3. *Then the surgery sequence of based sets is exact at the stably smoothable normal invariants*:

$$\mathcal{S}^s_{\mathrm{TOP0}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP0}}(X) \xrightarrow{\sigma_*} L_4^h(\mathbb{Z}[\pi]^\omega). \tag{3.4.2}$$

Corollary 3.5. *Let π be a free product of groups of the form*

$$\pi = \bigstar_{i=1}^n \Lambda_i$$

for some $n > 0$, where each Λ_i is a torsion-free lattice in either $\mathrm{Isom}(\mathbb{E}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{H}^{m_i})$ *or* $\mathrm{Isom}(\mathbb{CH}^{m_i})$ *for some $m_i > 0$. Suppose the orientation character ω is trivial. Then the surgery sequences* (3.4.1) *and* (3.4.2) *are exact.*

Corollary 3.6. *Suppose X is a* DIFF 4*-manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geqslant 0$, and each summand X_i is either $S^2 \times \mathbb{RP}^2$ or $S^2 \rtimes \mathbb{RP}^2$ or $\#_{S^1} n(\mathbb{RP}^4)$ for some $1 \leqslant n \leqslant 4$. Then the surgery sequences (3.4.1) *and* (3.4.2) *are exact.*

Corollary 3.7. *Suppose X is a* TOP 4*-manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geqslant 0$, and each summand X_i is the total space of a fiber bundle

$$H_i \longrightarrow X_i \longrightarrow S^1.$$

Here, we suppose H_i is a compact, connected 3*-manifold such that*:

(1) *H_i is S^3 or D^3, or*
(2) *H_i is irreducible with non-zero first Betti number.*

Moreover, if H_i is non-orientable, we assume that the quotient group $H_1(H_i; \mathbb{Z})_{(\alpha_i)_*}$ of coinvariants is 2*-torsionfree, where $\alpha_i : H_i \to H_i$ is the monodromy homeomorphism. Then the surgery sequence* (3.4.2) *is exact.*

Corollary 3.8. *Suppose X is a* TOP 4-*manifold of the form*

$$X = X_1 \# \cdots \# X_n \# r(S^2 \times S^2)$$

for some $n > 0$ and $r \geqslant 0$, and each summand X_i is the total space of a fiber bundle

$$\Sigma_i^f \longrightarrow X_i \longrightarrow \Sigma_i^b.$$

Here, we suppose the fiber and base are compact, connected 2-manifolds, $\Sigma_i^f \neq \mathbb{RP}^2$, and Σ_i^b has positive genus. Moreover, if X_i is non-orientable, we assume that the fiber Σ_i^f is orientable and that the monodromy action of $\pi_1(\Sigma_i^b)$ of the base preserves any orientation on the fiber. Then the surgery sequence (3.4.2) *is exact.*

4. Splitting of 5-manifolds

We generalize Cappell's 5-dimensional splitting theorem [3, Thm. 5, Remark], using the homological hypotheses developed in Sections 2–3. Our proof incorporates the possible non-vanishing of UNil_6. The DIFF and TOP cases are distinguished, and the results of this section are applied to the fibering problem in Section 5. The stable surgery version of the splitting theorem can be found in [7]. However, the stable splitting of 5-manifolds is not pursued here, since connecting sum a single fiber with $S^2 \times S^2$ destroys the fibering property over S^1.

Let $(Y, \partial Y)$ be a based, compact, connected CAT 5-manifold. Let $(Y_0, \partial Y_0)$ is a based, compact, connected CAT 4-manifold. Suppose Y_0 is an **incompressible**, **two-sided** submanifold of Y. That is, the induced homomorphism $\pi_1(Y_0) \to \pi_1(Y)$ is injective, and there is a separating decomposition

$$Y = Y_- \cup_{Y_0} Y_+ \quad \text{with } \partial Y = \partial Y_- \cup_{\partial Y_0} \partial Y_+$$

or, respectively, a non-separating decomposition

$$Y = \cup_{Y_0} Y_\infty \quad \text{with } \partial Y = \cup_{\partial Y_0} \partial Y_\infty.$$

The Seifert–van Kampen theorem identifies

$$\pi_1(Y) = \Pi = \Pi_- *_{\Pi_0} \Pi_+$$

as the corresponding injective, amalgamated free product of fundamental groups, or, respectively,

$$\pi_1(Y) = \Pi = *_{\Pi_0} \Pi_\infty$$

as the corresponding injective, HNN-extension[3] of fundamental groups.

A homotopy equivalence g to Y is CAT **splittable along** Y_0 if g is homotopic, relative to a CAT isomorphism ∂g, to a union $g_- \cup_{g_0} g_+$ (resp. $\cup_{g_0} g_\infty$) of homotopy equivalences from compact CAT manifolds to Y_-, Y_0, Y_+ (resp. Y_0, Y_∞) [1]. Under certain conditions, we show that the vanishing of high-dimensional obstructions in Nil_0 and UNil_6^s are sufficient for splitting. These two obstructions were formulated by Friedhelm Waldhausen (1960s) and Sylvain Cappell (1970s).

Theorem 4.1. *Let $(Y, \partial Y)$ be a finite, simple Poincaré pair of formal dimension* 5 [38, §2]. *Suppose ∂Y and Y_0 are compact* DIFF 4-*manifolds such that $(Y_0, \partial Y_0)$ is a connected, incompressible, two-sided Poincaré subpair of $(Y, \partial Y)$ with tubular neighborhood $Y_0 \times [-1, 1]$. If* CAT = DIFF, *assume Y_0 satisfies Hypothesis* 3.1 *or* 3.2 *and satisfies Hypothesis* 2.5. *If* CAT = TOP, *assume Y_0 satisfies Hypothesis* 3.1 *or* 3.2 *or* 3.3 *and satisfies Hypothesis* 2.1.

Suppose $g : (W, \partial W) \to (Y, \partial Y)$ is a homotopy equivalence for some compact DIFF 5-*manifold W such that the restriction $\partial g : \partial W \to \partial Y$ is a diffeomorphism. Then g is* CAT *splittable along Y_0 if and only if*

(1) *the cellular splitting obstruction, given by the image of the Whitehead torsion $\tau(g) \in \mathrm{Wh}_1(\Pi)$, vanishes*:

$$\mathrm{split}_K(g; Y_0) \in \mathrm{Wh}_0(\Pi_0) \oplus \widetilde{\mathrm{Nil}}_0\big(\mathbb{Z}[\Pi_0]; \mathbb{Z}[\Pi_- - \Pi_0], \mathbb{Z}[\Pi_+ - \Pi_0]\big)$$

or, respectively,

$$\mathrm{split}_K(g; Y_0) \in \mathrm{Wh}_0(\Pi_0) \oplus \widetilde{\mathrm{Nil}}_0\big(\mathbb{Z}[\Pi_0]; \mathbb{Z}\big[\Pi_\infty - \Pi_0^-\big], \mathbb{Z}\big[\Pi_\infty - \Pi_0^+\big], {}_-\mathbb{Z}[\Pi_\infty]_+, {}_+\mathbb{Z}[\Pi_\infty]_-\big)$$

and subsequently

(2) *the manifold splitting obstruction, given by the algebraic position of discs in the fundamental subdomains of the Π_0-cover, vanishes*:

$$\mathrm{split}_L(g; Y_0) \in \mathrm{UNil}_6^s\big(\mathbb{Z}[\Pi_0]^{\omega_0}; \mathbb{Z}[\Pi_- - \Pi_0]^{\omega_-}, \mathbb{Z}[\Pi_+ - \Pi_0]^{\omega_+}\big)$$

or, respectively,

$$\mathrm{split}_L(g; Y_0) \in \mathrm{UNil}_6^s\big(\mathbb{Z}[\Pi_0]^{\omega_0}; \mathbb{Z}\big[\Pi_\infty - \Pi_0^-\big]^{\omega_\infty}, \mathbb{Z}\big[\Pi_\infty - \Pi_0^+\big]^{\omega_\infty}\big).$$

[3] In the non-separating case, we write Π_0^-, Π_0^+ as the two monomorphic images of Π_0 in Π_∞.

Furthermore, if g is CAT *splittable along Y_0, then g is homotopic* $\operatorname{rel}\partial W$ *to a split homotopy equivalence $g': W \to Y$ such that the* CAT *inverse image $(g')^{-1}(Y_0)$ is* CAT *isomorphic to Y_0.*

Our theorem mildly generalizes [3, Theorem 5, Remark], which included: if Π_0 is a finite group of odd order, then $H_2(\Pi_0;\mathbb{Z}_2)=0$ and $L_5^h(\mathbb{Z}[\Pi_0])=0$.

