Free Transformations of $S^1 \times S^n$ of Square-free Odd Period

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ABSTRACT. Let n be a positive integer, and let $\ell > 1$ be square-free odd. We classify the set of equivariant homeomorphism classes of free C_ℓ-actions on the product $S^1 \times S^n$ of spheres, up to indeterminacy bounded in ℓ. The description is expressed in terms of number theory.

The techniques are various applications of surgery theory and homotopy theory, and we perform a careful study of h-cobordisms. The $\ell = 2$ case was completed by B. Jahren and S. Kwasik (2011). The new issues for the case of ℓ odd are the presence of nontrivial ideal class groups and a group of equivariant self-equivalences with quadratic growth in ℓ. The latter is handled by the composition formula for structure groups of A. Ranicki (2009).

1. INTRODUCTION

Let $\ell > 1$ be an integer. Consider the ℓ-periodic homeomorphism without fixed points:

$$T_\ell : S^1 \times S^n \longrightarrow S^1 \times S^n;\ (z, x) \longmapsto (\zeta_\ell z, x) \quad \text{where } \zeta_\ell := e^{i2\pi/\ell} \in \mathbb{C}.$$

Write $\mathcal{A}_\ell^n$ for the set of conjugacy classes (C) in $\mathrm{Homeo}(S^1 \times S^n)$ of those cyclic subgroups C of order ℓ without fixed points. B. Jahren and S. Kwasik classified the case $\ell = 2$ [JK11].

Recall the Euler totient function φ is the number of units modulo a given natural number. Let $d > 1$. A partition Q_d^k of $\mathbb{Z}_d^\times$ is given by $[q] = [q']$ if $a^k q \equiv \pm q' \pmod d$ for some a. The map $(g \longmapsto g^{-1})$ on the cyclic group C_d induces an involution ι on the projective class group $\mathrm{Wh}_0(C_d) := K_0(\mathbb{Z}C_d)/K_0(\mathbb{Z})$ with coinvariants $H_0(C_2; \mathrm{Wh}_0(C_d)) := \mathrm{Wh}_0(C_d)/(1 - \iota)$.

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Theorem 1.1 (Classification Theorem). *Let $\ell > 1$ be square-free odd. Then, $\mathcal{A}_\ell^{2k} = \{(T_\ell)\}$ for all $k > 0$ and $\mathcal{A}_\ell^1 = \{(T_\ell)\}$. Otherwise, for each $k > 1$, there is a finite-to-one surjection*

$$\coprod_{1<d \mid \ell} Q_d^k \times \mathbb{Z}^{(d-1)/2} \times H_0(C_2; \mathrm{Wh}_0(C_d)) \twoheadrightarrow \mathcal{A}_\ell^{2k-1} - \{(T_\ell)\}.$$

The d-indexed terms in the disjoint union have disjoint images. In the d-th image, each point-preimage has cardinality dividing $8\gcd(k, \varphi(d)/2)$, which has bounded growth in ℓ. In particular, the set $\mathcal{A}_\ell^{2k-1}$ of free C_ℓ-actions on $S^1 \times S^{2k-1}$ is countably infinite if $k > 1$.

Different preimages have different cardinalities (6.5). For $n = 3$, this theorem answers the existence part of [Sch85, Problem 6.14]; indeterminacy in the uniqueness is at most 16.

Corollary 1.2. *Let $p \neq 2$ be prime. Then, $\mathcal{A}_p^{2k} = \{(T_p)\}$ for all $k > 0$ and $\mathcal{A}_p^1 = \{(T_p)\}$. Otherwise, for any given $k > 1$, there is a finite-to-one surjection*

$$Q_p^k \times \mathbb{Z}^{(p-1)/2} \times H_0(C_2; \mathrm{Cl}_p) \twoheadrightarrow \mathcal{A}_p^{2k-1} - \{(T_p)\}.$$

Each preimage has cardinality dividing $8\gcd(k, (p-1)/2)$, which is bounded in p.

Here, Cl_p is the ideal class group of $\mathbb{Z}[\zeta_p]$; the involution ι is induced by $(\zeta_p \longmapsto \zeta_p^{-1})$. The three parts are understood by using the quotient manifold M of the free C_p-action, specifically, invariants of the infinite cyclic cover $\bar{M}$, as follows. The Q_p^k-part is the first Postnikov invariant of $\bar{M}$. The $\mathbb{Z}^{(p-1)/2}$-part is a projective ρ-invariant of $\bar{M}$. The Cl_p-part is the Siebenmann end obstruction of $\bar{M}$. The indeterminacy $8\gcd(k, (p-1)/2)$ is due to ineffective action of the group (quadratic growth in p) of self-homotopy equivalences of M.

Remark 1.3. Consider the ideal class group Cl_p^+ of the real subring $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ of $\mathbb{Z}[\zeta_p]$. Write G for the Galois group of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$. The induced $\mathbb{Z}[G]$-module map $\mathrm{Cl}_p^+ \longrightarrow \mathrm{Cl}_p$ is injective ([Was97, Theorem 4.14]). The norm map $N := 1 + \iota : \mathrm{Cl}_p \longrightarrow \mathrm{Cl}_p^+$ is surjective ([Was97, Proof 10.2]). Since the fixed field of the automorphism $\iota \in G$ is $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, ι induces the identity on Cl_p^+. Then, ι induces negative the identity on $\mathrm{Cl}_p^- := \mathrm{Cl}_p / \mathrm{Cl}_p^+$, since

$$\iota(I) = N(I) - I \equiv -I \pmod{\mathrm{Cl}_p^+}.$$

Therefore, we obtain an exact sequence of $\mathbb{Z}[G/\iota]$-modules:

$${}_2(\mathrm{Cl}_p^-) \xrightarrow{\widetilde{1-\iota}} \mathrm{Cl}_p^+ \longrightarrow H_0(C_2; \mathrm{Cl}_p) \longrightarrow \mathrm{Cl}_p^- /2 \longrightarrow 0.$$

Here, ${}_2A := \{a \in A \mid 2a = 0\}$ denotes the exponent-two subgroup of any abelian group A, and $\widetilde{1-\iota} := (1-\iota) \circ s$ is a well-defined homomorphism via a setwise section $s : \mathrm{Cl}_p^- \longrightarrow \mathrm{Cl}_p$.

Remark 1.4. The $\mathrm{Cl}_p^-/2$ are only known for $p < 500$ [Sch98]. Even worse, the Cl_p^+ are only known for $p \leqslant 151$. The Cl_p^+ are *conditionally known* for $157 \leqslant p \leqslant 241$ [Mil15], which we denote by *, under the Generalized Riemann Hypothesis for the zeta function of the Hilbert class field of $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. We list these new results of R. Schoof and J. C. Miller:

TABLE 1.1. Cl_p^+ derives from [Mil15, Theorem 1.1]. $\mathrm{Cl}_p^-/2$ derives from Table 4.4 in [Sch98]. For $H_0(C_2; \mathrm{Cl}_p)$, this group vanishes* for the 46 primes $p \leqslant 241$ not listed.

p	Cl_p^+	$\mathrm{Cl}_p^-/2$	$H_0(C_2; \mathrm{Cl}_p)$
29	0	$(2,2,2)$	$(2,2,2)$
113	0	$(2,2,2)$	$(2,2,2)$
163	$(2,2)*$	$(2,2)$	$4 \leqslant \text{order} \leqslant 16*$
191	$(11)*$	0	$(11)*$
197	$0*$	$(2,2,2)$	$(2,2,2)*$
229	$(3)*$	0	$(3)*$
239	$0*$	$(2,2,2)$	$(2,2,2)*$

Theorem 1.1 follows from Theorems 1.6 and 1.7 below. Consider complex coordinates

$$S^{2k-1} = \{u \in \mathbb{C}^k \mid u \cdot \bar{u} = 1\}.$$

For any q coprime to any $d > 1$, there is a linear isometry of S^{2k-1} giving a free C_d-action:

$$\Phi_{d,q} : S^{2k-1} \longrightarrow S^{2k-1};\ (u_1, u_2, \ldots, u_k) \longmapsto (\zeta_d^q u_1, \zeta_d u_2, \ldots, \zeta_d u_k).$$

Note that the quotient manifold $L_{d,q}^{2k-1} := S^{2k-1}/\Phi_{d,q}$ is called *the lens space of type* $(d; q, 1, \ldots, 1)$.

Remark 1.5. The products of S^1 with the classical lens spaces

$$\Lambda \quad \text{of type } (p; q_1, \ldots, q_k),$$
$$\Lambda' \quad \text{of type } (p; q_1', \ldots, q_k'),$$

are distinguished in Corollary 1.2, first by homotopy type in the first factor, and then by homeomorphism type in the other factors, as follows. First, note that Λ

has the homotopy type of $L_{p,q}$, where $q := q_1 \cdots q_k$, and similarly for Λ' with $q' := q'_1 \cdots q'_k$. Furthermore, these types are equal if and only if $[q] = [q']$ in the set Q_p^k [Coh73, (29.4)]. Now, assume $[q] = [q']$, so there exists a homotopy equivalence $f : \Lambda' \longrightarrow \Lambda$.

Second, assume $0 = \rho[\Lambda', f] = \rho(\Lambda') - \rho(\Lambda)$, which is independent of the choice of f. Indeed, ρ is an invariant of the h-bordism class of (Λ', f) [AS68, 7.5]. Then, $[\Lambda', f] = [S^1 \times \Lambda', \mathrm{id}_{S^1} \times f]$ in $\mathcal{S}^s_{\mathrm{TOP}}(S^1 \times \Lambda)$ maps to zero in $\mathcal{S}^h_{\mathrm{TOP}}(S^1 \times \Lambda) \cong \mathbb{Z}^{(p-1)/2}$ (see Corollary 3.6). This kernel is identified with the kernel of $\tilde{L}^h_{2k}(C_p) \longrightarrow \tilde{L}^p_{2k}(C_p)$, which is further identified with the following cokernel $\mathcal{H}(C_p)$ arising in the Ranicki-Rothenberg sequence [Bak78]:

$$\mathcal{H}(C_p) := \mathrm{Cok}(\hat{H}_0(C_2; K_0(\mathbb{Z}C_p, \mathbb{Q}C_p)) \longrightarrow \hat{H}_0(C_2; \mathrm{Cl}_p)).$$

Thus, the structure $[\Lambda', f]$ lies in the subquotient $\mathcal{H}(C_p)$ of the third factor, that is, $H_0(C_2; \mathrm{Cl}_p)$.

Third, assume the given two-torsion element $[\Lambda', f]$ of $\mathcal{S}^h_{\mathrm{TOP}}(\Lambda)$ vanishes in $H_0(C_2; \mathrm{Cl}_p)$. Then, $f : \Lambda' \longrightarrow \Lambda$ is h-bordant to the identity map. In particular, Λ' is h-cobordant to Λ. Therefore, they are isometric [Mil66, 12.12]; equivalently, Λ and Λ' are homeomorphic.

For any closed manifold X, consider the set $\mathcal{M}^{h/s}_{\mathrm{TOP}}(X)$ of closed topological manifolds M homotopy equivalent to X up to homeomorphism. The calculation of $\mathcal{A}_\ell$ reduces to $\mathcal{M}$.

Theorem 1.6. *Let ℓ be square-free odd. Then, $\mathcal{A}^{2k}_\ell = \{(T_\ell)\}$ for all $k > 0$ and $\mathcal{A}^1_\ell = \{(T_\ell)\}$. Otherwise, for all $k > 1$, passage to orbit spaces induces a bijection*

$$\mathcal{A}^{2k-1}_\ell - \{(T_\ell)\} \xrightarrow[\approx]{} \coprod_{1<d|\ell} \coprod_{[q]\in Q^k_d} \mathcal{M}^{h/s}_{\mathrm{TOP}}(S^1 \times L^{2k-1}_{d,q}).$$

We calculate these $\mathcal{M}$ by methods of surgery theory, and express them with K-theory.