Corollary 4.2 *(Cappell). Suppose $g: W \to Y$ is a homotopy equivalence of closed* DIFF *5-manifolds. Assume*:

(1) *Y_0 is orientable,*
(2) *$H_2(\Pi_0;\mathbb{Z}_2)=0$,*
(3) *Π_0 is square-root closed*[4] *in Π, and*
(4) *the following surgery obstruction map is surjective* (*cf. Hypothesis* 2.5):

$$\sigma_* : \mathcal{N}_{\mathrm{DIFF}}\big(X\times\Delta^1\big) \longrightarrow L_5^h\big(\mathbb{Z}[\Pi_0]\big).$$

Then g is DIFF *splittable along Y_0 if and only if the above image* $\operatorname{split}_K(g;Y_0)$ *of the Whitehead torsion $\tau(g)\in \mathrm{Wh}_1(\Pi)$ vanishes.*

Define a decoration subgroup $B\subseteq \mathrm{Wh}_1(\Pi)$ as the image of $\mathrm{Wh}_1(\Pi_-)\oplus\mathrm{Wh}_1(\Pi_+)$, respectively $\mathrm{Wh}_1(\Pi_\infty)$, under the homomorphism induced by inclusion. Recall that the **structure set** $\mathcal{S}^B_{\mathrm{CAT}}(Y)$ is defined as the set of equivalence classes of homotopy equivalences $g:(W,\partial W)\to(Y,\partial Y)$ such that W is a compact CAT manifold and $\partial g:\partial W\to\partial Y$ is a CAT isomorphism and g has Whitehead torsion $\tau(g)\in B$, under the equivalence relation $g\sim g'$ if there exists a CAT isomorphism $h: W\to W'$ such that $g'\circ h$ is homotopic to g. The **split structure set** $\mathcal{S}^{\mathrm{split}}_{\mathrm{CAT}}(Y;Y_0)$ is defined as the subset of $\mathcal{S}^B_{\mathrm{CAT}}(Y)$ whose elements are represented by homotopy equivalences CAT splittable along Y_0. The abelian group UNil_6^s depends only on the fundamental groups Π_-, Π_0, Π_+ (resp. Π_0, Π_∞) with orientation character ω. UNil_6^s is algebraically defined and has zero decoration in $\widetilde{\mathrm{Nil}}_0$ [2].

Definition 4.3. Let $(Y,\partial Y)$ be a compact DIFF manifold. Define the **smoothable structure set** $\mathcal{S}_{\mathrm{TOP}+}(Y)$ as the image of $\mathcal{S}_{\mathrm{DIFF}}(Y)$ under the forgetful map to $\mathcal{S}_{\mathrm{TOP}}(Y)$. That is, $\mathcal{S}_{\mathrm{TOP}+}(Y)$ is the subset of $\mathcal{S}_{\mathrm{TOP}}(Y)$ consisting of the elements representable by homotopy equivalences $g:(W,\partial W)\to(Y,\partial Y)$ such that W admits a DIFF structure extending the DIFF structure on ∂W induced by ∂g.

A more succinct statement illuminates the method of proof in higher dimensions: Sylvain Cappell's "nilpotent normal cobordism construction" [1,3].

Theorem 4.4. *Let $(Y,\partial Y)$ be a finite, simple Poincaré pair of formal dimension* 5 [38, §2]. *Suppose ∂Y and Y_0 are compact* DIFF *4-manifolds such that $(Y_0,\partial Y_0)$ is a connected, incompressible, two-sided Poincaré subpair of $(Y,\partial Y)$ with tubular neighborhood $Y_0\times[-1,1]$.*

(1) *Assume Y_0 satisfies Hypothesis* 3.1 *or* 3.2 *and satisfies Hypothesis* 2.5. *Then there is a bijection*

$$\mathrm{nncc}^s : \mathcal{S}^B_{\mathrm{DIFF}}(Y) \longrightarrow \mathcal{S}^{\mathrm{split}}_{\mathrm{DIFF}}(Y;Y_0)\times \mathrm{UNil}_6^s$$

such that composition with projection onto the first factor is a subset retraction, and composition with projection onto the second factor is the manifold splitting obstruction split_L. *Furthermore, g and $\mathrm{nncc}^s(g)$ have equal image in $\mathcal{N}_{\mathrm{DIFF}}(Y)$.*
(2) *Assume Y_0 satisfies Hypothesis* 3.1 *or* 3.2 *or* 3.3 *and satisfies Hypothesis* 2.1. *Then there is an injection*

$$\mathrm{nncc}^s_+ : \mathcal{S}^B_{\mathrm{TOP}+}(Y) \longrightarrow \mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}(Y;Y_0)\times \mathrm{UNil}_6^s$$

such that composition with projection onto the first factor restricts to a subset inclusion $\mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}+}(Y;Y_0)\subseteq\mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}(Y;Y_0)$, and composition with projection onto the second factor is the manifold splitting obstruction split_L. *Furthermore, g and $\mathrm{nncc}^s(g)$ have equal image in $\mathcal{N}_{\mathrm{TOP}}(Y)$.*

4.1. Proof by cobordism

We simply extend Cappell's modification [3, Chapter V] of the Cappell–Shaneson proof [7, Theorems 4.1, 5.1] of 5-dimensional splitting as to include the non-vanishing of UNil_6^s. Our homological conditions eschew the performance of surgery on the 4-manifold Y_0. Examples are given in Section 5.

[4] **Square-root closed**: if $g\in\Pi$, then $g^2\in\Pi_0$ implies $g\in\Pi_0$.

Remark 4.5. Friedhelm Waldhausen had shown that $\widetilde{\mathrm{Nil}}_0$ is a summand of $\mathrm{Wh}_1(\Pi)$ and that there is an exact sequence of abelian groups [37]:

$$\begin{aligned}&\mathrm{Wh}_1(\Pi_0) \xrightarrow{i_- - i_+} \mathrm{Wh}_1(\Pi_-) \oplus \mathrm{Wh}_1(\Pi_+) \xrightarrow{j_- + j_+} \mathrm{Wh}_1(\Pi)/\widetilde{\mathrm{Nil}}_0\\ &\xrightarrow{\partial} \mathrm{Wh}_0(\Pi_0) \xrightarrow{i_- - i_+} \mathrm{Wh}_0(\Pi_-) \oplus \mathrm{Wh}_0(\Pi_+) \xrightarrow{j_- + j_+} \mathrm{Wh}_0(\Pi)\end{aligned}$$

or, respectively,

$$\mathrm{Wh}_1(\Pi_0) \xrightarrow{i_- - i_+} \mathrm{Wh}_1(\Pi_\infty) \xrightarrow{j_\infty} \mathrm{Wh}_1(\Pi)/\widetilde{\mathrm{Nil}}_0 \xrightarrow{\partial} \mathrm{Wh}_0(\Pi_0) \xrightarrow{i_- - i_+} \mathrm{Wh}_0(\Pi_\infty) \xrightarrow{j_\infty} \mathrm{Wh}_0(\Pi).$$

Waldhausen showed that the cellular splitting obstruction is algebraically defined as the image $\mathrm{split}_K(g; Y_0) \in \mathrm{Wh}_0(\Pi_0) \oplus \widetilde{\mathrm{Nil}}_0$ of the Whitehead torsion $\tau(g) \in \mathrm{Wh}_1(\Pi)$. It vanishes if and only if g is CW splittable along Y_0 [36, erratum].

Remark 4.6. Sylvain Cappell had shown that UNil_6^s is a summand of $L_6^B(\Pi)$ and that there is an exact sequence of abelian groups [2]:

$$L_6^h(\Pi_0) \xrightarrow{i_- - i_+} L_6^h(\Pi_-) \oplus L_6^h(\Pi_+) \xrightarrow{j_- + j_+} L_6^B(\Pi)/\mathrm{UNil}_6^s \xrightarrow{\partial} L_5^h(\Pi_0) \xrightarrow{i_- - i_+} L_5^h(\Pi_-) \oplus L_5^h(\Pi_+) \xrightarrow{j_- + j_+} L_5^B(\Pi)$$

or, respectively,

$$L_6^h(\Pi_0) \xrightarrow{i_- - i_+} L_6^h(\Pi_\infty) \xrightarrow{j_\infty} L_6^B(\Pi)/\mathrm{UNil}_6^s \xrightarrow{\partial} L_5^h(\Pi_0) \xrightarrow{i_- - i_+} L_5^h(\Pi_\infty) \xrightarrow{j_\infty} L_5^B(\Pi).$$

If the cellular splitting obstruction vanishes, then Cappell showed that the manifold splitting obstruction is algebraically defined as $\mathrm{split}_L(g; Y_0) \in \mathrm{UNil}_6^s$. It vanishes if g is CAT splittable along Y_0 [1]. We shall investigate the converse.

Proof of Theorem 4.1. (Necessity) Suppose g is CAT splittable along Y_0. Then $\mathrm{split}_K(g; Y_0) = 0$ and $\mathrm{split}_L(g; Y_0) = 0$ vanish by Remarks 4.5 and 4.6.

(Sufficiency) Suppose $\mathrm{split}_K(g; Y_0) = 0$ and $\mathrm{split}_L(g; Y_0) = 0$. Then g is CW splittable along Y_0 and $g \in \mathcal{S}^B_{\mathrm{DIFF}}(Y)$ (resp. $g \in \mathcal{S}^B_{\mathrm{TOP}+}(Y)$) by Remark 4.5. Since Y_0 satisfies the hypotheses in Sections 2–3 for exactness of the CAT surgery sequence, by Theorem 4.4, it follows that $\mathrm{nncc}^s(g) = (g, 0)$. In other words, g is CAT splittable along Y_0.

Furthermore, the normal bordisms over Y_0 in the proof of Theorem 4.4 depend only on the homotopy self-equivalences and normal self-bordisms of [21, Proposition 3.5] and Section 2. Therefore $g: W \to Y$ is CAT normally bordant to a split homotopy equivalence $g' = g^4$ such that the CAT restriction $g': (g')^{-1}(Y_0) \to Y_0$ is a homotopy self-equivalence. □

Proof of Theorem 4.4. (Definition, I) Let $g: (W, \partial W) \to (Y, \partial Y)$ be a homotopy equivalence with Whitehead torsion $\tau(g) \in B$ and ∂g a CAT isomorphism, representing an element of $\mathcal{S}^B_{\mathrm{CAT}}(Y)$. Our principal goal is to define a CAT normal bordism G' over $Y \times \Delta^1$ from g to a homotopy equivalence $g': (W', \partial W') \to (Y, \partial Y)$ such that h is CAT split along Y_0 and that G' has surgery obstruction

$$\sigma_*(G') \in \mathrm{UNil}_6^s \subseteq L_6^B(\Pi).$$

Define

$$\mathrm{nncc}^s(g) := \big(g', \sigma_*(G')\big) \in \mathcal{S}^{\mathrm{split}}_{\mathrm{CAT}}(Y; Y_0) \times \mathrm{UNil}_6^s.$$

(Well-definition; Projection properties) Note that $\sigma_*(G')$ depends only on the normal bordism class of G' relative to $\partial G' = g \sqcup g'$, and that $\sigma_*(G')$ lies in L_6^B since $\tau(g), \tau(g') \in B$. Let $Z := \mathbb{CP}^4 \# 2(S^3 \times S^5)$ be the closed CAT 8-manifold with Euler characteristic $\chi(Z) = 1$ and signature $\sigma^*(Z) = 1$ used by Weinberger for decorated periodicity [39]. Cappell has shown

$$\sigma_*(G' \times \mathbf{1}_Z) = \mathrm{split}_L(g \times \mathbf{1}_Z; Y_0 \times Z)$$

for 13-dimensional homotopy equivalences [3]. Note $\sigma_*(G') = \sigma_*(G' \times \mathbf{1}_Z)$, by Kwun–Szczarba's torsion product formula and Ranicki's surgery product formula [28, Prop. 8.1(ii)]. Also note $\mathrm{split}_L(g \times \mathbf{1}_Z; Y_0 \times Z) = \mathrm{split}_L(g; Y_0)$, since these splitting obstructions in UNil_6^s coincide [29, Prop. 7.6.2A] with the codimension-one quadratic signatures [29, Prop. 7.2.2] of g and $g \times \mathbf{1}_Z$ in the codimension-one Poincaré embedding groups LS_4 and LS_{12}, and since $\times \mathbf{1}_Z: LS_4 \to LS_{12}$ is an isomorphism [38, Cor. 11.6.1]. So $\sigma_*(G') = \mathrm{split}_L(g; Y_0)$. Suppose G'' is another such CAT normal bordism from g to some split g''. Then $\sigma_*(G'') = \mathrm{split}_L(g; Y_0)$. So $G' \cup_g -G''$ is a normal bordism from g' to g'' with surgery obstruction $0 \in L_6^B$. By the 6-dimensional CAT s-cobordism theorem, it follows that g' and g'' are CAT isomorphic. Therefore $\mathrm{nncc}^s(g) = (g', \sigma_*(G'))$ is well defined and satisfies the asserted projection properties.

(Bijectivity) Consider Wall's action [38, Thm. 5.8] of the abelian group L_6^B on the 5-dimensional structure set $\mathcal{S}^B_{\mathrm{CAT}}(Y)$. It follows from the properties defining nncc^s that the restriction of Wall's action is the inverse function of nncc^s:

$$\mathrm{act}: \mathcal{S}^{\mathrm{split}}_{\mathrm{CAT}}(Y; Y_0) \times \mathrm{UNil}_6^s \longrightarrow \mathcal{S}^B_{\mathrm{CAT}}(Y).$$

(Definition, II) *First, we normally*[5] *cobord* g *to a degree one normal map* g^1 *so that the restriction to* $(g^1)^{-1}(Y_0)$ *is a homotopy equivalence.* By general position, we may assume that $g: W \to Y$ is DIFF transversal to Y_0. Consider the degree one normal map

$$g_0 := g|_{W_0} : W_0 \longrightarrow Y_0$$

where the DIFF 4-manifold $W_0 := g^{-1}(Y_0)$ is the transverse inverse image of Y_0. Denote $\widehat{Y} = \widehat{Y}_- \cup_{Y_0} \widehat{Y}_+$ as the based cover of Y corresponding to the subgroup Π_0. The $\mathbb{Z}[\Pi_0]$-submodule $P := K_6(\widehat{W}_- \times Z)$ is a finitely generated, projective Lagrangian of the $\mathbb{Z}[\Pi_0]$-equivariant intersection form on the surgery kernel

$$K_6(W_0 \times Z) = \mathrm{Ker}\big(g \times \mathbf{1}_Z : H_6(W_0 \times Z) \longrightarrow H_6(Y_0 \times Z)\big),$$

where we can homotope $g \times \mathbf{1}_Z : W \times Z \to Y \times Z$ so that $g|_{g^{-1}(Y_0 \times Z)}$ is 6-connected, degree one, normal map between 12-dimensional manifolds [3, Lemma II.1]. Furthermore, the projective class $[P] \in \mathrm{Wh}_0(\Pi_0)$ satisfies $[P^*] = -[P]$ [3, Lemma II.2] and is the homomorphic image of the Whitehead torsion $\tau(g) = \tau(g \times \mathbf{1}_Z) \in \mathrm{Wh}_1(\Pi)$ under Waldhausen's connecting map ∂ (see Remark 4.5). But $\tau(g) \in B$ implies that $[P] = \partial(\tau(g)) = 0$. Therefore the stably free surgery obstruction vanishes by decorated periodicity:

$$\sigma_*(g_0) = \sigma_*(g_0 \times \mathbf{1}_Z) = 0 \in L_4^h(\Pi_0).$$

Then, by Theorem 3.4, there exists a CAT normal bordism G_0^1 from g_0 to a homotopy equivalence $g_0^1 : W_0^1 \to Y_0$. So the union

$$G^1 := \big(g \times [0,1]\big) \cup_{g_0 \times 1 \times [-1,1]} \big(G_0^1 \times [-1,1]\big)$$

is a CAT normal bordism from the homotopy equivalence $g: W \to Y$ to a degree one normal map $g^1 : W^1 \to Y$ with transversal restriction $g_0^1 = (g^1)|_{(g^1)^{-1}(Y_0)}$ a homotopy equivalence.