Theorem 1.7. *Let d be square-free odd, q coprime to d, and $k > 1$. There is a surjection*

$$\mathbb{Z}^{(d-1)/2} \times H_0(C_2; \mathrm{Wh}_0(C_d)) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{M}^{h/s}_{\mathrm{TOP}}(S^1 \times L^{2k-1}_{d,q}).$$

Any preimage has cardinality dividing $8\gcd(k, \varphi(d)/2)$, *which has bounded growth in d.*

Theorem 1.6 and Theorem 1.7 are proven in Section 2 and Section 6, respectively. The difficulty in generalizing Theorem 1.1 to all odd ℓ comes from the proof of Theorem 1.6. When $d > 1$ is *not square-free*, say $d = p^2$, the groups $NK_1(\mathbb{Z}[C_{p^2}])$ are huge: they are closely related to infinitely generated modules over the Verschiebung algebra of $\mathbb{F}_p[t]$. Nonetheless, there would be two difficulties in handling elements of NK_1 in this paper: topologically, there would be a "relaxation" obstruction to making Proposition 2.2 work, and algebraically, there would be a "homothety" obstruction to making Lemma 4.1 (1) work.

2. Classification of Homotopy Types

The first stage is the homotopy classification of orbit spaces, then analysis of conjugacy.

Proposition 2.1. *Let $S^1 \times S^n$ be an ℓ-fold regular cyclic cover of a topological space M, with $n \geqslant 1$ and odd $\ell > 1$. Then, M is homotopy equivalent to $S^1 \times S^n$ or $S^1 \times L^n_{d,q}$ with $d|\ell$.*

The degree ℓ must be odd, or else the Klein bottle $M = \mathbb{RP}^2 \# \mathbb{RP}^2$ is a counterexample.

Proof. The regular covering map $S^1 \times S^n \longrightarrow M$ has degree $\ell > 1$. Since ℓ is odd, the quotient manifold M is oriented. If $n = 1$, then M must be homeomorphic to the torus $S^1 \times S^1$. If $n = 2$, then M must be homotopy equivalent to $S^1 \times S^2$. Thus, we now assume $n \geqslant 3$.

The covering map $S^1 \times S^n \longrightarrow M$ has covering group C_ℓ. Write $\Gamma := \pi_1(M)$ for the fundamental group of the quotient space. The exact sequence of homotopy groups contains

$$1 \longrightarrow C_\infty \xrightarrow{\iota} \Gamma \xrightarrow{\varphi} C_\ell \longrightarrow 1.$$

Write $T \in \Gamma$ for the image under ι of a generator of C_∞. Select an element $S \in \Gamma$ such that S maps under φ to a generator s of C_ℓ. Define a setwise section

$$\sigma : C_\ell \longrightarrow \Gamma;\ s^b \longmapsto S^b \quad \text{for all } 0 \leqslant b < \ell.$$

In general, for a group extension equipped with a setwise section, one has that $\Gamma = (\operatorname{Im}\iota)(\operatorname{Im}\sigma)$. Then, for each $x \in \Gamma$, we obtain the normal form $x = T^a S^b$ for some $a \in \mathbb{Z}$ and $0 \leqslant b < \ell$. Note $S^{-1}TS \in \{T, T^{-1}\}$. If $S^{-1}TS = T^{-1}$, then $S^{-\ell}TS^\ell = T^{(-1)^\ell}$, but $S^\ell \in \operatorname{Ker}\varphi = \operatorname{Im}\iota$ and ℓ is odd, so $T = T^{-1}$, a contradiction. Hence, $TS = ST$; therefore, Γ is abelian. Hence, we have that $\pi_1(M) = \Gamma \cong C_\infty \times C_d$ for some divisor d of ℓ (this includes the case of $d = 1$).

There exists a corresponding infinite cyclic cover $\bar{M}$ with covering translation $t : \bar{M} \longrightarrow \bar{M}$. There is a bundle sequence $\mathbb{R} \longrightarrow \mathrm{Torus}(t) \longrightarrow M$, with total space the mapping torus of t.

Observe that $\bar{M}$ is a PD_n-complex, since the PD_n-complex $\mathbb{R} \times S^n$ is its universal cover with finite covering group $\pi_1(\bar{M}) = C_d$. Also, for any PD_n-complex X with $n \geqslant 3$ and $\tilde{X} \simeq S^n$, Wall showed that the first Postnikov invariant $k_1(X) : K(\pi_1 X, 1) \to K(\mathbb{Z}, n+1)$ is a generator of abelian group $H^{n+1}(\pi_1 X; \mathbb{Z})$, and that the oriented homotopy type of X is uniquely determined by the orbit $[k_1(X)]$ under action of the group $\mathrm{Out}(\pi_1 X)$ [Wal67, Theorem 4.3].

If $d = 1$, then $\bar{M}$ is homotopy equivalent to S^n. Otherwise, we assume $d > 1$. Recall the cohomology ring $H^*(C_d;\mathbb{Z}) = \mathbb{Z}[\iota]/(d\iota)$, where ι has degree 2; in particular, $K(C_d,1)$ has 2-periodic cohomology. However, C_d acts freely on $\mathbb{R}\times S^n \simeq S^n$, so a standard argument with the Leray-Serre spectral sequence shows that $K(C_d,1)$ has $(n+1)$-periodic cohomology. Hence, $n = 2k-1$ for some $k > 1$. Write $q\iota^k \in H^{2k}(C_d;\mathbb{Z}) = \mathbb{Z}/d$ for the first Postnikov invariant of $\bar{M}$; we have $\gcd(d,q) = 1$. The lens space $L(d;q,1,\dots,1)$ also has first Postnikov invariant q, so $\bar{M}$ must be homotopy equivalent to $L^{2k-1}_{d,q} = L(d;q,1,\dots,1)$.

In any of these cases of d and q, there exist a closed n-manifold L and a homotopy equivalence $h : L \longrightarrow \bar{M}$. Select a homotopy inverse $\bar{h} : \bar{M} \longrightarrow L$ for h; consider the oriented homotopy equivalence $\alpha := \bar{h}\circ t\circ h : L \longrightarrow L$. By cyclic permutation of factors,

$$\mathrm{Torus}(\alpha) \simeq \mathrm{Torus}(h\circ\bar{h}\circ t) \simeq \mathrm{Torus}(t) \simeq M.$$

Then, on fundamental groups we have $C_d \rtimes_{\alpha_\#} C_\infty \cong C_d \times C_\infty$, where $\alpha_\# \in \mathrm{Out}(C_d)$ is the induced automorphism on $\pi_1(L)$. Hence, $\alpha_\# = \mathrm{id}$, and therefore, $\alpha \simeq \mathrm{id}$ [Coh73, (29.5A)]. □

The linking form on the $(k-1)$-st homology group of the infinite cyclic cover $\bar{M}$ is the 1×1 matrix $[q/p] \in \mathbb{Q}/\mathbb{Z}$ [ST80, Section 77: p. 290], which recovers the Postnikov invariant $q\iota^k$.

In the sequel, we shall fix $k > 1$ and consider the latter, closed $2k$-dimensional manifold

$$X_{d,q} := S^1 \times L^{2k-1}_{d,q}.$$

The following definition generalizes the homeomorphism of Jahren-Kwasik [JK11, Section 4]. Write t and s for the usual generators of C_∞ and C_d, respectively. Note $(t^k,s^j) \longmapsto (t^k,s^{k+j})$ in $\mathrm{Aut}(C_\infty\times C_d)$ is induced by the well-defined self-homeomorphism (like a Dehn twist):

$$\begin{aligned} &\varepsilon : X_{d,q} \longrightarrow X_{d,q};\\ &(z,[u_1:u_2:\ldots:u_k]) \longmapsto (z,[z^{q/d}u_1 : z^{1/d}u_2 : \cdots : z^{1/d}u_k]). \end{aligned} \tag{2.1}$$

This is multiplication by the path

$$[0,2\pi] \longrightarrow \mathrm{GL}_k(\mathbb{C});\ \theta \longmapsto \mathrm{diag}(e^{\theta i q/d}, e^{\theta i/d},\dots,e^{\theta i/d}).$$

Proposition 2.2. *Let $f : M \longrightarrow X_{d,q}$ be a homotopy equivalence with M a closed manifold. There exists $\delta \in \mathrm{Homeo}(M)$ satisfying a homotopy commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & X_{d,q} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\varepsilon^2} \\ M & \xrightarrow{f} & X_{d,q}. \end{array}$$

Later, in Section 4, we prove Proposition 2.2 based on surgery-theoretic calculations.

Notice that $\pi_1(X_{d,q}) = C_\infty \times C_d$ does *not* have a *unique* infinite cyclic subgroup Z of index d; rather, there are exactly d such subgroups (generated by ts^r with $0 \leqslant r < d$). Although each Z is normal, none is characteristic: $\mathrm{Aut}(C_\infty \times C_d)$ acts transitively on them.

Corollary 2.3. *Let M be a closed manifold in the homotopy type of $X_{d,q}$. Let Z and Z' be infinite cyclic subgroups of index d in $\pi_1(M)$. Then, $\delta'_\#(Z) = Z'$ for some $\delta' \in \mathrm{Homeo}(M)$.*

Proof. Select a homotopy equivalence $f : M \longrightarrow X_{d,q}$. There are integers a and b such that $f_\#(Z)$ and $f_\#(Z')$ are generated by ts^a and ts^b, respectively, in $\pi_1(X_{d,q})$. By Proposition 2.2, there is $\delta \in \mathrm{Homeo}(M)$ with $f \circ \delta \simeq \varepsilon^2 \circ f$. Define $\delta' := \delta^{(b-a)(1-d)/2} \in \mathrm{Homeo}(M)$. Note

$$\begin{aligned} \delta'_\#(f_\#^{-1}(ts^a)) &= f_\#^{-1}(\varepsilon_\#^{(b-a)(1-d)}(ts^a)) \\ &= f_\#^{-1}(ts^{(b-a)(1-d)}s^a) = f_\#^{-1}(ts^b). \end{aligned}$$ ◻

Proof of Theorem 1.6. Conjugate subgroups of $\mathrm{Homeo}(S^1 \times S^n)$ give homeomorphic orbit spaces. Then, by Proposition 2.1, we can define a function Φ given by homeomorphism classes of homotopy types of orbit spaces:

$$\Phi : \mathcal{A}_\ell^n \longrightarrow \mathcal{M}_{\mathrm{TOP}}^{h/s}(S^1 \times S^n) \sqcup \begin{cases} \emptyset & \text{if } n = 1 \text{ or } n = 2k, \\ \displaystyle\coprod_{1<d|\ell} \coprod_{[q]\in Q_d^k} \mathcal{M}_{\mathrm{TOP}}^{h/s}(X_{d,q}) & \\ & \text{if } n = 2k-1 \geqslant 3. \end{cases}$$

Note $\Phi\{(T_\ell)\} = \{[S^1 \times S^n]\} = \mathcal{M}_{\mathrm{TOP}}^{h/s}(S^1 \times S^n)$, where the latter equality follows from classification of surfaces if $n = 1$, Thurston's Geometrization Conjecture if $n = 2$ (see [And04]), and the topological surgery sequence [KS77] if $n \geqslant 3$ (use [FQ90] if $n = 3$).

First, suppose $n = 1$. Then, as noted above, Φ is constant, and hence surjective. (Since ℓ is odd, only the torus $S^1 \times S^1$ has ℓ-fold cover $S^1 \times S^1$. That is, $\Phi(\mathcal{A}_\ell^1) = \{[S^1 \times S^1]\}$.)

Let $(C) \in \mathcal{A}^1_\ell$. There exists a choice of homeomorphism

$$h : (S^1 \times S^1)/C \longrightarrow S^1 \times S^1.$$

Under the quotient map $S^1 \times S^1 \longrightarrow (S^1 \times S^1)/C$ composed with h, the image of the fundamental group of $S^1 \times S^1$ is a subgroup Z of index ℓ in $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$. There exists a nontrivial homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}/\ell$ such that $Z = \mathrm{Ker}(\phi)$. Write $a := \phi(1,0)$ and $b := \phi(0,1)$. Post-composition with an automorphism of $\mathbb{Z}/\ell$ preserves the kernel Z, so we may assume that either $a = 1$ or $(a,b) = (0,1)$. If $a = 1$ then define $A := \left[\begin{smallmatrix} 1 & 0 \\ -b & 1 \end{smallmatrix}\right]$. If $(a,b) = (0,1)$ then define $A := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. In any case, the unimodular matrix $A \in \mathrm{GL}_2(\mathbb{Z}/\ell)$ carries (a,b) to $(1,0)$. Observe $(1,0)$ corresponds to the index ℓ subgroup $\ell\mathbb{Z} \times \mathbb{Z}$. There is $\delta' \in \mathrm{Homeo}(S^1 \times S^1)$ inducing A on fundamental group. Write $h' := \delta' \circ h$. Then, by the lifting property of covering spaces, there exists a commutative diagram

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\ \widehat{h'}\ } & S^1 \times S^1 \\ \big\downarrow /C & & \big\downarrow /T_\ell \\ (S^1 \times S^1)/C & \xrightarrow{\ h'\ } & S^1 \times S^1. \end{array}$$

The element $\widehat{h'} \in \mathrm{Homeo}(S^1 \times S^1)$ conjugates T_ℓ into C. Therefore, Φ is injective.