Second, we normally cobord g^1 *relative* $(g^1)^{-1}(Y - Y_0)$ *to a degree one normal map* g^2 *so that the restriction to* $(g^2)^{-1}(Y - Y_0)$ *has vanishing surgery obstruction.* Since $g_\pm^1 = (g^1)|_{(g^1)^{-1}(Y_- \sqcup Y_+)}$ (resp. $g_\infty^1 = (g^1)|_{(g^1)^{-1}(Y_\infty)}$) restricts to a homotopy equivalence $g_0^1 \times \{-1,1\}$ on the boundary, and since G^1 is a normal bordism of source and target from $g_\pm^1$ (resp. g_∞^1) to the B-torsion homotopy equivalence g over the reference space $K(\Pi,1)$, the image of surgery obstruction vanishes [38, §9]:

$$(j_- + j_+)\big(\sigma_*\big(g_\pm^1\big)\big) = 0 \in L_5^B(\Pi)$$

or, respectively,

$$(j_\infty)\big(\sigma_*\big(g_\infty^1\big)\big) = 0 \in L_5^B(\Pi).$$

Therefore, by Cappell's L-theory Mayer–Vietoris exact sequence (Remark 4.6), there exists $a \in L_5^h(\Pi_0)$ such that

$$(i_- - i_+)(a) = -\sigma_*\big(g_\pm^1\big) \in L_5^h(\Pi_-) \oplus L_5^h(\Pi_+)$$

or, respectively,

$$(i_- - i_+)(a) = -\sigma_*\big(g_\infty^1\big) \in L_5^h(\Pi_\infty).$$

Then, by Theorem 2.2 if CAT = TOP, or by Hypothesis 2.5 if CAT = DIFF, there exists a CAT normal bordism G_0^2 from the homotopy equivalence g_0^1 to itself realizing this surgery obstruction: $\sigma_*(G_0^2) = a \in L_5^h(\Pi_0)$. So the union

$$G^2 := \big(g^1 \times [0,1]\big) \cup_{g_0^1 \times 1 \times [-1,1]} \big(G_0^2 \times [-1,1]\big)$$

is a CAT normal bordism from the degree one normal map $g^1 : W^1 \to Y$ to another degree one normal map $g^2 : W^2 \to Y$ such that the transversal restriction $g_0^2 = g_0^1 : W_0^1 \to Y_0$ is a homotopy equivalence and the transversal restriction $g_\pm^2 : W_\pm^2 \to Y_\pm$ (resp. $g_\infty^2 : W_\infty^2 \to Y_\infty$) has vanishing surgery obstruction.

Third, we normally cobord g^2 *relative* $(g^2)^{-1}(Y_0)$ *to a degree one normal map* g^3 *so that the restriction to* $(g^3)^{-1}(Y - Y_0)$ *is also a homotopy equivalence.* Since $\sigma_*(g_\pm^2) = 0 \in L_5^h(\Pi_\pm)$ (resp. $\sigma_*(g_\infty^2) = 0 \in L_5^h(\Pi_\infty)$), by exactness of the 5-dimensional surgery exact sequence at the CAT normal invariants [38, Thm. 10.3], there exists a CAT normal bordism $G_\pm^3$ (resp. G_∞^3) relative $g_2 \times \{-1,1\}$ from the degree one, CAT normal map $g_\pm^2$ (resp. g_∞^2) to a homotopy equivalence $g_\pm^3$ (resp. g_∞^3). So the union

$$G^3 := \big(g^2 \times [0,1] \times [-1,1]\big) \cup_{g_0^2 \times [0,1] \times \{-1,1\}} G_\pm^3$$

or, respectively,

$$G^3 := \big(g^2 \times [0,1] \times [-1,1]\big) \cup_{g_0^2 \times [0,1] \times \{-1,1\}} G_\infty^3$$

[5] The stable normal CAT microbundle on Y and on Y_0 is $(\bar{g})^*(\nu_W)$ and $(\bar{g})^*(\nu_W) \mid Y_0$, lifting the Spivak normal spherical fibration on Y, where $\bar{g} : Y \to W$ is a homotopy inverse of $g : W \to Y$.

is a CAT normal bordism from the degree one normal map $g^2 : W^2 \to Y$ to homotopy equivalence $g^3 = g^3_- \cup_{g^3_0} g^3_+ : W^3 \to Y$ (resp. $g^3 = \cup_{g^3_0} g^3_\infty : W^3 \to Y$) which is CAT split along Y_0.

Finally, we normally cobord the split homotopy equivalence g^3 to another split homotopy equivalence $g^4 = g'$ so that the normal bordism $G' = G^1 \cup G^2 \cup G^3 \cup G^4$ from g to g' has surgery obstruction in the subgroup UNil^s_6 *of the abelian group L^B_6.* Consider the surgery obstruction of the CAT normal bordism from g to g^3:

$$b := -\sigma_*\big(G^1 \cup G^2 \cup G^3\big) \in L^B_6(\Pi).$$

Let $c := \partial(b) \in L^h_5(\Pi_0)$ be the image in Cappell's L-theory Mayer–Vietoris exact sequence (Remark 4.6). By Theorem 2.2 if CAT = TOP, or by Hypothesis 2.5 if CAT = DIFF, there exists a CAT normal bordism $G^{3.5}_0$ from the homotopy equivalence g^3_0 to itself realizing this surgery obstruction: $\sigma_*(G^{3.5}_0) = c \in L^h_5(\Pi_0)$. Then

$$0 = (i_- - i_+)(c) = \sigma_*\big(g^3_\pm \cup G^{3.5}_0 \times \{-1, 1\}\big) \in L^h_5(\Pi_-) \oplus L^h_5(\Pi_+)$$

or, respectively,

$$0 = (i_- - i_+)(c) = \sigma_*\big(g^3_\infty \cup G^{3.5}_0 \times \{-1, 1\}\big) \in L^h_5(\Pi_\infty).$$

Therefore, by exactness of the 5-dimensional surgery exact sequence at the CAT normal invariants [38, Thm. 10.3], there exists a CAT normal bordism $G^{3.5}_\pm$ (resp. $G^{3.5}_\infty$) relative $g^3_0 \times \{-1, 1\}$ from the degree one normal map $g^3_\pm \cup G^{3.5}_0 \times \{-1, 1\}$ (resp. $g^3_\infty \cup G^{3.5}_0 \times \{-1, 1\}$) to a homotopy equivalence $g^{3.5}_\pm : W^{3.5}_\pm \to Y_\pm$ (resp. $g^{3.5}_\infty : W^{3.5}_\infty \to Y_\infty$). So the union

$$G^{3.5} := \big(G^{3.5}_0 \times [-1, 1]\big) \cup_{G^{3.5}_0 \times \{-1,1\}} G^{3.5}_\pm$$

or, respectively,

$$G^{3.5} := \big(G^{3.5}_0 \times [-1, 1]\big) \cup_{G^{3.5}_0 \times \{-1,1\}} G^{3.5}_\infty$$

is a CAT normal bordism from the split homotopy equivalence $g^3 : W^3 \to Y$ to another split homotopy equivalence $g^{3.5} : W^{3.5} \to Y$ such that

$$\sigma_*\big(G^1 \cup G^2 \cup G^3 \cup G^{3.5}\big) = j(d) \oplus e \in L^B_6(\Pi)$$

for some $d \in L^h_6(\Pi_-) \oplus L^h_6(\Pi_+)$ (resp. $d \in L^h_6(\Pi_\infty)$) and $e \in \mathrm{UNil}^s_6$.

By Wall realization on 5-dimensional CAT structure sets [38, Thm. 10.5], there exists a CAT normal bordism $G^4_\pm$ (resp. G^4_∞) relative $g^{3.5}_0 \times \{-1, 1\}$ from the homotopy equivalence $g^{3.5}_\pm$ (resp. $g^{3.5}_\infty$) to another homotopy equivalence $g^4_\pm$ (resp. g^4_∞) such that $\sigma_*(G^4_\pm) = -d$ (resp. $\sigma_*(G^4_\infty) = -d$). So the union

$$G^4 := G^{3.5} \cup_{g^{3.5} \times 0} \big(g^{3.5}_0 \times [0, 1] \times [-1, 1] \cup G^4_\pm\big)$$

or, respectively,

$$G^4 := G^{3.5} \cup_{g^{3.5} \times 0} \big(g^{3.5}_0 \times [0, 1] \times [-1, 1] \cup G^4_\infty\big)$$

is a CAT normal bordism from the split homotopy equivalence $g^{3.5} : W^{3.5} \to Y$ to another split homotopy equivalence $g^4 : W^4 \to Y$ such that

$$\sigma_*\big(G^1 \cup G^2 \cup G^3 \cup G^4\big) = e \in \mathrm{UNil}^s_6.$$

Thus the definition of nncc^s is complete. □

5. Fibering and splitting over the circle

We approach the problem of fibering a 5-manifold W over the circle S^1 from Farrell's point of view [9], which involves a finite domination of the infinite cyclic cover $\overline{W}$, the covering translation $t : \overline{W} \to \overline{W}$, and a certain mapping torus.

Gluing the ends of a self h-cobordism $(Y; X, X)$ by a self homeomorphism $\alpha : X \to X$ yields an h-**block bundle** $\cup_\alpha Y$ **over** S^1 [4, p. 306]. This is classically known as a *pseudo-fibering over* S^1 [10, Defn. 3.1] [34, Defn. 4.2]. Consider the zero-torsion version of h-block bundles over S^1.

Definition 5.1. We call E the total space of a CAT s-**block bundle over** S^1 **with homotopy fiber** X if E is the compact CAT manifold obtained by gluing the ends of a self CAT s-cobordism $(Y; X, X) \operatorname{rel} \partial X$ by a CAT automorphism $\alpha : X \to X$. We write $E = \cup_\alpha Y := Y/(x, 0) \sim (\alpha(x), 1)$, and a special case is the mapping torus $X \rtimes_\alpha S^1 := \cup_\alpha X \times \Delta^1$. The induced continuous map $E \to S^1$ is called a CAT s-**block bundle projection**, which is unique up to homotopy.

Fiber bundles are special cases of s-block bundles, and the converse is true if the s-cobordism theorem holds for the fiber in the given manifold category.

Definition 5.2. (Quinn, compare [26, §2.3].) Let X be a compact CAT manifold. The **block homotopy automorphism space** $\widetilde{\mathrm{G}}^s(X)$ is the geometric realization of the Kan Δ-set whose k-simplices are simple homotopy self-equivalences $e : X \times \Delta^k \to X \times \Delta^k$ of $(k+2)$-ads which restrict to the identity over ∂X. Note that $\mathrm{hAut}^s(X) = \pi_0 \widetilde{\mathrm{G}}^s(X)$. The basepoint is the identity $\mathbf{1}_X : X \to X$.

Similarly, the **block structure space** $\widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X)$ is the geometric realization of the Kan Δ-set whose k-simplices are simple homotopy equivalences $Y \to X \times \Delta^k$ of CAT manifold $(k+2)$-ads in $\mathbb{R}^\infty$ which restrict to CAT isomorphisms over ∂X. Note the CAT s-bordism structure set is $\mathcal{S}^s_{\mathrm{CAT}}(X) = \pi_0 \widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X)$. We define the decoration CAT = TOP+ for block structures to mean that the TOP manifolds Y are smoothable, that is, without a preference of DIFF structure.

Naturally, there is a simplicial inclusion $\widetilde{\mathrm{G}}^s(X) \hookrightarrow \widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X)$.

An assembly-type function over S^1 is described as follows.

Definition 5.3. Define an α-**twisted simplicial loop in** $(\widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X), \widetilde{\mathrm{G}}^s(X))$ as a simple homotopy equivalence $(h; h_0, h_1) : (Y; X, X) \to X \times (\Delta^1; 0, 1)$ of CAT manifold triads such that the simple homotopy self-equivalences $h_i : X \to X$ satisfy $h_1 = \alpha \circ h_0$ and that h restricts to a CAT isomorphism over ∂X. We define the α-**twisted fundamental set** $\pi_1^\alpha(\widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X), \widetilde{\mathrm{G}}^s(X))$ as the set of homotopy classes of these α-twisted loops. Note, if α is the identity automorphism, then this set is the first homotopy set of the pair. Define the **union function**

$$\cup : \pi_1^\alpha\big(\widetilde{\mathbb{S}}^s_{\mathrm{CAT}}(X), \widetilde{\mathrm{G}}^s(X)\big) \longrightarrow \mathcal{S}^s_{\mathrm{CAT}}\big(X \rtimes_\alpha S^1\big)$$

as follows. Let $(h; h_0, h_1) : (Y; X, X) \to X \times (\Delta^1; 0, 1)$ be an α-twisted simplicial loop. Then there is an induced simple homotopy equivalence, well-defined on homotopy classes of loops:

$$\cup(h; h_0, h_1) : \cup_{\mathbf{1}_X} Y \longrightarrow X \rtimes_\alpha S^1; \quad [y] \longmapsto \big[h(y)\big].$$

5.1. Statement of results

Our theorems below are crafted as to eliminate any algebraic K- or L-theory obstructions to splitting and fibering over the circle.

The first splitting theorem (5.4) is a special case of the general splitting theorem (4.1). Here, for any homotopy self-equivalence $h : (X, \partial X) \to (X, \partial X)$, the mapping torus $Y = X \rtimes_h S^1$ is a Poincaré pair with X a two-sided Poincaré subpair [30, Prop. 24.4]. This level of abstraction is required to prove the fibering theorem (5.6).

Theorem 5.4. *Let X be any of the 4-manifolds Q, S, F defined in (1.1, 1.2, 1.4). Let $h : X \to X$ be a homotopy equivalence which restricts to a diffeomorphism $\partial h : \partial X \to \partial X$. Suppose M is a compact DIFF 5-manifold and $g : M \to X \rtimes_h S^1$ is a homotopy equivalence which restricts to a diffeomorphism $\partial g : \partial M \to \partial X \rtimes_{\partial h} S^1$.*

Then g is homotopic to a map g' which restricts to a simple homotopy equivalence $g' : X' \to X$ such that the TOP inverse image $X' := (g')^{-1}(X)$ is homeomorphic to X and the exterior M' of X' in M is a smoothable TOP self s-cobordism of X.

The second splitting theorem (5.5) connects homotopy TOP structures on mapping tori to smoothable s-cobordisms, homotopy self-equivalences, and the stable smoothing invariant of Kirby and Siebenmann [23].

Theorem 5.5. *Let X be any of the 4-manifolds Q, S, F defined in (1.1, 1.2, 1.4). Let $\alpha : X \to X$ be a diffeomorphism. Then there is an exact sequence of based sets:*

$$\pi_1^\alpha\big(\widetilde{\mathbb{S}}^s_{\mathrm{TOP}+}(X), \widetilde{\mathrm{G}}^s(X)\big) \xrightarrow{\cup} \mathcal{S}^h_{\mathrm{TOP}}\big(X \rtimes_\alpha S^1\big) \xrightarrow{\mathrm{ks}} \mathbb{F}_2 \oplus H_1(X; \mathbb{F}_2)_\alpha.$$

Our fibering theorem (5.6) is proven using a key strategy of Tom Farrell [9], except we do not require 4-dimensional Siebenmann ends on the infinite cyclic cover $\overline{M}$ to exist. A **connected manifold band** (M, f) consists of a connected manifold M and a continuous map $f : M \to S^1$ such that the infinite cyclic cover $\overline{M} := f^*(\mathbb{R})$ is connected (i.e. $f_* : \pi_1(M) \to \pi_1(S^1)$ surjective) and is dominated by a finite CW complex [19, Defn. 15.3]. Observe that the manifold $\overline{M}$ is a strong deformation retract of the homotopy fiber hofiber(f). If $f : M \to S^1$ is homotopic to a fiber bundle projection, say with fiber X', then hofiber(f) is homotopy equivalent to X'.

Theorem 5.6. *Let X be any of the 4-manifolds Q, S, F defined in (1.1, 1.2, 1.4). Let (M, f) be a connected DIFF 5-manifold band such that $\partial X \to \partial M \xrightarrow{\partial f} S^1$ is a DIFF fiber bundle and the homotopy equivalence $\partial X \to$ hofiber(∂f) extends to a homotopy equivalence $X \to$ hofiber(f). Then $f : M \to S^1$ is homotopic rel ∂M to the projection of a smoothable TOP s-block bundle with fiber X.*

Remark 5.7. Topological splitting and fibering of 5-manifolds W over the circle S^1 with fibers like T^4 or $Kl \times S^2$ can be established by Weinberger's splitting theorem [39], since the fundamental groups $\mathbb{Z}^4$ and $\mathbb{Z} \rtimes \mathbb{Z}$ have subexponential growth. A fortiori, simply-connected topological 4-manifolds X are allowable fibers in Weinberger's fibering theorem. These 4-manifolds X are classified, by Milnor in the PL case and Freedman and Quinn in the TOP case [12], up to homotopy equivalence by their intersection form. Unfortunately, the smooth splitting and fibering problems for such W remain unsolved, even as DIFF s-block bundle maps.

Now let us state the smooth result promised in the Introduction.

Theorem 5.8. *Consider the closed, non-orientable, smooth* 4*-manifolds* Q (1.1)*. Let* (M, f) *be a connected* DIFF 5*-manifold band such that* Q *is homotopy equivalent to* hofiber(f)*. Then* $f : M \to S^1$ *is homotopic to the projection of a* DIFF *s-block bundle with fiber* Q*.*

5.2. Vanishing of lower Whitehead groups

We start by showing that every homotopy equivalence under consideration has zero Whitehead torsion. Comparable results are found by J. Hillman [17, §6.1] and intensely use [37, §19].