Now, suppose $n > 1$ and that the orbit space of $(C) \in \mathcal{A}^n_\ell$ is homeomorphic to $S^1 \times S^n$, say by a homeomorphism h. Since $\pi_1(S^1 \times S^n) = C_\infty$ has a unique subgroup of index ℓ, by the lifting property of covering spaces, there exists a commutative diagram

$$\begin{array}{ccc} S^1 \times S^n & \xrightarrow{\ \hat{h}\ } & S^1 \times S^n \\ \big\downarrow /C & & \big\downarrow /T_\ell \\ (S^1 \times S^n)/C & \xrightarrow{\ h\ } & S^1 \times S^n. \end{array}$$

In other words, there is $\hat{h} \in \mathrm{Homeo}(S^1 \times S^n)$ that conjugates T_ℓ into C. Thus, Φ restricts to

$$\Phi : \mathcal{A}^n_\ell - \{(T_\ell)\} \longrightarrow \begin{cases} \emptyset & \text{if } n = 1 \text{ or } n = 2k, \\ \displaystyle\coprod_{1<d|\ell} \ \coprod_{[q]\in Q^k_d} \mathcal{M}^{h/s}_{\mathrm{TOP}}(X_{d,q}) & \text{if } n = 2k-1 \geqslant 3. \end{cases}$$

Next, we show that Φ is surjective if $n = 2k - 1 \geqslant 3$. Let M be a closed manifold in the homotopy type of some example $X_{d,q}$, say by a homotopy equivalence

f. There is a pullback diagram of covering spaces

$$\begin{CD} \hat{M} @>{\hat{f}}>> S^1 \times S^n \\ @VVV @VV{/T_{d,q}}V \\ M @>{f}>> X_{d,q} \end{CD}$$

Let $T \neq \mathrm{id}$ be a covering transformation of $\hat{M}$. Since

$$\mathcal{M}^{h/s}_{\mathrm{TOP}}(S^1 \times S^n) = \{[S^1 \times S^n]\},$$

there is a homeomorphism $h : \hat{M} \longrightarrow S^1 \times S^n$. Then, $T_M := h \circ T \circ h^{-1}$ is an element of $\mathrm{Homeo}(S^1 \times S^n)$ of order d without fixed points. Hence, $M = \Phi(T_M)$ and Φ is surjective.

Finally, we show that Φ is injective if $n = 2k - 1 \geqslant 3$. Let $(C), (C') \in \mathcal{A}^n_\ell$ have orbit spaces M, M' in the homotopy type of some example $X_{d,q}$. Suppose there is a homeomorphism $h : M' \longrightarrow M$. Write $\Pi := \pi_1(S^1 \times S^n)$. Consider the lifting problem

$$\begin{CD} S^1 \times S^n @>{}>{\text{- - -}}> S^1 \times S^n \\ @V{p'}VV @VV{p}V \\ M' @>{h}>> M. \end{CD}$$

By Corollary 2.3, there exists $\delta' \in \mathrm{Homeo}(M)$ such that $\delta'_\#((h \circ p')_\#(\Pi)) = p_\#(\Pi)$. Note $h' := \delta' \circ h : M' \longrightarrow M$ satisfies $(h' \circ p')_\#(\Pi) = p_\#(\Pi)$. Then, by the lifting property, there is $\widehat{h'} \in \mathrm{Homeo}(S^1 \times S^n)$ covering h' that conjugates C' to C. Therefore, Φ is injective. ◻

See [Tha10] for the homotopy types of free C_p-actions on products of 1-connected spheres.

3. Classification of h-cobordism Types

For the second stage, consider the subgroup $\mathrm{SI}(X)$ of $\mathrm{Wh}_1(\pi_1 X)$ consisting of the Whitehead torsions of *strongly inertial h-cobordisms*, that is, the torsion $\tau(W \twoheadrightarrow X)$ of any h-cobordism $(W; X, X')$ such that the map $X' \hookrightarrow W \twoheadrightarrow X$ is homotopic to a homeomorphism.

Theorem 3.1. *Let M and X be closed connected topological manifolds of dimension $n \geqslant 4$. If $n = 4$, then assume $\pi_1 X$ is good in the sense of Freedman-Quinn* [FQ90]*. If M is homotopy equivalent to X, then $\mathrm{SI}(M) \cong \mathrm{SI}(X)$ as subgroups of $\mathrm{Wh}_1(\pi_1 M) \cong \mathrm{Wh}_1(\pi_1 X)$.*

This theorem is an affirmative answer to a question raised by Jahren-Kwasik [JK15, Section 7]. Later, in Section 5, we shall develop the techniques needed to prove this theorem.

Next, for any compact manifold X, write $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ for the set of pairs (M, f), where M is a compact topological manifold and $f : M \longrightarrow X$ is a homotopy equivalence that restricts to a homeomorphism $\partial f : \partial M \longrightarrow \partial X$, taken up to s-bordism relative to ∂X. Assuming that the s-cobordism theorem applies, then $[M, f] = [M', f']$ if and only if f' is homotopic to $f \circ h$ relative to ∂X for some homeomorphism $h : M' \longrightarrow M$. Then, observe

$$\mathcal{M}^{h/s}_{\mathrm{TOP}}(X) = \mathrm{hMod}(X) \setminus \mathcal{S}^{h/s}_{\mathrm{TOP}}(X).$$

Here, $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ has a canonical left action by the group $\mathrm{hMod}(X)$, which consists of homotopy equivalences $X \to X$ restricting to the identity on ∂X, taken up to homotopy rel ∂X.

The first step in proving Theorem 1.7 is an observation of Jahren-Kwasik [JK15, Section 3]. In the definition of $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$, weaken the equivalence relation "s-bordism" to "h-bordism." Then, the resulting set $\mathcal{S}^{h}_{\mathrm{TOP}}(X)$ has the structure of an abelian group, according to Ranicki [Ran92]. Hence, $\mathcal{S}^{h}_{\mathrm{TOP}}(X)$ is more calculable; it also has a left setwise action of $\mathrm{hMod}(X)$.

Proposition 3.2 (Jahren-Kwasik). *Let X be a closed connected topological manifold of dimension $n \geqslant 4$. If $n = 4$, then assume $\pi_1 X$ is good in the sense of Freedman-Quinn* [FQ90]. *The set $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ has a canonical right action of the Whitehead group $\mathrm{Wh}_1(\pi_1 X)$, so that*

$$\mathcal{S}^{h}_{\mathrm{TOP}}(X) = \mathcal{S}^{h/s}_{\mathrm{TOP}}(X) / \mathrm{Wh}_1(\pi_1 X).$$

The isotropy group of any element $[M, f]$ in $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ is the subgroup $f_* \mathrm{SI}(M)$. The forgetful map $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X) \longrightarrow \mathcal{S}^{h}_{\mathrm{TOP}}(X)$ is equivariant with respect to the left action of $\mathrm{hMod}(X)$.

Only the isotropy group of $[M, f] = [X, \mathrm{id}]$ is proven in [JK15, Section 3]; we prove the others.

Proof. Recall the canonical left action. Let

$$\gamma \in \mathrm{hMod}(X) \qquad \text{and} \qquad [M, f] \in \mathcal{S}^{h/s}_{\mathrm{TOP}}(X).$$

Define $\gamma \cdot [M, f] := [M, \gamma \circ f]$. The left action on $\mathcal{S}^{h}_{\mathrm{TOP}}(X)$ has the same formula, so the forgetful map is equivariant.

Next, we recall the canonical right action. Let $[M, f] \in \mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ and let $\alpha \in \mathrm{Wh}_1(\pi_1 X)$. By realization, there is an h-cobordism $(W; M, M')$ with torsion $\tau(W \twoheadrightarrow M) = f_*^{-1}(\alpha)$. Define

$$[M, f] \cdot \alpha := [M', f \circ (M \twoheadleftarrow W \leftarrow M')].$$

This is well defined in $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X)$ since $(W; M, M')$ is unique up to homeomorphism rel M. Thus, the forgetful map induces a function

$$\mathcal{S}^{h/s}_{\mathrm{TOP}}(X) / \mathrm{Wh}_1(\pi_1 X) \longrightarrow \mathcal{S}^{h}_{\mathrm{TOP}}(X),$$

a bijection.

Finally, we determine isotropy groups of the right action. Clearly, $f_* \,\mathrm{SI}(M)$ fixes $[M, f]$. Suppose $[M, f] \cdot \alpha = [M, f]$. Abbreviate the homotopy equivalence $g_\alpha := (M' \hookrightarrow W \twoheadrightarrow M)$. Then, $f \circ g_\alpha$ is s-bordant to f. By the s-cobordism theorem, there exists a homeomorphism $h : M' \longrightarrow M$ such that $f \circ g_\alpha$ is homotopic to $f \circ h$. By post-composition with a homotopy inverse $\bar{f} : X \longrightarrow M$ of f, we have g_α is homotopic to h. Therefore, $f_*^{-1}(\alpha) \in \mathrm{SI}(M)$. ◻

In general, when $X = S^1 \times Y$, the Ranicki-Shaneson decomposition for L^h-groups [Ran73a] induces a corresponding decomposition for the h-structure groups [Ran92, C1].

Proposition 3.3 (Ranicki). *Let Y be a topological space, and let m be an integer. There is a functorial isomorphism of algebraic structure groups:*

$$\mathcal{S}^h_m(S^1 \times Y) \cong \mathcal{S}^h_m(Y) \oplus \mathcal{S}^p_{m-1}(Y).$$

Further, suppose that Y is a closed connected topological manifold of dimension $n-1$. The total surgery obstruction of Ranicki [Ran92, Theorem 18.5] gives the identifications

$$\mathcal{S}^h_{\mathrm{TOP}}(S^1 \times Y) \xrightarrow[\approx]{s} \mathcal{S}^h_{n+1}(S^1 \times Y),$$

and

$$\mathcal{S}^h_{\mathrm{TOP}}(I \times Y) \xrightarrow[\approx]{s} \mathcal{S}^h_{n+1}(Y).$$

Since s exists for all dimensions n, by the Five Lemma applied to the 4-dimensional surgery sequence [FQ90, Section 11.3], we also have these bijections when $n = 4$ and $\pi_1 Y$ is finite.

The next two lemmas determine certain $\mathcal{S}_*(Y)$ when Y is a lens space of odd order.

Lemma 3.4. *Let $d > 1$ be odd, select q coprime to d, and let $k > 1$. Then, $\mathcal{S}^{s,h}_{2k+1}(L^{2k-1}_{d,q}) = 0$.*

Proof. Write $L^n := L^{2k-1}_{d,q}$. Consider the s- or h-algebraic surgery exact sequence [Ran92]:

$$L^{s,h}_{2k+1}(C_d) \longrightarrow \mathcal{S}^{s,h}_{2k+1}(L^n) \longrightarrow H_{2k}(L^n; \mathbb{L}\langle 1 \rangle) \xrightarrow{\sigma^{s,h}_{2k}} L^{s,h}_{2k}(C_d).$$

First, since d is odd, $L^{s,h}_{2k+1}(C_d) = 0$ by Bak's vanishing result [Bak75]. Next, we apply the Atiyah-Hirzebruch spectral sequence to the homological version of the normal invariants:

$$E^2_{i,j} = H_i(L^n; L\langle 1\rangle_j) \Longrightarrow H_{i+j}(L^n; \mathbb{L}\langle 1\rangle).$$

The coefficient group $L\langle 1\rangle_j$ vanishes for $j \leqslant 0$ or j odd. Otherwise, it either is $\mathbb{Z}$ if $j \equiv 0 \pmod 4$ or is $\mathbb{Z}/2$ if $j \equiv 2 \pmod 4$. Note that $\tilde{H}_{\mathrm{even}}(L^n;\mathbb{Z}) = 0$, and, since d is odd, that $\tilde{H}_{\mathrm{even}}(L^n;\mathbb{Z}/2) = 0$. Thus, the diagonal entries $i + j =$ even are zero except along $i = 0$. Also note that $H_{\mathrm{odd}}(L^n;\mathbb{Z}) \in \{0, \mathbb{Z}/d, \mathbb{Z}\}$, and, since d is odd, that $H_{\mathrm{odd}}(L^n;\mathbb{Z}/2) = 0$. Therefore, since the image of an odd-order group in either $\mathbb{Z}$ or $\mathbb{Z}/2$ is zero, in summary we obtain

$$H_{2k}(L^n; \mathbb{L}\langle 1\rangle) = E^\infty_{0,2k} = E^2_{0,2k} = L\langle 1\rangle_{2k} = L_{2k}(1). \tag{3.1}$$

Thus, the assembly map is injective, $\sigma^{s,h}_{2k} : L_{2k}(1) \longrightarrow L^{s,h}_{2k}(C_d)$. Hence,

$$S^{s,h}_{2k+1}(L^n) = 0.$$

∎

Lemma 3.5. *Let $d > 1$ be odd, select q coprime to d, and let $k > 1$. Then, $S^p_{2k}(L^{2k-1}_{d,q})$ is free abelian of rank $(d-1)/2$. Moreover, $\tilde{L}^p_{2k}(C_d) \longrightarrow S^p_{2k}(L^{2k-1}_{d,q})$ is injective with finite index.*

Proof. Write $L^n := L^{2k-1}_{d,q}$; consider the p-algebraic surgery sequence [Ran92]:

$$H_{2k}(L^n;\mathbb{L}\langle 1\rangle) \xrightarrow{\sigma^p_{2k}} L^p_{2k}(C_d) \longrightarrow S^p_{2k}(L^n) \longrightarrow H_{2k-1}(L^n;\mathbb{L}\langle 1\rangle) \xrightarrow{\sigma^p_{2k-1}} L^p_{2k-1}(C_d).$$

From the proof of Lemma 3.4, the edge map $L_{2k}(1) \longrightarrow H_{2k}(L^n;\mathbb{L}\langle 1\rangle)$ is an isomorphism, so σ^p_{2k} is split injective. Also, σ^p_{2k-1} is zero, since it factors through $L^h_{2k-1}(C_d) = 0$ above. We thus obtain an exact sequence of abelian groups:

$$0 \longrightarrow \tilde{L}^p_{2k}(C_d) \longrightarrow S^p_{2k}(L^n) \longrightarrow H_{2k-1}(L^n;\mathbb{L}\langle 1\rangle) \longrightarrow 0.$$

Since $\mathbb{R}C_d = \mathbb{R} \times \prod^{(d-1)/2} \mathbb{C}$ as rings, the reduced L-group $\tilde{L}^p_{2k}(C_d)$ is free abelian of rank $(d-1)/2$, and it is detected by the projective multi-signature [Bak78]. From the same Atiyah-Hirzebruch spectral sequence as in the proof of Lemma 3.4, since d is odd, note the following:

$$E^2_{i,j} = H_i(L^n; L\langle 1\rangle_j) = \begin{cases} \mathbb{Z} & \text{if } i = 2k-1, \text{ and 4 divides } j > 0,\\ \mathbb{Z}/d & \text{if } 0 < i < 2k-1 \text{ odd, 4 divides } j > 0,\\ 0 & \text{otherwise,}\end{cases}$$
$$\longrightarrow H_{i+j}(L^n;\mathbb{L}\langle 1\rangle).$$

Then, each $E^\infty_{i,j}$ is either zero or $\mathbb{Z}/\delta$ with $\delta | d$.

Thus, it follows that $H_{2k-1}(L^n; \mathbb{L}\langle 1\rangle)$ is a finite abelian group of odd order.[1] Therefore, it remains to show that $\mathcal{S}^p_{2k}(L^n)$ has no odd torsion.

The function $\mathcal{S}^s_{\mathrm{TOP}}(L^n) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} R^{(-1)^k}_{\hat{G}}$, defined by the difference of ρ-invariants, was shown by Wall to be injective [Wal99, Theorem 14E.7].[2] Later, Macko-Wegner promoted this function to a homomorphism of abelian groups and re-proved its injectivity [MW11, Theorem 5.2]. Therefore, $\mathcal{S}^s_{\mathrm{TOP}}(L^n)$ is free abelian. Then, by the Ranicki-Rothenberg exact sequences [Ran92, p. 327], $\mathcal{S}^s_{2k}(L^n) \longrightarrow \mathcal{S}^h_{2k}(L^n)$ and $\mathcal{S}^h_{2k}(L^n) \longrightarrow \mathcal{S}^p_{2k}(L^n)$ have kernels and cokernels of exponent two, and so $\mathcal{S}^p_{2k}(L^n)$ has no odd torsion; it is free abelian of rank $(d-1)/2$. ◻

Corollary 3.6. *Let $d > 1$ be odd, select q coprime to d, and let $k > 1$. Then, the group $\mathcal{S}^h_{\mathrm{TOP}}(S^1 \times L^{2k-1}_{d,q})$ is free abelian of rank $(d-1)/2$. Moreover, the component homomorphism $\tilde{L}^p_{2k}(C_d) \longrightarrow L^h_{2k+1}(\pi_1 X_{d,q}) \longrightarrow \mathcal{S}^h_{\mathrm{TOP}}(X_{d,q})$ of Wall realization is injective with finite index.*

Proof. This is immediate from Proposition 3.3, Lemma 3.4, and Lemma 3.5. ◻

4. Application to the "Dehn Twist" Homeomorphism

Fix $n = 2k - 1 \geqslant 3$. Recall the self-homeomorphism ε of $X_{d,q} = S^1 \times L^n_{d,q}$ in equation (2.1).

Lemma 4.1. *Let $d > 1$ be an odd integer, and select q coprime to d. We have the following:*

(1) *The self-map ε induces the identity map on $\mathrm{Wh}_1(\pi_1 X_{d,q})$ if d is square-free.*
(2) *The self-map ε induces the identity map on $\mathcal{S}^h_{\mathrm{TOP}}(X_{d,q})$.*
(3) *The self-map ε^2 induces the identity map on $\mathcal{S}^s_{\mathrm{TOP}}(X_{d,q})$.*

The $d = 2$ case for part (2) was a key technical assertion of Jahren-Kwasik [JK11, Section 4].

Remark 4.2. Milnor [Mil66, 1.6] falsely claimed $SK_1(\mathbb{Z}G) = 0$ for all finite abelian groups G; when $G = C_{p^2} \times C_{p^2}$, this SK_1-group is isomorphic to $(\mathbb{Z}/p)^{p-1}$ [Oli88, 9.8 (ii)]. However, it holds for all finite cyclic groups $G = C_n$ by Bass-Milnor-Serre [Bas68, XI:7.3], and so the determinant map $K_1(\mathbb{Z}C_n) \longrightarrow (\mathbb{Z}C_n)^\times$ is an isomorphism. By a theorem of Higman [Bas68, XI:7.1a], the torsion subgroup of $(\mathbb{Z}C_n)^\times$ is $\pm C_n$. Hence, $\mathrm{Wh}_1(C_n)$ is free abelian. Consequently, the proof of [Mil66, Lemma 6.7] still holds in this case, so that the group-ring involution $(g \longmapsto g^{-1})$ induces the identity on the Whitehead group $\mathrm{Wh}_1(C_n)$.

[1] A more detailed analysis can show furthermore that $H_{2k-1}(L^n; \mathbb{L}\langle 1\rangle) \to H_{2k-1}(L^n; ko[\frac{1}{2}])$ is an isomorphism.

[2] For the case of $k = 2$, we use the *simple homology structure set* of the three-dimensional lens space $L^3 = L(d, q)$.

Proof of Lemma 4.1 (1). On the fundamental group $\pi_1(X_{d,q}) = C_\infty \times C_d$, recall that ε induces $(t^k, s^j) \longmapsto (t^k, s^{k+j})$; it is the identity on the subgroup C_d, which is generated by s. Then, by Proposition 5.2(1), we obtain a commutative diagram whose rows are split exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Wh}_1(C_d) & \longrightarrow & \mathrm{Wh}_1(\pi_1 X_{d,q}) & \xrightarrow{\partial} & \mathrm{Wh}_0(C_d) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \\ 0 & \longrightarrow & \mathrm{Wh}_1(C_d) & \longrightarrow & \mathrm{Wh}_1(\pi_1 X_{d,q}) & \xrightarrow{\partial} & \mathrm{Wh}_0(C_d) & \longrightarrow & 0 \end{array}$$

Here, note that $R := \mathbb{Z}[C_d]$, and $\varepsilon : R[t, t^{-1}] \longrightarrow R[t, t^{-1}]$ restricts to ring maps $\varepsilon : R[t^{\pm 1}] \longrightarrow R[t^{\pm 1}]$.

Now, the splitting of the epimorphism ∂ of Bass-Heller-Swan [Bas68, XII:7.4] is

$$h : \mathrm{Wh}_0(C_d) \longrightarrow \mathrm{Wh}_1(\pi_1 X_{d,q}); [P] \longmapsto [t : P[t, t^{-1}] \to P[t, t^{-1}]].$$

Here, P is a finitely generated projective R-module. Then, note

$$\varepsilon_*[P] = (\varepsilon_* \circ \partial \circ h)[P] = (\partial \circ \varepsilon_*)[t : P[t, t^{-1}] \to P[t, t^{-1}]].$$

Since $\varepsilon(t) = st$, and since $\varepsilon(s) = s$ implies

$$(R \hookrightarrow R[t, t^{-1}] \xrightarrow{\varepsilon} R[t, t^{-1}]) = (R \hookrightarrow R[t, t^{-1}]),$$

we have

$$\varepsilon_*[t : P[t, t^{-1}] \to P[t, t^{-1}]] = [st : P[t, t^{-1}] \to P[t, t^{-1}]].$$

Recall [Bas68, IX:6.3] the map ∂ in the localization sequence for $R[t] \to R[t, t^{-1}]$:

$$\varepsilon_*[P] = \partial[st : P[t, t^{-1}] \to P[t, t^{-1}]] = [\mathrm{Cok}(st : P[t] \to P[t])] = [P].$$

Thus, $\varepsilon_* = \mathrm{id}$ on $\mathrm{Wh}_0(C_d)$. Moreover, in $\mathrm{Wh}_1(\pi_1 X_{d,q})$ note

$$\begin{aligned} \varepsilon_*(h[P]) - h[P] &= [s : P \to P] \in \mathrm{Wh}_1(C_d), \\ d \cdot [s : P \to P] &= [s^d = 1 : P \to P] = 0. \end{aligned}$$

Thus, since $\mathrm{Wh}_1(C_d)$ is torsion-free by Remark 4.2, we obtain

$$\varepsilon_* = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \quad \text{on } \mathrm{Wh}_1(\pi_1 X_{d,q}) = \mathrm{Wh}_1(C_d) \oplus \mathrm{Wh}_0(C_d).$$

Therefore, ε induces the identity automorphism on $\mathrm{Wh}_1(\pi_1 X_{d,q})$. ◻