Proposition 5.9. *Let* X *be any of the* 4*-manifolds* Q, S, F *defined in* (1.1, 1.2, 1.4)*. Let* $h : X \to X$ *be a homotopy self-equivalence which restricts to a diffeomorphism* $\partial h : \partial X \to \partial X$*. Suppose* $Y = X \rtimes_h S^1$ *is the total space of a mapping torus fibration* $X \to Y \to S^1$ *with monodromy* h*. Then the Whitehead groups vanish for all* $* \leqslant 1$:

$$\mathrm{Wh}_*(\pi_1 X) = \mathrm{Wh}_*(\pi_1 Y) = 0.$$

Note if h is a diffeomorphism, then Y is the total space of a DIFF fiber bundle.

Proof. Each case of fiber shall be handled separately.

Case of fiber Q (1.1). Let Q be any of the non-orientable 4-manifolds listed. There are no non-identity automorphisms of the cyclic group $\pi_1(Q) = C_2$ of order two, so $\pi_1(Y) = C_2 \times C_\infty$. S. Kwasik has shown that $\mathrm{Wh}_1(\pi_1 Q) = \mathrm{Wh}_1(\pi_1 Y) = 0$, using the Rim square for the ring $\mathbb{Z}[C_\infty][C_2]$, the Bass–Heller–Swan decomposition, and assorted facts [24, pp. 422–423].

Case of fiber S (1.2). There are three types of S.

Type 1. Suppose S is the total space of a fiber bundle $S^2 \to S \to \Sigma$ such that Σ is a compact, connected, possibly non-orientable 2-manifold of positive genus. Then, by [37, Theorem 19.5(5)], the fundamental group $\pi_1(S) = \pi_1(\Sigma)$ is a member of Waldhausen's class Cl of torsion-free groups. So, by [37, Proposition 19.3], the HNN-extension $\pi_1(Y) = \pi_1(S) \rtimes C_\infty$ is a member of Cl. Therefore, by [37, Theorem 19.4], we obtain that $\mathrm{Wh}_*(\pi_1 S) = \mathrm{Wh}_*(\pi_1 Y) = 0$.

Type 2. Suppose S is the total space of a fiber bundle $H \to S \to S^1$ such that H is a closed, connected, hyperbolic 3-manifold. By Mostow rigidity, we may select the monodromy diffeomorphism $H \to H$ to be an isometry [25] up to smooth isotopy [14]. Since $H \to S \to S^1$ has isometric monodromy implies that S is isometrically covered by $\mathbb{H}^3 \times \mathbb{E}^1$, the curvature matrix is constant, hence S is A-regular. By hypothesis, h is homotopic rel ∂S to an isometry of $S - \partial S$. Then $Y - \partial Y$ is isometrically covered by $\mathbb{H}^3 \times \mathbb{E}^2$, hence $Y - \partial Y$ is A-regular. Therefore, by [11, Lemma 0.12], we obtain that $\mathrm{Wh}_*(\pi_1 S) = \mathrm{Wh}_*(\pi_1 Y) = 0$ for all $* \leqslant 1$.

Type 3. Suppose the interior $S - \partial S$ admits a complete, finite volume metric of euclidean type (resp. real hyperbolic or complex hyperbolic). Since $S - \partial S$ is isometrically covered by $\mathbb{E}^4$ (resp. $\mathbb{H}^4$ or $\mathbb{CH}^2$), the curvature matrix is constant, hence $S - \partial S$ is A-regular. By hypothesis, h is homotopic rel ∂S to an isometry of $S - \partial S$. Then $Y - \partial Y$ is isometrically covered by $\mathbb{E}^4 \times \mathbb{E}^1$ (resp. $\mathbb{H}^4 \times \mathbb{E}^1$ or $\mathbb{CH}^2 \times \mathbb{E}^1$), the curvature matrix is constant, hence $Y - \partial Y$ is A-regular. Therefore, by [11, Lemma 0.12], we obtain that $\mathrm{Wh}_*(\pi_1 S) = \mathrm{Wh}_*(\pi_1 Y) = 0$ for all $* \leqslant 1$.

Case of fiber F (1.4). There are two types of connected summands F_i.

Type 1. Suppose F_i is the total space of a fiber bundle $H_i \to F_i \to S^1$ such that the compact, connected, irreducible, orientable 3-manifold H_i either is S^3 or D^3 or has non-zero first Betti number. Then, by [37, Proposition 19.5(6,8)], we obtain that $\pi_1(H_i)$ is a member of Cl. So, by [37, Proposition 19.3], the HNN-extension $\pi_1(F_i) = \pi_1(H_i) \rtimes C_\infty$ is a member of Cl.

Type 2. Suppose F_i is the total space of a fiber bundle $\Sigma_i^f \to F_i \to \Sigma_i^b$ such that the fiber and base are compact, connected, orientable 2-manifolds of positive genus. Then, by a theorem of J. Hillman [16, Thm. 1], we obtain that $\pi_1(F_i)$ is a member of Cl. Indeed, the direct algebraic proof of Cavicchioli, Hegenbarth and Spaggiari [5, Thm. 3.12] uses a Mayer–Vietoris argument for a connected-sum decomposition of the base Σ_i^b, which extends to aspherical, compact, possibly non-orientable surfaces with possibly non-empty boundary.

Conclusion. Now, since $\pi_1(F_i)$ is a member of Cl, by [37, Proposition 19.3], the fundamental group $\pi_1(F) = \bigstar_{i=1}^n \pi_1(F_i)$ of the connected sum $F = F_1 \# \cdots \# F_n$ is a member of Cl. So the HNN-extension $\pi_1(Y) = \pi_1(F) \rtimes C_\infty$ is a member of Cl. Therefore, by [37, Theorem 19.4], we obtain $\mathrm{Wh}_*(\pi_1 F) = \mathrm{Wh}_*(\pi_1 Y) = 0$. □

5.3. Proof of main theorems over the circle

Proof of Theorem 5.4. By the general splitting theorem (Theorem 4.1), it suffices to show that the following conditions hold, in order:

(1) X satisfies Hypothesis 3.1 or 3.2 or 3.3 and Hypothesis 2.1,
(2) the obstructions $\mathrm{split}_K(g;X)$ and $\mathrm{split}_L(g;X)$ vanish, and
(3) the h-cobordism $(M';X',X')$ and homotopy equivalence $g':X'\to X$ have zero Whitehead torsion.

Condition (1). *Case of* $X=Q$. By Corollary 3.6, Q satisfies Hypothesis 3.2. Since [38, Theorem 13A.1] implies $L_5^h(\mathbb{Z}[C_2]^-)=0$, Q fulfills Theorem 2.2.

Case of $X=S$. There are three types of S. If S is non-orientable, then, since the abelianization $H_1(S;\mathbb{Z})$ is 2-torsionfree, there exists a lift $\hat{\omega}:\pi_1(S)\to\mathbb{Z}$ of the orientation character $\omega:\pi_1(S)\to\mathbb{Z}^\times$.

Type 1. By Corollary 2.4, S satisfies Hypothesis 2.1. If S is orientable, then, by Corollary 3.5, S satisfies Hypothesis 3.1. Otherwise, if S is non-orientable, then, by Corollary 3.8, S satisfies Hypothesis 3.3.

Type 2. Since H is irreducible, by Corollary 2.3, S satisfies Hypothesis 2.1. Recall that S can be isometrically covered by $\mathbb{H}^3\times\mathbb{E}^1$, by Mostow rigidity [25,14]. Then, by [11, Proposition 0.10] and the argument of Corollary 3.5, it follows that κ_2 is injective. If S is orientable, then S satisfies Hypothesis 3.1. Otherwise, if S is non-orientable, then, since the lift $\hat{\omega}$ exists, S satisfies Hypothesis 3.3.

Type 3. By Corollary 2.3, S satisfies Hypothesis 2.1. If S is orientable, then, by Corollary 3.5, S satisfies Hypothesis 3.1. Otherwise, suppose S is non-orientable. The argument in Corollary 3.5 for the injectivity of κ_2 is the same as if S were orientable. Then, since $\hat{\omega}$ exists, S satisfies Hypothesis 3.3.

Case of $X=F$. There are two types of connected summands F_i. By Corollary 2.3, F_i satisfies Hypothesis 2.1. Note that F_i satisfies Hypothesis 3.1, by Corollary 3.7 for *Type* 1 and by Corollary 3.8 for *Type* 2.