Proof of Lemma 4.1 (2). By Corollary 3.6, it suffices to show that $\varepsilon_* = \mathrm{id}$ on $L^p_{2k}(C_d)$. Its definition is $\varepsilon_* := B \circ \varepsilon_* \circ \bar{B}$, which is in terms of the induced automorphism $\varepsilon_* : L^h_{2k+1}(C_\infty \times C_d) \longrightarrow L^h_{2k+1}(C_\infty \times C_d)$, the epimorphism

$$B : L^h_{2k+1}(C_\infty \times C_d) \longrightarrow L^p_{2k}(C_d),$$

and its algebraic splitting $\bar{B} : L^p_{2k}(C_d) \longrightarrow L^h_{2k+1}(C_\infty \times C_d)$ (see Theorem 1.1 in [Ran73a]). Then, heavily using Ranicki's notation and slightly modifying his proof of splitness [Ran73b, p. 134], we note

$$\begin{aligned}
\varepsilon_*[Q,\varphi] &= (B \circ \varepsilon_* \circ \bar{B})[Q,\varphi] \\
&= B\Big[(Q_t \oplus Q_t, \varphi \oplus -\varphi) \oplus \mathcal{H}_\pm(-Q_t); \\
&\qquad \Delta_{(Q_t,\varphi)} \oplus -Q_t, \left(\begin{smallmatrix} 1 & 0 \\ 0 & st \end{smallmatrix}\right) \Delta_{(Q_t,\varphi)} \oplus -Q_t\Big] \\
&= \Big[B_1^+\Big(\Delta_{(Q,\varphi)} \oplus \Delta^*_{(Q^*,\psi)}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & st \end{smallmatrix}\right)(\Delta_{(Q,\varphi)} \oplus \Delta^*_{(Q^*,\psi)})\Big), \varphi \oplus -\varphi\Big] \\
&\qquad \oplus [\mathcal{H}_\pm(-Q)] \\
&= [B_1^+(Q \oplus Q, Q \oplus stQ), \varphi \oplus -\varphi] \oplus [\mathcal{H}_\pm(-Q)] \\
&= [Q,\varphi] \in L^p_{2k}(C_d).
\end{aligned}$$

Here, the equivalence classes are of various quadratic forms and formations. We have only used that the $\mathbb{Z}[C_d]$-algebra map $\varepsilon_\# : \mathbb{Z}[C_d][t,t^{-1}] \longrightarrow \mathbb{Z}[C_d][t,t^{-1}]$ is graded of degree 0. ◻

Proof of Lemma 4.1 (3). Observe ε_* respects the Ranicki-Rothenberg exact sequence

$$\begin{aligned}
\hat{H}^{n+3}(C_2; \mathrm{Wh}_1\, X_{d,q}) \longrightarrow S^s_{\mathrm{TOP}}(X_{d,q}) \longrightarrow S^h_{\mathrm{TOP}}(X_{d,q}) \\
\longrightarrow \hat{H}^{n+2}(C_2; \mathrm{Wh}_1\, X_{d,q}).
\end{aligned}$$

In particular, by Corollary 3.6, this restricts to an exact sequence

$$0 \longrightarrow H \longrightarrow S^s_{\mathrm{TOP}}(X_{d,q}) \longrightarrow K \longrightarrow 0 \tag{4.1}$$

with H finite abelian and K free abelian. By Lemma 4.1 (1)–(2), $\varepsilon_* = \mathrm{id}$ on H and K. Hence,

$$\varepsilon_* = \begin{pmatrix} \mathrm{id}_H & \nu \\ 0 & \mathrm{id}_{\iota K} \end{pmatrix} \quad \text{on } S^s_{\mathrm{TOP}}(X_{d,q}) = H \oplus \iota K,$$

where $\nu : K \longrightarrow H$ is a component of ε_* and $\iota : K \longrightarrow S^s_{\mathrm{TOP}}(X_{d,q})$ is a choice of the right-inverse of $S^s_{\mathrm{TOP}}(X_{d,q}) \longrightarrow K$. Since $2H = 0$, note $2\nu = 0$. Hence, $\varepsilon_*^2 = \mathrm{id}$ on $S^s_{\mathrm{TOP}}(X_{d,q})$. ◻

We show that the homotopy-theoretic order of ε divides $2d^2$ (see more in the proof of Corollary 6.4).

Lemma 4.3. *The homeomorphism ε^{2d^2} is homotopic to the identity on $X_{d,q} = S^1 \times L^n$.*

Proof. Observe that the d-th power of ε induces the identity on the fundamental group:

$$\begin{aligned}\varepsilon^d : S^1 \times L^n &\longrightarrow S^1 \times L^n; (z, [u_1 : u_2 : \dots : u_k]) \\ &\longmapsto (z, [z^q u_1 : z u_2 : \dots : z u_k]).\end{aligned}$$

Each $1 \leqslant j \leqslant k$ has an isotopy of diffeomorphisms that lifts the generator of $\pi_1(SO_3) = C_2$:

$$\begin{aligned}\rho_j : S^1 \times L^n &\longrightarrow S^1 \times L^n; (z, [u_1 : \dots :_j : \dots :_k]) \\ &\longmapsto (z, [u_1 : \dots : z u_j : \dots : u_k]).\end{aligned}$$

In the proof of [HJ83, Proposition 3.1], Hsiang-Jahren showed that each homotopy class $[\rho_j]$ has order $2d$ in the group $\pi_1(\mathrm{Map}\, L^n, \mathrm{id})$. As S^1 is a co-H-space and $\mathrm{Diff}\, L^n$ is an H-space, the two multiplications on $\pi_1(\mathrm{Diff}\, L^n, \mathrm{id})$ are equal (and abelian), so

$$[\varepsilon^d] = [\rho_1^q \circ \rho_2 \circ \cdots \circ \rho_k] = [\rho_1]^q * [\rho_2] * \cdots * [\rho_k] \in \pi_1(\mathrm{Diff}\, L^n, \mathrm{id}).$$

Therefore,

$$[\varepsilon^{2d^2}] = [\varepsilon^d]^{2d} = [\rho_1]^{2dq} [\rho_2]^{2d} \cdots [\rho_k]^{2d} = 1 \quad \text{in } \pi_1(\mathrm{Map}\, L^n, \mathrm{id}). \qquad \square$$

Structure sets quantify homeomorphism types within a homotopy type, so we can start, as follows.

Proof of Proposition 2.2. Consider the homotopy equivalence

$$\alpha := \bar{f} \circ \varepsilon^2 \circ f : M \longrightarrow M,$$

where $\bar{f}$ denotes a homotopy inverse for f. By the composition formula for Whitehead torsion [Mil66, Lemma 7.8], by topological invariance [Cha74], and by Lemma 4.1 (1),

$$\begin{aligned}\tau(\alpha) &= \tau(\bar{f}) + \bar{f}_*(\tau(\varepsilon^2) + \varepsilon^2_* \tau(f)) \\ &= -f_*^{-1}\tau(f) + f_*^{-1}(0 + \tau(f)) = 0 \in \mathrm{Wh}_1(\pi_1 M).\end{aligned}$$

That is, α is a simple homotopy equivalence, and hence it defines an element $[M, \alpha] \in \mathcal{S}^s_{\mathrm{TOP}}(M)$.

On the other hand, by Lemma 4.1 (3) and Lemma 4.3, note

$$\alpha_* = \bar{f}_* \circ \varepsilon_*^2 \circ f_* = \bar{f}_* \circ \mathrm{id} \circ f_* = \mathrm{id} : \mathcal{S}^s_{\mathrm{TOP}}(M) \longrightarrow \mathcal{S}^s_{\mathrm{TOP}}(M)$$
$$\alpha^{d^2} \simeq \bar{f} \circ \varepsilon^{2d^2} \circ f \simeq \bar{f} \circ \mathrm{id} \circ f \simeq \mathrm{id} : M \longrightarrow M.$$

Then, by Ranicki's composition formula for simple structure groups [Ran09], note

$$\begin{aligned} d^2[M,\alpha] &= \sum_{j=0}^{d^2-1} [M,\alpha] = \sum_{j=0}^{d^2-1} (\alpha_*)^j [M,\alpha] \\ &= [M,\alpha^{d^2}] = [M,\mathrm{id}] = 0 \in \mathcal{S}^s_{\mathrm{TOP}}(M). \end{aligned}$$

By equation (4.1) and Corollary 3.6, $\mathcal{S}^s_{\mathrm{TOP}}(M) \cong \mathcal{S}^s_{\mathrm{TOP}}(X_{d,q})$ is a sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}$. Thus, since d is odd, we must have $[M,\alpha] = 0$. That is, α is s-bordant to the identity. Therefore, by the s-cobordism theorem, α is homotopic to a self-homeomorphism δ. □

5. Classification of Homeomorphism Types

We resume with the calculation of the isotropy subgroups $\mathrm{SI}(M)$ from Proposition 3.2. Understood in the context of an abelian group A with involution $*$, we consider subgroups

$$\begin{aligned} (-1)^n\text{-symmetrics} &:= \{a \in A \mid a = (-1)^n a^*\}, \\ (-1)^n\text{-evens} &:= \{b + (-1)^n b^* \mid b \in A\}. \end{aligned}$$

Furthermore, for use later, we abbreviate symmetrics and evens as, respectively,

$$(+1)\text{-symmetrics} \qquad \text{and} \qquad (+1)\text{-evens},$$

and skew-symmetrics and skew-evens as, respectively,

$$(-1)\text{-symmetrics} \qquad \text{and} \qquad (-1)\text{-evens}.$$

Proposition 5.1. *Let M be a closed connected topological manifold of dimension $n \geqslant 4$. If $n = 4$, then assume $\pi_1 M$ is good in the sense of Freedman-Quinn* [FQ90].

(1) *With respect to the standard involution on $\mathrm{Wh}_1(\pi_1 M)$ given by $(g \longmapsto g^{-1})$,*

$$(-1)^n\text{-evens} \leqslant \mathrm{SI}(M) \leqslant (-1)^n\text{-symmetrics}.$$

Hence, $\mathrm{SI}(M)/(-1)^n$-evens $\leqslant \hat{H}^n(C_2; \mathrm{Wh}_1(\pi_1 M))$, which is a sum of copies of $\mathbb{Z}/2$.

(2) *This quotient is expressible in structure groups* (*add by stacking in the* I-*coordinate*):

$$\mathrm{Cok}(S^s_{\mathrm{TOP}}(M \times I) \to S^h_{\mathrm{TOP}}(M \times I)) \xrightarrow[\cong]{\text{tors}} \frac{\mathrm{SI}(M)}{(-1)^n\text{-evens}}.$$

This quantification generalizes a specific argument given by Jahren-Kwasik [JK15, Section 7]. Our structure sets are "rel ∂" (homeomorphism on the unspecified boundary [Wal99, Section 0]).

Proof of Proposition 5.1 (1). Let $\alpha \in \mathrm{SI}(M)$. There is a strongly inertial h-cobordism $(W; M, M')$ such that $\alpha = \tau(W \twoheadrightarrow M)$. By the composition formula [Mil66, Lemma 7.8],

$$0 = \tau(\mathrm{id}_M) = \tau(M \hookrightarrow W \twoheadrightarrow M) = \tau(W \twoheadrightarrow M) + (W \twoheadrightarrow M)_* \tau(M \hookrightarrow W).$$

Next, by Milnor duality [Mil66, Section 10], note

$$\tau(M' \hookrightarrow W) = (-1)^n \tau(M \hookrightarrow W)^*.$$

Finally, since the h-cobordism is strongly inertial, by Chapman's topological invariance of Whitehead torsion [Cha74], by the composition formula again, and by substitution, note

$$\begin{aligned} 0 = \tau(M' \hookrightarrow W \twoheadrightarrow M) &= \tau(W \twoheadrightarrow M) + (W \twoheadrightarrow M)_* \tau(M' \hookrightarrow W) \\ &= \alpha + (-1)^n (W \twoheadrightarrow M)_* \tau(M \hookrightarrow W)^* = \alpha - (-1)^n \alpha^*. \end{aligned}$$

Thus, $\mathrm{SI}(M) \leqslant (-1)^n$-symmetrics in $\mathrm{Wh}_1(\pi_1 M)$.