Condition (2). Recall the notation $\Pi_0:=\pi_1(X)$ and $\Pi:=\pi_1(X\rtimes_h S^1)\cong\pi_1(M)$. The K-theory obstruction $\mathrm{split}_K(g;X)$ lies in

$$\mathrm{Wh}_0(\Pi_0)\oplus\widetilde{\mathrm{Nil}}_0\big(\mathbb{Z}[\Pi_0];0,0,{}_-\mathbb{Z}[\Pi_0]_+,{}_+\mathbb{Z}[\Pi_0]_-\big).$$

By [37, Theorem 2], the second factor is a summand of $\mathrm{Wh}_1(\Pi)$. Both $\mathrm{Wh}_0(\Pi_0)$ and $\mathrm{Wh}_1(\Pi)$ vanish by Proposition 5.9. The L-theory obstruction $\mathrm{split}_L(g;X)$ lies in $\mathrm{UNil}_6^s(\mathbb{Z}[\Pi_0];0,0)$, which vanishes by definition [2, §1].

Condition (3). The torsions of the h-cobordism $(M';X',X')$ and the homotopy equivalence $g':X'\to X$ lie in $\mathrm{Wh}_1(\Pi_0)$, which vanishes by Proposition 5.9. □

Proof of Theorem 5.5. Let $[g]\in\mathcal{S}^h_{\mathrm{TOP}}(X\rtimes_\alpha S^1)$. Then $[g]$ is an h-bordism class of some homotopy equivalence $g:M\to X\rtimes_\alpha S^1$ such that M is a compact TOP 5-manifold and the restriction $\partial g:\partial M\to\partial X\rtimes_{\partial\alpha}S^1$ is a homeomorphism. Since $\alpha:X\to X$ is a diffeomorphism implies that the mapping torus $X\rtimes_\alpha S^1$ is a DIFF 5-manifold, we have

$$\mathrm{ks}[g]=g_*\mathrm{ks}(M)\in\mathbb{F}_2\oplus H_1(X;\mathbb{F}_2)_\alpha\cong H^4\big(X\rtimes_\alpha S^1,\partial X\rtimes_{\partial\alpha}S^1;\mathbb{F}_2\big).$$

Here, the isomorphism is obtained from Poincaré duality and the Wang sequence. Note, by definition, that the basepoints are respected by $\cup$ and ks, and that the composite $\mathrm{ks}\circ\cup$ vanishes.

Suppose $\mathrm{ks}[g]=0$. Since ∂g induces a DIFF structure on ∂M and $\mathrm{ks}(M)=0$ and $\dim(M)>4$, by a consequence [23, Thm. IV.10.1] of Milnor's Lifting Criterion and the Product Structure Theorem, the DIFF structure on ∂M extends to a DIFF structure on M. So, by Theorem 5.4, we obtain that g is homotopic to a TOP split homotopy equivalence $g':M\to X\rtimes_\alpha S^1$ such that the restriction $g':(g')^{-1}(X)\to X$ is a homotopy self-equivalence. Moreover, the restriction of g' to the exterior of X yields a smoothable TOP s-cobordism $(M';X,X)$ and an α-twisted simplicial loop $(g'_\infty;g'_0,g'_1):(M';X,X)\to X\times(\Delta^1;0,1)$ in $(\widetilde{\mathbb{S}}^s_{\mathrm{TOP}+}(X),\widetilde{\mathrm{G}}^s(X))$. Therefore $[g]=[g']=\cup[g'_\infty,g'_0,g'_1]$. Thus exactness is proven at $\mathcal{S}^h_{\mathrm{TOP}}(X\rtimes_\alpha S^1)$. □

Proof of Theorem 5.6. The hypothesis gives a homotopy equivalence $d:(X,\partial X)\to(\overline{M},\partial\overline{M})$ of pairs with homotopy inverse $u:(\overline{M},\partial\overline{M})\to(X,\partial X)$ such that $\partial u\circ\partial d=\mathbf{1}_{\partial X}$. In particular, d is a domination of $(\overline{M},\partial\overline{M})$ by $(X,\partial X)$. That is, there is a homotopy $H:[0,1]\times\overline{M}\to\overline{M}$ such that $H_0=d\circ u$ and $H_1=\mathbf{1}_{\overline{M}}$.

Recall $\overline{M}=f^*(\mathbb{R})$. Let $\overline{f}:\overline{M}\to\mathbb{R}$ be the sub-projection covering $f:M\to S^1$. Let $t:\overline{M}\to\overline{M}$ be the unique covering transformation such that $\overline{f}t(x)=\overline{f}(x)+1$. Then the following composite is a homotopy self-equivalence of pairs:

$$h:=u\circ t\circ d:(X,\partial X)\longrightarrow(X,\partial X).$$

So, by cellular approximation of h and [30, Proposition 24.4], the mapping torus $X\rtimes_h S^1=X\times[0,1]/(x,0)\sim(h(x),1)$ is a finite Poincaré pair of formal dimension 5, such that $X\cong X\times[1/2]$ is a two-sided Poincaré subpair with tubular neighborhood $X\times[-1,1]\cong X\times[1/3,2/3]$. Furthermore, since $\partial d:\partial X\to\partial\overline{M}$ and $\partial u:\partial\overline{M}\to\partial X$ are the 0-section and projection from $\partial\overline{M}=\partial X\times\mathbb{R}$, we obtain that the homotopy self-equivalence $\partial h:\partial X\to\partial X$ is in fact a self-diffeomorphism on each connected component. In particular, $\partial M=\partial X\rtimes_{\partial h}S^1$.

Observe that the Borel construction fits into a fiber bundle $\mathbb{R}\to\overline{M}\times_{\mathbb{Z}}\mathbb{R}\to M$ and similarly for ∂M. Then the projection $g_1:\overline{M}\times_{\mathbb{Z}}\mathbb{R}=\overline{M}\rtimes_t S^1\to M$ is a homotopy equivalence of manifold pairs. Note $\{H_s\circ t\circ d\}_{s\in[0,1]}$ is a homotopy from $d\circ h$ to $t\circ d:X\to\overline{M}$. Define a continuous map

$$g_2:X\rtimes_h S^1\longrightarrow\overline{M}\rtimes_t S^1;\quad [x,s]\longmapsto\big[H\big(s,td(x)\big),s\big].$$

By cyclic permutation of the composition factors of h, and by the adjunction lemma (see [34]), the map g_2 is a homotopy equivalence of manifold pairs. Let $\bar{g}_i$ be a homotopy inverse of g_i for $i = 1, 2$. Then we obtain a homotopy equivalence

$$g := \bar{g}_2 \circ \bar{g}_1 : (M, \partial M) \longrightarrow \big(X \rtimes_h S^1, \partial X \rtimes_{\partial h} S^1\big).$$

Furthermore, since $\partial X \to \partial M \to S^1$ is already a fiber bundle, the homotopy inverse $\partial\bar{g} = \partial g_1 \circ \partial g_2$ is homotopic to the above diffeomorphism $\partial X \rtimes_{\partial h} S^1 \to \partial M$. By Theorem 5.4, the homotopy equivalence g is homotopic rel ∂M to a map g' such that the TOP transverse restriction $g' : X' := (g')^{-1}(X) \to X$ is a simple homotopy equivalence and there is a homeomorphism $X' \approx X$. Moreover, $M = \cup_{\mathbf{1}_{X'}} M'$ is obtained by gluing the ends of the smoothable TOP self s-cobordism $M' := M - X' \times (-1, 1)$ by the identity map.