We let $\beta \in \mathrm{Wh}_1(\pi_1 M)$. There exists an h-cobordism $(W'; M, M'')$ with $\beta = \tau(W' \twoheadrightarrow M)$. Consider the *untwisted double*

$$W := W' \cup_{M''} -W'.$$

To avoid confusion, we denote $\partial W =: -M_0 \sqcup M_1$ with the canonical homeomorphisms $M_i \approx M$ understood. Note that $(W; M_0, M_1)$ is a strongly inertial h-cobordism, since $(W' \twoheadrightarrow M'' \hookrightarrow W')$ is homotopic to the identity:

$$\begin{aligned} (M_1 \hookrightarrow W \twoheadrightarrow M_0) &= (M_1 \hookrightarrow -W' \twoheadrightarrow M'' \hookrightarrow W' \twoheadrightarrow M_0) \\ &\simeq (M_1 \hookrightarrow -W' \overset{\text{flip}}{\approx} W' \twoheadrightarrow M_0) \\ &\simeq (M_1 \overset{\text{id}}{\approx} M_0). \end{aligned}$$

Using the above techniques, this doubled h-cobordism has Whitehead torsion

$$\begin{aligned}\tau(W \twoheadrightarrow M_0) &= \tau(W \twoheadrightarrow W' \twoheadrightarrow M_0)\\ &= \tau(W' \twoheadrightarrow M_0) + (W' \twoheadrightarrow M_0)_* \tau(W \twoheadrightarrow W')\\ &= \beta + (M'' \hookrightarrow W' \twoheadrightarrow M_0)_* \tau(-W' \twoheadrightarrow M'')\\ &= \beta + (-1)^n \tau(-W' \twoheadrightarrow M_1)^*\\ &= \beta + (-1)^n \beta^*.\end{aligned}$$

In the third step, we could excise $\mathring{W}'$ since $W \twoheadrightarrow W'$ is the identity on W', whose mapping cone consists of elementary expansions. Thus, $\mathrm{SI}(M) \geqslant (-1)^n$-evens in $\mathrm{Wh}_1(\pi_1 M)$. ◻

Proof of Proposition 5.1 (2). Let $f : (W; M_0, M_1) \longrightarrow M \times (I; 0, 1)$ be a homotopy equivalence of manifold triads such that the restriction $\partial f : \partial W \longrightarrow M \times \partial I$ is a homeomorphism. Since $f : W \to M \times I$ represents the retraction $W \twoheadrightarrow M_0$, the h-cobordism $(W; M_0, M_1)$ is strongly inertial. Then, assuming the identification $\partial_0 f : M_0 \longrightarrow M$, we have

$$\tau(f) = \tau(W \twoheadrightarrow M_0) \in \mathrm{SI}(M).$$

Now, we suppose that $F : (V; W, W') \longrightarrow M \times I \times (I; 0, 1)$ is an h-bordism, relative to $M \times \partial I \times I$, existing from f to another such homotopy equivalence $f' : (W'; M_0', M_1') \longrightarrow M \times (I; 0, 1)$ of triads. By the composition formula [Mil66, Lemma 7.8], note

$$\begin{aligned}\tau(M_0 \hookrightarrow W \hookrightarrow V) &= \tau(W \hookrightarrow V) + (W \hookrightarrow V)_* \tau(M_0 \hookrightarrow W),\\ \tau(M_0' \hookrightarrow W' \hookrightarrow V) &= \tau(W' \hookrightarrow V) + (W' \hookrightarrow V)_* \tau(M_0' \hookrightarrow W').\end{aligned}$$

As above, $\tau(f') = \tau(W' \twoheadrightarrow M_0')$. Since $\tau(\mathrm{id}_M) = 0$, by [Mil66, Lemma 7.8] again, note

$$\begin{aligned}\tau(M_0 \hookrightarrow W) &= -(M_0 \hookrightarrow W)_* \tau(f),\\ \tau(M_0' \hookrightarrow W') &= -(M_0' \hookrightarrow W')_* \tau(f').\end{aligned}$$

By Milnor duality [Mil66, Section 10], note

$$\tau(W' \hookrightarrow V) = (-1)^{n+1} \tau(W \hookrightarrow V)^*.$$

Then, since $M_0 \approx M_0'$ and since

$$(M_0 \hookrightarrow W \hookrightarrow V) \text{ is homotopic to } (M_0' \hookrightarrow W \hookrightarrow V),$$

note

$$\tau(W \hookrightarrow V) - (M_0 \hookrightarrow V)_* \tau(f) = (-1)^{n+1}\tau(W \hookrightarrow V)^* - (M_0' \hookrightarrow V)_* \tau(f'),$$
$$\tau(f) - \tau(f') = (M_0 \hookrightarrow V)_*^{-1}(1 + (-1)^n *)\tau(W \hookrightarrow V).$$

Thus, we obtain a well-defined homomorphism of abelian groups, where addition in this relative structure set is given by stacking homotopy equivalences in the I-coordinate:

$$S^h_{\mathrm{TOP}}(M \times I) \xrightarrow{\text{tors}} \frac{\mathrm{SI}(M)}{(-1)^n\text{-evens}}; \ [f] \longmapsto [\tau(f)].$$

Let $\alpha \in \mathrm{SI}(M)$. Then, there exists an h-cobordism $(W; M, M')$ with torsion $\tau(W \twoheadrightarrow M) = \alpha$ such that $(M' \hookrightarrow W \twoheadrightarrow M)$ is homotopic to a homeomorphism. By first mapping $W \twoheadrightarrow M \times \{\frac{1}{2}\}$, and then applying the Homotopy Extension Property with regard to a choice of the above homotopy to a homeomorphism $M' \to M$ and a choice of homotopy of $(M \hookrightarrow W \twoheadrightarrow M)$ to the identity on M, we obtain a homotopy equivalence $f : (W; M, M') \longrightarrow M \times (I; 0, 1)$ such that $\partial f : \partial W \longrightarrow M \times \partial I$ is the prescribed homeomorphism and $f : W \longrightarrow M \times I$ represents $W \twoheadrightarrow M$. Then, $[f] \in S^h_{\mathrm{TOP}}(M \times I)$ and $\tau(f) = \alpha$. Therefore, tors is surjective.

Finally, $\mathrm{tors}[f] = 0$ if and only if $f : W \longrightarrow M \times I$ is h-bordant to a simple homotopy equivalence (as was done in the proof of Proposition 5.1 (1)). Thus, the kernel of tors is the image of $S^s_{\mathrm{TOP}}(M \times I)$. ◻

The homotopy invariance of the subgroup $\mathrm{SI}(X) \leqslant \mathrm{Wh}_1(\pi_1 X)$ is now a corollary.

Proof of Theorem 3.1. The function tors is a homomorphism with respect to Ranicki's abelian group structure on the structure sets. This follows from the commutative diagram with exact rows (using Proposition 5.1 and Theorem 18.5 of [Ran92]):

$$\begin{array}{ccccc}
S^s_{\mathrm{TOP}}(X \times I) & \longrightarrow & S^h_{\mathrm{TOP}}(X \times I) & \xrightarrow{\text{tors}} & \mathrm{SI}(X)/(-1)^n\text{-evens} \\
{\scriptstyle s}\downarrow{\scriptstyle \approx} & & {\scriptstyle s}\downarrow{\scriptstyle \approx} & & \downarrow \\
S^s_{n+2}(X \times I) & \longrightarrow & S^h_{n+2}(X \times I) & \xrightarrow{\text{tors}} & \hat{H}^n(C_2; \mathrm{Wh}_1(\pi_1 X)) \\
{\scriptstyle \partial}\uparrow & & {\scriptstyle \partial}\uparrow & & \uparrow{\scriptstyle \cong} \\
L^s_{n+2}(X \times I) & \longrightarrow & L^h_{n+2}(X \times I) & \xrightarrow{\text{tors}} & \hat{H}^{n+2}(C_2; \mathrm{Wh}_1(\pi_1 X))
\end{array}$$

The bottom two squares consist of *homotopy-invariant* functors from the category of spaces to the category of abelian groups; that is, if continuous functions of spaces are homotopic, then these functors induce equal homomorphisms of abelian groups.

Consider the homotopy class of any continuous function $f : M \longrightarrow X$, which induces a homomorphism $f_* : \mathrm{Wh}_1(\pi_1 M) \longrightarrow \mathrm{Wh}_1(\pi_1 X)$. By the functoriality of the upper-right corner of the diagram, the induced map

$$f_* : \hat{H}^n(C_2; \mathrm{Wh}_1(\pi_1 M)) \longrightarrow \hat{H}^n(C_2; \mathrm{Wh}_1(\pi_1 X))$$

restricts to a map $f_* : \mathrm{SI}(M)/(-1)^n$-evens $\longrightarrow \mathrm{SI}(X)/(-1)^n$-evens of subgroups. Therefore, the induced map $f_* : \mathrm{Wh}_1(\pi_1 M) \longrightarrow \mathrm{Wh}_1(\pi_1 X)$ restricts to a map $f_* : \mathrm{SI}(M) \longrightarrow \mathrm{SI}(X)$. If f is a homotopy equivalence, then all of these induced maps are isomorphisms. ◻

The following proposition is not original; it is merely a record. Recall that $X_{d,q} = S^1 \times L^{2k-1}_{d,q}$.

Proposition 5.2. *Let $d > 1$ be a square-free odd integer. Select an integer q coprime to d. We have the following:*

(1) *There is a canonical identification*

$$\mathrm{Wh}_1(\pi_1 X_{d,q}) = \mathrm{Wh}_1(C_d) \oplus \mathrm{Wh}_0(C_d).$$

(2) *The standard involution ($g \longmapsto g^{-1}$) on $\mathrm{Wh}_1(\pi_1 X_{d,q})$ restricts to the standard involution on $\mathrm{Wh}_1(C_d)$ and to negative the standard involution on $\mathrm{Wh}_0(C_d)$.*

(3) *Furthermore, with respect to these restricted involutions,*

$$\frac{\mathrm{Wh}_1(C_d)}{\text{symmetrics}} = 0 \quad \text{and} \quad \frac{\mathrm{Wh}_0(C_d)}{\text{skew-evens}} = H_0(C_2; \mathrm{Wh}_0(C_d)).$$

Proof.

Part (1) is the fundamental theorem of algebraic K-theory [Bas68, XII:7.3, 7.4b] combined with the vanishing of $NK_1(Z[C_d])$ for d square-free [Har87].

Part (2) is the analysis of the restriction of the overall involution done in page 21 of [Ran73b].

For (3), by Remark 4.2, the group-ring involution ($g \longmapsto g^{-1}$) on $\mathbb{Z}[C_d]$ induces the identity on $\mathrm{Wh}_1(C_d)$. Therefore, $\mathrm{Wh}_1(C_d)/\text{symmetrics} = 0$. The assertion about $\mathrm{Wh}_0(C_d)$ is simply the definition of $H_0(C_2; \mathrm{Wh}_0(C_d))$. ◻

Corollary 5.3. *Let $d > 1$ be square-free odd, select an integer q coprime to d, and let $k > 1$. Let M be any closed topological manifold in the homotopy type of $X_{d,q}$. We can identify*

$$\frac{\mathrm{Wh}_1(\pi_1 X_{d,q})}{\mathrm{SI}(M)} = H_0(C_2; \mathrm{Wh}_0(C_d)).$$

Proof. By Theorem 3.1, $\mathrm{SI}(M) = \mathrm{SI}(X_{d,q})$ as subgroups of $\mathrm{Wh}_1(\pi_1 X_{d,q})$.

The surgery exact sequence for $X_{d,q} \times I \operatorname{rel} \partial$ admits forgetful maps of decorations. Consider the commutative diagram with exact rows, which we write schematically:

$$\begin{array}{ccccccccc} H_{2k+2} & \longrightarrow & L^s_{2k+2} & \longrightarrow & \mathcal{S}^s & \longrightarrow & H_{2k+1} & \longrightarrow & L^s_{2k+1} \\ \| & & \downarrow & & \downarrow & & \| & & \downarrow \\ H_{2k+2} & \longrightarrow & L^h_{2k+2} & \longrightarrow & \mathcal{S}^h & \longrightarrow & H_{2k+1} & \longrightarrow & L^h_{2k+1}. \end{array} \tag{5.1}$$

By Ranicki's version of Shaneson's thesis [Ran73a], Bak's vanishing result [Bak75], and Bak-Kolster's vanishing result [BK82, Corollary 4.7], note the computations:

$$\begin{aligned} L^s_{2k+2}(C_\infty \times C_d) &= L^s_{2k+2}(C_d) \oplus L^h_{2k+1}(C_d) = L^s_{2k+2}(C_d), \\ L^h_{2k+2}(C_\infty \times C_d) &= L^h_{2k+2}(C_d) \oplus L^p_{2k+1}(C_d) = L^h_{2k+2}(C_d), \\ L^s_{2k+1}(C_\infty \times C_d) &= L^s_{2k+1}(C_d) \oplus L^h_{2k}(C_d) = L^h_{2k}(C_d), \\ L^h_{2k+1}(C_\infty \times C_d) &= L^h_{2k+1}(C_d) \oplus L^p_{2k}(C_d) = L^p_{2k}(C_d). \end{aligned}$$

Substituting, we may now consider the following commutative diagram of groups:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^s_{2k+2}(C_d)/H_{2k+2} & \longrightarrow & \mathcal{S}^s & \longrightarrow & H_{2k+1}/L_{2k}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & L^h_{2k+2}(C_d)/H_{2k+2} & \longrightarrow & \mathcal{S}^h & \longrightarrow & H_{2k+1}/L_{2k}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{H}^{2k+2}(C_2; \mathrm{Wh}_1(C_d)) & \longrightarrow & \mathcal{S}^h/\mathcal{S}^s & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{5.2}$$

Clearly, the right column of (5.2) is exact. Next, the work of Bass-Milnor-Serre showed that $\mathrm{Wh}_1(C_d)$ is a free abelian group and that the group-ring involution $(g \longmapsto g^{-1})$ on $\mathbb{Z}[C_d]$ induces the identity on $\mathrm{Wh}_1(C_d)$ (refer to Remark 4.2). Then, the subgroup of skew-symmetrics in $\mathrm{Wh}_1(C_d)$ is zero, and therefore, $\hat{H}^{2k+3}(C_2; \mathrm{Wh}_1(C_d)) = 0$. Recall the vanishing result above: $L^s_{2k+1}(C_d) = 0$. Therefore, by the Rothenberg sequence, the left column of (5.2) is exact. Then, finally, a diagram chase in (5.1) shows that the middle column of (5.2) is exact.

The generalized homology of a space cross a circle admits a canonical decomposition:

$$H_{2k+1} = H_{2k+1}(X_{d,q}; \mathbb{L}\langle 1\rangle) = H_{2k+1}(L^{2k-1}_{d,q}; \mathbb{L}\langle 1\rangle) \oplus H_{2k}(L^{2k-1}_{d,q}; \mathbb{L}\langle 1\rangle).$$

By naturality, the assembly map $H_{2k+1} \longrightarrow L^{s,h}_{2k+1}$ for $X_{d,q}$ is the direct sum of the assembly maps $H_{2k+1} \longrightarrow L^{s,h}_{2k+1} = 0$ and $H_{2k} = L_{2k}(1) \longrightarrow L^{h,p}_{2k}$ for $L^{2k-1}_{d,q}$ by (3.1). Thus, the kernel of the assembly map $H_{2k+1} \longrightarrow L^{s,h}_{2k+1}$ for $X_{d,q}$ is the summand $H_{2k+1}(L^{2k-1}_{d,q}; \mathbb{L}\langle 1\rangle) \cong H_{2k+1}/L_{2k}(1)$. Therefore, by exactness of rows in (5.1), the top and middle rows of (5.2) are exact.

Thus, by the Nine Lemma, the bottom row of (5.2) is exact. Then, by Proposition 5.1,

$$\frac{\mathrm{SI}(X_{d,q})}{\text{evens}} = \hat{H}^{2k+2}(C_2; \mathrm{Wh}_1(C_d)) = \frac{\text{symmetrics in } \mathrm{Wh}_1(C_d)}{\text{evens in } \mathrm{Wh}_1(C_d)}.$$

Therefore, we obtain the formula

$$\mathrm{SI}(X_{d,q}) = \text{symmetrics in } \mathrm{Wh}_1(C_d) \oplus \text{skew-evens in } \mathrm{Wh}_0(C_d).$$

The calculation of $\mathrm{Wh}_1(X_{d,q})/\,\mathrm{SI}(X_{d,q})$ now follows from Proposition 5.2. ◻

Remark 5.4. Proposition 3.2, Corollary 3.6, and Corollary 5.3 produce a based bijection

$$\mathbb{Z}^{(d-1)/2} \times H_0(C_2; \mathrm{Wh}_0(C_d)) \xrightarrow[\approx]{\quad} S^{h/s}_{\mathrm{TOP}}(X_{d,q}).$$

6. Computation of the Action of the Group of Self-Equivalences

For any topological space Z, write $\mathrm{Map}(Z)$ for the topological monoid of continuous self-maps $Z \longrightarrow Z$. Recall that $\mathrm{hMod}(Z) \subset \pi_0 \mathrm{Map}(Z)$ is the group of homotopy classes of self-homotopy equivalences. A pair (X_1, X_2) of based topological spaces satisfies the *Induced Equivalence Property* if

$$[f] \in \mathrm{hMod}(X_1 \times X_2) \Rightarrow [p_j \circ f \circ i_j] \in \mathrm{hMod}(X_j)$$

for both $j = 1, 2$, with based inclusion $i_j : X_j \longrightarrow X_1 \times X_2$ and with projection $p_j : X_1 \times X_2 \longrightarrow X_j$. We slightly simplify the following result of P. I. Booth and P. R. Heath [BH90, Corollary 2.8]. Write $[-,-]_0$ for the set of the based homotopy classes of maps preserving basepoint.

Theorem 6.1 (Booth-Heath). *Let X be a connected CW complex equipped with a co-H-space structure, and let Y be a based connected CW complex such that*

$[Y, X]_0 = 0 = [X \wedge Y, X]_0$. *If (X, Y) satisfies the Induced Equivalence Property, there is a split exact sequence of groups:*

$$1 \longrightarrow [X, \mathrm{Map}(Y)]_0 \longrightarrow \mathrm{hMod}(X \times Y) \longrightarrow \mathrm{hMod}(X) \times \mathrm{hMod}(Y) \longrightarrow 1.$$

Corollary 6.2. *Let Y be a nonempty connected CW complex. Suppose that $\pi_1(Y)$ is finite. Then, there is a natural decomposition of groups:*

$$\mathrm{hMod}(S^1 \times Y) = \pi_1 \mathrm{Map}(Y) \rtimes (\mathrm{hMod}\, S^1 \times \mathrm{hMod}\, Y).$$

Hence, each element of $\mathrm{hMod}(S^1 \times Y)$ *is* splittable: *it restricts to a self-equivalence of* $1 \times Y$.

This is false without the hypothesis, since $\mathrm{hMod}(S^1 \times S^1) = \mathrm{GL}_2(\mathbb{Z}) \not\cong \mathbb{Z} \rtimes (\{\pm 1\} \times \mathbb{Z})$.

Proof of Corollary 6.2. The circle $X = S^1$ is a co-H-space, and it is a model of $K(\mathbb{Z}, 1)$. Note that $[Y, X]_0 = H^1(Y; \mathbb{Z}) = 0$ and

$$[X \wedge Y, X]_0 = H^1(S^1 \wedge Y; \mathbb{Z}) \cong \tilde{H}_0(Y; \mathbb{Z}) = 0.$$

By Theorem 6.1, it remains to show that (S^1, Y) satisfies the Induced Equivalence Property. Let $f : S^1 \times Y \longrightarrow S^1 \times Y$ be a based homotopy equivalence.

On the one hand, to prove that $p_1 \circ f \circ i_1 : S^1 \longrightarrow S^1$ is a homotopy equivalence, we must show that induced map on the Hopfian group $\pi_1(S^1) = C_\infty$ is surjective. Since $f_\#$ is surjective, there exists $(a, b) \in \pi_1(S^1) \times \pi_1(Y)$ such that $f_\#(a, b) = (t, 1)$, where t generates $\pi_1(S^1)$. Then, since $\mathrm{Hom}(\pi_1 Y, \pi_1 S^1) = 1$, note $(p_1)_\#(f_\#(1, b)) = 1$. Thus, $(p_1)_\#(f_\#(a, 1)) = t$.

On the other hand, f induces an isomorphism on $\pi_n(S^1 \times Y) = \pi_n(Y)$ for all $n > 1$. Since Y is a CW complex, by the Whitehead theorem, it remains to show that $p_2 \circ f \circ i_2$ is injective on the co-Hopfian group $\pi_1(Y)$. For all $b \in \pi_1(Y)$, recall $(p_1)_\#(f_\#(1, b)) = 1$. Then, $(p_2 \circ f \circ i_2)_\#(b) = 1$ if and only if $f_\#(1, b) = 1$, if and only if $b = 1$, since $f_\#$ is injective. ◻

Remark 6.3. The corollary below is parallel to $p = 2$; Jahren-Kwasik [JK11, 3.5] showed

$$\mathrm{hMod}(S^1 \times \mathbb{RP}^{2k-1}) = \begin{cases} C_2 \times (C_2)^2 & \text{if } k \equiv 0 \pmod 2 \\ C_2 \times C_4 & \text{if } k \equiv 1 \pmod 2 \end{cases} \times (C_2 \times C_2).$$

Unlike below, the first factor (the C_2 on the left) is not represented by a diffeomorphism. The very last C_2 factor is represented by the diffeomorphism that reflects $\mathbb{RP}^n$ in $\mathbb{RP}^{n-1}$.

Corollary 6.4. *Let $d > 1$ be odd, q coprime to d, and $k > 1$. We have a metabelian group*

$$\mathrm{hMod}(S^1 \times L^{2k-1}_{d,q}) = A \rtimes (C_2 \times B),$$

where A is abelian of order $2d^2$, and B is the exponent $e := \gcd(2k, \varphi(d))$ subgroup of $\mathrm{Aut}(C_d)$.[3] *Furthermore, the subgroup $A \rtimes C_2$ is generated by the three diffeomorphisms*

$$\begin{aligned} \rho : (z,[u]) &\longmapsto (z,[zu_1 : u_2 : \ldots : u_k]) \\ \varepsilon : (z,[u]) &\longmapsto (z,[z^{q/d}u_1 : z^{1/d}u_2 : \ldots : z^{1/d}u_k]) \\ {}^{-} \times \mathrm{id}_{L^n} : (z,[u]) &\longmapsto (\bar{z},[u]). \end{aligned}$$

Proof. Since the fundamental group $\pi_1(L^n) = C_d$ is finite, by Corollary 6.2, we have

$$\mathrm{hMod}(S^1 \times L^n) = \pi_1 \mathrm{Map}(L^n) \rtimes (\mathrm{hMod}\, S^1 \times \mathrm{hMod}\, L^n).$$

The subgroup $\mathrm{hMod}(S^1)$ is generated by the homotopy class of the diffeomorphism ${}^{-} \times \mathrm{id}_{L^n}$. Since d is odd, by [Coh73, (29.5)], any homotopy equivalence $h : L^n \longrightarrow L^n$ is classified uniquely by the induced automorphism $h_\# : s \longmapsto s^a$ on $\pi_1(L^n)$ where $a^k \equiv \deg(h) \pmod d$ and $\deg(h) = \pm 1$; any a with $a^k \equiv \pm 1 \pmod d$ is induced by an equivalence $h_a : L^n \longrightarrow L^n$. That is, since $a^k \equiv \pm 1 \pmod d$ if and only if $a^{2k} \equiv 1 \pmod d$, the homomorphism

$$\# : \mathrm{hMod}(L^n) \longrightarrow \mathrm{Out}(\pi_1 L^n) = \mathrm{Out}(C_d)$$

is injective with image the subgroup B of exponent e.

Consider then the fibration sequence $\mathrm{Map}_0(L^n) \longrightarrow \mathrm{Map}(L^n) \longrightarrow L^n$, where $\mathrm{Map}_0 \subseteq \mathrm{Map}$ is the topological submonoid of basepoint-preserving self-maps. Since $\pi_2(L^n) = 0$, and since any unbased homotopy between two based self-maps of a connected CW complex is relatively homotopic to a based homotopy, there is an exact sequence of abelian groups:

$$1 \longrightarrow \pi_1 \mathrm{Map}_0(L^n) \longrightarrow \pi_1 \mathrm{Map}(L^n) \longrightarrow \pi_1(L^n) \longrightarrow 1.$$

On the one hand, Hsiang-Jahren [HJ83, Proposition 3.1] showed that the forgetful map $\pi_1 \mathrm{Diff}_0(L^n) \longrightarrow \pi_1 \mathrm{Map}_0(L^n)$ is surjective with image of order $2d$ generated by the based homotopy class $[\rho]_0$ of the diffeomorphism ρ. On the other hand, since $\varepsilon_\#(t) = ts$, the unbased homotopy class $[\varepsilon]$ of the diffeomorphism ε maps to the generator s of $\pi_1(L^n)$. Therefore, $\pi_1 \mathrm{Map}(L^n)$ is an abelian group of order $2d^2$ generated by $[\rho]_0$ and $[\varepsilon]$. ◻

[3] Classically, it is known that $\mathrm{Aut}(C_d)$ has order $\varphi(d)$. If d is an odd-prime power, then $\mathrm{Aut}(C_d)$ is cyclic. Conversely, $\mathrm{Aut}(C_d)$ contains a product of copies of C_2, one for one for each odd-prime factor of d, such as $\mathrm{Aut}(C_{15}) = C_2 \times C_4$.