Define quotient maps

$$q : X \rtimes_h S^1 \longrightarrow S^1; \quad [x, s] \longmapsto [s],$$
$$q' : M \longrightarrow S^1; \quad q' := q \circ g'.$$

Note $\partial q' = \partial f : \partial M \to S^1$ is the fiber bundle projection. Therefore, by obstruction theory, the continuous map $f : M \to S^1$ and the TOP s-block bundle projection $q' : M \to S^1$ are homotopic rel ∂M if and only if they determine the same kernel subgroup of $\pi_1(M)$. Then, by covering space theory, it suffices to show that the isomorphism $g_* : \pi_1(M) \to \pi_1(X \rtimes_h S^1)$ maps the subgroup $\operatorname{Ker}(f_*) = p_*\pi_1(\overline{M})$ onto the subgroup $\operatorname{Ker}(q_*) = p'_*\pi_1(X \times \mathbb{R})$. Here, $p : \overline{M} \to M$ and $p' : X \times \mathbb{R} \to X \rtimes_h S^1$ are the infinite cyclic covers. Observe that the π_1-isomorphism induced by the split homotopy equivalence $g_2 : X \rtimes_h S^1 \to \overline{M} \rtimes_t S^1$ maps the subgroup $\operatorname{Ker}(q_*) = \pi_1(X)$ onto $\pi_1(\overline{M})$, and that the π_1-isomorphism induced by the homotopy equivalence $g_1 : \overline{M} \rtimes_t S^1 \to M$ maps the subgroup $\pi_1(\overline{M})$ onto $\operatorname{Ker}(f_*)$. So, since $g_1 \circ g_2 = \bar{g}$ is the homotopy inverse of g, we are done. □

Proof of Theorem 5.8. The proof of Theorem 5.6 constructs homotopy equivalences $h : Q \to Q$ and $g : M \to Q \rtimes_h S^1$. Observe Corollary 3.6 implies that Q satisfies Hypothesis 3.2, and Remark 2.6 implies that Q satisfies Hypothesis 2.5. Recall that Conditions (2) and (3) of Proof of Theorem 5.4 hold. Then, by Theorem 4.1, the homotopy equivalence g is homotopic to a map g' such that the DIFF transverse restriction $g' : Q' := (g')^{-1}(Q) \to Q$ is a simple homotopy equivalence and there is a diffeomorphism $Q' \approx Q$. Moreover, the DIFF 5-manifold $M = \cup_{\mathbf{1}_{Q'}} M'$ is obtained by gluing the ends of the DIFF self s-cobordism $M' := M - Q' \times (-1, 1)$ by the identity map. The remainder of Proof of Theorem 5.6 shows that $f : M \to S^1$ is homotopic to the DIFF s-block bundle projection $q' : \cup_{\mathbf{1}_Q}(M'; Q, Q) \to S^1$ obtained from g'. □

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References

[1] Sylvain E. Cappell, Manifolds with fundamental group a generalized free product, Bull. Amer. Math. Soc. 80 (1974) 1193–1198.
[2] Sylvain E. Cappell, Unitary nilpotent groups and Hermitian K-theory, Bull. Amer. Math. Soc. 80 (1974) 1117–1122.
[3] Sylvain E. Cappell, A splitting theorem for manifolds, Invent. Math. 33 (2) (1976) 69–170.
[4] T.A. Chapman, Approximation results in Hilbert cube manifolds, Trans. Amer. Math. Soc. 262 (2) (1980) 303–334.
[5] Alberto Cavicchioli, Friedrich Hegenbarth, Fulvia Spaggiari, Manifolds with poly-surface fundamental groups, Monatsh. Math. 148 (3) (2006) 181–193.
[6] Frank Connolly, Andrew Ranicki, On the calculation of UNil, Adv. Math. 195 (1) (2005) 205–258.
[7] Sylvain E. Cappell, Julius L. Shaneson, On four-dimensional surgery and applications, Comment. Math. Helv. 46 (1971) 500–528.
[8] Sylvain E. Cappell, Julius L. Shaneson, Some new four-manifolds, Ann. of Math. (2) 104 (1) (1976) 61–72.
[9] F. Thomas Farrell, The obstruction to fibering a manifold over a circle, in: Actes du Congrès International des Mathématiciens, Tome 2, Nice, 1970, Gauthier-Villars, Paris, 1971, pp. 69–72.
[10] F.T. Farrell, The obstruction to fibering a manifold over a circle, Indiana Univ. Math. J. 21 (1971/1972) 315–346.
[11] F.T. Farrell, L.E. Jones, Rigidity for aspherical manifolds with $\pi_1 \subset \mathrm{GL}_m(\mathbb{R})$, Asian J. Math. 2 (2) (1998) 215–262.
[12] Michael H. Freedman, Frank Quinn, Topology of 4-Manifolds, Princeton Math. Ser., vol. 39, Princeton University Press, Princeton, NJ, 1990.
[13] Ronald Fintushel, Ronald J. Stern, An exotic free involution on S^4, Ann. of Math. (2) 113 (2) (1981) 357–365.
[14] David Gabai, G. Robert Meyerhoff, Nathaniel Thurston, Homotopy hyperbolic 3-manifolds are hyperbolic, Ann. of Math. (2) 157 (2) (2003) 335–431.
[15] John Hempel, 3-Manifolds, Ann. of Math. Stud., vol. 86, Princeton University Press, Princeton, NJ, 1976.
[16] Jonathan A. Hillman, On 4-manifolds homotopy equivalent to surface bundles over surfaces, Topology Appl. 40 (3) (1991) 275–286.
[17] J.A. Hillman, Four-Manifolds, Geometries and Knots, Geom. Topol. Monogr., vol. 5, Geometry & Topology Publications, Coventry, 2002.
[18] Ian Hambleton, Matthias Kreck, Peter Teichner, Nonorientable 4-manifolds with fundamental group of order 2, Trans. Amer. Math. Soc. 344 (2) (1994) 649–665.
[19] Bruce Hughes, Andrew Ranicki, Ends of Complexes, Cambridge Tracts in Math., vol. 123, Cambridge University Press, Cambridge, 1996.
[20] Qayum Khan, On connected sums of real projective spaces, Indiana University, PhD dissertation, 2006. Thesis advisor: James F. Davis.
[21] Qayum Khan, On smoothable surgery for 4-manifolds, Algebr. Geom. Topol. 7 (2007) 2117–2140.
[22] Vyacheslav S. Krushkal, Frank Quinn, Subexponential groups in 4-manifold topology, Geom. Topol. 4 (2000) 407–430 (electronic).

[23] Robion C. Kirby, Laurence C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Ann. of Math. Stud., vol. 88, Princeton University Press, Princeton, NJ, 1977. With notes by John Milnor and Michael Atiyah.
[24] Sławomir Kwasik, On low-dimensional s-cobordisms, Comment. Math. Helv. 61 (3) (1986) 415–428.
[25] G.D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 53–104.
[26] Andrew J. Nicas, Induction theorems for groups of homotopy manifold structures, Mem. Amer. Math. Soc. 39 (267) (1982), vi+108.
[27] A.A. Ranicki, Algebraic L-theory. III. Twisted Laurent extensions, in: Algebraic K-Theory, III: Hermitian K-Theory and Geometric Application, Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972, in: Lecture Notes in Math., vol. 343, Springer, Berlin, 1973, pp. 412–463.
[28] Andrew Ranicki, The algebraic theory of surgery. II. Applications to topology, Proc. London Math. Soc. (3) 40 (2) (1980) 193–283.
[29] Andrew Ranicki, Exact Sequences in the Algebraic Theory of Surgery, Math. Notes, vol. 26, Princeton University Press, Princeton, NJ, 1981.
[30] Andrew Ranicki, High-Dimensional Knot Theory, Springer Monogr. Math., Springer-Verlag, New York, 1998. Algebraic surgery in codimension 2. With an appendix by Elmar Winkelnkemper.
[31] Sayed K. Roushon, L-theory of 3-manifolds with nonvanishing first Betti number, Int. Math. Res. Not. 3 (2000) 107–113.
[32] Sayed K. Roushon, Vanishing structure set of Haken 3-manifolds, Math. Ann. 318 (3) (2000) 609–620.
[33] Julius L. Shaneson, Non-simply-connected surgery and some results in low dimensional topology, Comment. Math. Helv. 45 (1970) 333–352.
[34] L.C. Siebenmann, A total Whitehead torsion obstruction to fibering over the circle, Comment. Math. Helv. 45 (1970) 1–48.
[35] Laurence Taylor, Bruce Williams, Surgery spaces: Formulae and structure, in: Algebraic Topology, Waterloo, 1978, Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978, in: Lecture Notes in Math., vol. 741, Springer, Berlin, 1979, pp. 170–195.
[36] Friedhelm Waldhausen, Whitehead groups of generalized free products, 1969. Unpublished paper with erratum, electronically available from http://www.maths.ed.ac.uk/~aar/surgery/.
[37] Friedhelm Waldhausen, Algebraic K-theory of generalized free products. I, II, III, IV, Ann. of Math. (2) 108 (1–2) (1978) 135–256.
[38] C.T.C. Wall, Surgery on Compact Manifolds, second ed., Math. Surveys Monogr., vol. 69, Amer. Math. Soc., Providence, RI, 1999. Edited and with a foreword by A.A. Ranicki.
[39] Shmuel Weinberger, On fibering four- and five-manifolds, Israel J. Math. 59 (1) (1987) 1–7.