To find $\mathcal{M}^{h/s}_{\mathrm{TOP}}(X_{d,q})$, we now compute the action of the group $\mathrm{hMod}(X_{d,q})$ on $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{d,q})$.

Proof of Theorem 1.7. First, we show the order d^2 subgroup of $\mathrm{hMod}(X_{d,q})$ acts trivially. By the proof of Corollary 6.4, this subgroup is generated by the classes $[\rho^2]$ and $[\varepsilon^2]$ of diffeomorphisms. Let $[M, f] \in \mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{d,q})$, and write $\bar{f} : X_{d,q} \longrightarrow M$ for a homotopy inverse of $f : M \longrightarrow X_{d,q}$. Then, for any element $[\phi] \in \mathrm{hMod}(X_{d,q})$, consider the *pullback* $f^*[\phi] := [\bar{f} \circ \phi \circ f] \in \mathrm{hMod}(M)$. Recall, by Proposition 2.2, that each pullback $f^*[\varepsilon^2]$ is represented by a homeomorphism. Thus, $[\varepsilon^2]$ acts trivially on the hybrid structure set $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{d,q})$.

The overall argument for $[\rho^2]$ is similar to but slightly simpler than that of $[\varepsilon^2]$ in Section 4. By the composition formula for Whitehead torsion, by Lemma 7.8 of [Mil66], and since $\rho_\# = \mathrm{id}$,

$$\begin{aligned}\tau(f^*\rho) &= \tau(\bar{f}) + \bar{f}_*(\tau(\rho) + \rho_*\tau(f))\\ &= -f_*^{-1}\tau(f) + f_*^{-1}(0 + \tau(f)) = 0 \in \mathrm{Wh}_1(\pi_1 M).\end{aligned}$$

Thus, $[M, f^*\rho] \in \mathcal{S}^s_{\mathrm{TOP}}(M)$. Much as in Proposition 3.3, there is a direct sum decomposition

$$\mathcal{S}^s_{\mathrm{TOP}}(X_{d,q}) \cong \mathcal{S}^s_{\mathrm{TOP}}(I \times L^n) \oplus \mathcal{S}^h_{\mathrm{TOP}}(L^n).$$

Since ρ restricts to id on $1 \times L^n \subset S^1 \times L^n$, there is an induced commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}^s_{\mathrm{TOP}}(I \times L^n) & \xrightarrow{\text{glue}} & \mathcal{S}^s_{\mathrm{TOP}}(X_{d,q}) & \xrightarrow{\text{split}} & \mathcal{S}^h_{\mathrm{TOP}}(L^n) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle (\rho|)_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle [\rho_*]} & & \\
0 & \longrightarrow & \mathcal{S}^s_{\mathrm{TOP}}(I \times L^n) & \xrightarrow{\text{glue}} & \mathcal{S}^s_{\mathrm{TOP}}(X_{d,q}) & \xrightarrow{\text{split}} & \mathcal{S}^h_{\mathrm{TOP}}(L^n) & \longrightarrow & 0.
\end{array}$$

The decomposition is compatible with those of $L^s_*(\pi_1 X_{d,q})$ and $H_*(X_{d,q}; \mathbb{L}\langle 1\rangle)$, inducing

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{L}^h_{2k}(C_d) & \longrightarrow & \mathcal{S}^h_{\mathrm{TOP}}(L^n) & \longrightarrow & H_{2k-1}(L^n; \mathbb{L}\langle 1\rangle) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle [(\rho_\#)_*]=\mathrm{id}} & & \downarrow{\scriptstyle [\rho_*]} & & \downarrow & & \\
0 & \longrightarrow & \tilde{L}^h_{2k}(C_d) & \longrightarrow & \mathcal{S}^h_{\mathrm{TOP}}(L^n) & \longrightarrow & H_{2k-1}(L^n; \mathbb{L}\langle 1\rangle) & \longrightarrow & 0.
\end{array}$$

Recall from the proof of Lemma 3.5 that $H_{2k-1}(L^n; \mathbb{L}\langle 1\rangle)$ is an abelian group annihilated by a power of d. An argument similar to that proof shows that $\mathcal{S}^h_{\mathrm{TOP}}(L^n)$

has no "d-torsion."[4] Thus, $[\rho_*] = \mathrm{id}$ on $\mathcal{S}^h_{\mathrm{TOP}}(L^n)$. But $\mathcal{S}^s_{\mathrm{TOP}}(I \times L^n) = 0$ by Lemma 3.4. Therefore, $\rho_* = \mathrm{id}$ on $\mathcal{S}^s_{\mathrm{TOP}}(X_{d,q})$, and then,

$$(f^*\rho^2)_* = \bar{f}_* \circ (\rho^2)_* \circ f_* = \bar{f}_* \circ \mathrm{id} \circ f_* = \mathrm{id} : \mathcal{S}^s_{\mathrm{TOP}}(M) \longrightarrow \mathcal{S}^s_{\mathrm{TOP}}(M),$$
$$(f^*\rho^2)^d \simeq \bar{f} \circ \rho^{2d} \circ f \simeq \bar{f} \circ \mathrm{id} \circ f \simeq \mathrm{id} : M \longrightarrow M.$$

Then, by Ranicki's composition formula for simple structure groups [Ran09], note

$$\begin{aligned} d[M, f^*\rho^2] &= \sum_{j=0}^{d-1} [M, f^*\rho^2] = \sum_{j=0}^{d-1} (f^*\rho^2)^j_* [M, f^*\rho^2] \\ &= [M, (f^*\rho^2)^d] = 0 \in \mathcal{S}^s_{\mathrm{TOP}}(M). \end{aligned}$$

By equation (4.1) and Corollary 3.6, $\mathcal{S}^s_{\mathrm{TOP}}(M) \cong \mathcal{S}^s_{\mathrm{TOP}}(X_{d,q})$ is a sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}$. Thus, $[M, f^*\rho^2] = 0$ since d is odd. That is, $f^*\rho^2$ is s-bordant to id. By the s-cobordism theorem, $f^*\rho^2$ is homotopic to a homeomorphism, and so $[\rho^2]$ acts trivially on $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{d,q})$. Therefore, from Corollary 6.4, the order d^2 subgroup of $\mathrm{hMod}(X_{d,q})$ acts trivially.

Now, this induces a left action of the quotient group $C_2 \times C_2 \times B$ on the set $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{d,q})$. Thus, by Remark 5.4, we are done, since this group has order $4e = 8\gcd(k, \varphi(d)/2)$. ◻

Remark 6.5. Let $p \neq 2$ be prime. This quotient group *does not act with uniform isotropy,* unlike the order p^2 subgroup. To conclude, we discuss the three generators of $C_2 \times C_2 \times C_e$.

(1) The above methods demonstrate that post-composition with ρ^p is the identity on the h-cobordism structure group. There may be a "cross-effect" on the s-cobordism structure group, that is, a nonzero component of ρ^p_* from the free part of $\mathcal{S}^s_{\mathrm{TOP}}(X_{p,q})$ to the 2-torsion part. The author is unaware of the effect within $H_0(C_2; \mathrm{Cl}_p)$-orbits.

(2) Since complex conjugation $^-$ reverses orientation on the symmetric Poincaré complex $\sigma^*(S^1) \in L^1(C_\infty)$, post-composition with the diffeomorphism $^- \times \mathrm{id}_{L_{p,q}}$ is negation[5] on the h-cobordism structure group

$$\mathcal{S}^h_{\mathrm{TOP}}(X_{p,q}) \xleftarrow{\cong} \mathcal{S}^p_{\mathrm{TOP}}(L_{p,q}) = \mathbb{Z}^{(p-1)/2}.$$

[4] This lack of "d-torsion" is true for the h-structure group, despite that $\tilde{L}^h_{2k}(C_d)$ may now have some 2-torsion.

[5] [JK11, Lemma 3.7] falsely implies that $^- \times \mathrm{id}_{\mathbb{RP}^n}$ induces the identity on $\mathcal{S}_{\mathrm{TOP}}(S^1 \times \mathbb{RP}^n)$, rather than negation. The proof's error is that Ranicki's $\mathbb{L}^\cdot$-orientation of a manifold is preserved by tangential homotopy equivalences. Call a manifold w_1*-oriented* if an orientation is chosen on the $\mathrm{Ker}(w_1)$-cover [Wal67, p. 216]. The correction is that the $\mathbb{L}^\cdot$-orientation of a w_1-oriented manifold is preserved by w_1-oriented tangential homotopy equivalences [Ran92, 16.16, Appendix A]. For example, the diffeomorphism $^- \times \mathrm{id}_{\mathbb{RP}^n}$ is tangential with $\mu = +1$ but reverses w_1-orientation.

Then, $^{-} \times \mathrm{id}_{L_{p,q}}$ must act freely away from the $H_0(C_2;\mathrm{Cl}_p)$-orbit of the basepoint $[X_{p,q}, \mathrm{id}]$ of $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{p,q})$. But $^{-} \times \mathrm{id}_{L_{p,q}}$ must fix $[X_{p,q}, \mathrm{id}]$, since any two homeomorphisms $M \longrightarrow X_{p,q}$ are s-bordant.[6] Thus, $^{-} \times \mathrm{id}_{L_{p,q}}$ acts non-uniformly on $\mathcal{S}^{h/s}_{\mathrm{TOP}}(X_{p,q})$.

(3) Let a be a primitive e-th root of unity in the field $\mathbb{F}_p$. Recall, from the proof of Corollary 6.4, that the homotopy equivalence $h_a : L_{p,q} \longrightarrow L_{p,q}$ uniquely induces $s \longmapsto s^a$ on fundamental group. Note $\mathrm{id}_{S^1} \times h_a : X_{p,q} \longrightarrow X_{p,q}$ has zero Whitehead torsion, by the product formula, but the author suspects that $\mathrm{id}_{S^1} \times h_a$ is often non-representable by a homeomorphism of $X_{p,q}$.[7] On the other hand, the automorphism of $\mathcal{S}^h_{\mathrm{TOP}}(X_{p,q})$ induced by $\mathrm{id}_{S^1} \times h_a$ is identified with the automorphism of $\mathcal{S}^p_{\mathrm{TOP}}(L_{p,q}) \cong \mathbb{Z}^{(p-1)/2}$ induced by h_a, given by a permutation matrix Π_a of order $e/2$ determined by a. Both these issues complicate the systematic use of Ranicki's composition formula:

$$[(\mathrm{id}_{S^1} \times h_a) \circ (f : M \longrightarrow X_{p,q})] = [\mathrm{id}_{S^1} \times h_a] + \Pi_a[f] \in \mathcal{S}^h_{\mathrm{TOP}}(X_{p,q}) \cong \mathbb{Z}^{(p-1)/2}.$$

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[6] Suppose there exists $[\alpha] \neq 0 \in H_0(C_2;\mathrm{Cl}_p)$, for example if $p = 29$ by Remark 1.4. It is unlikely that $^{-} \times \mathrm{id}_{L_{p,q}}$ fixes $[X_{p,q}, \mathrm{id}] \cdot [\alpha]$, since the h-cobordism W_α on $X_{p,q}$ with torsion $\alpha \in \mathrm{Wh}_1(C_\infty \times C_p)$ has projection $\alpha \neq 0 \in \mathrm{Wh}_0(C_p) = \mathrm{Cl}_p$. Thus, the h-cobordism is unlikely splittable along $1 \times L_{p,q}$; compare with [FH73, 6.1, 6.3].

[7] Using a splitting argument along $1 \times L_{p,q}$, if $\mathrm{id}_{S^1} \times h_a$ is homotopic to a homeomorphism, then h_a is h-bordant to a homeomorphism, if and only if the Whitehead torsion $\tau(h_a)$ is divisible by two in $\mathrm{Wh}_1(C_p) \cong \mathbb{Z}^{(p-3)/2}$. Note h_a is homotopic to a homeomorphism if and only if $\tau(h_a) = 0$ [Coh73, Section 31], if and only if $e = 2$ [Coh73, (30.1)].

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