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Topological rigidity and H_1–negative involutions on tori

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We show, for $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, there is precisely one equivariant homeomorphism class of C_2–manifolds (N^n, C_2) for which N^n is homotopy equivalent to the n–torus and $C_2 = \{1, \sigma\}$ acts so that $\sigma_*(x) = -x$ for all $x \in H_1(N)$. If $n \equiv 2, 3 \pmod 4$ and $n > 3$, we show there are infinitely many such C_2–manifolds. Each is smoothable with exactly 2^n fixed points.

The key technical point is that we compute, for all $n \geq 4$, the equivariant structure set $\mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ for the corresponding crystallographic group Γ_n in terms of the Cappell UNil–groups arising from its infinite dihedral subgroups.

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1 Introduction

1.1 Statement of results

Our goal here is to analyze topological rigidity for a sequence of crystallographic groups containing 2–torsion. For each n, we define the group $\Gamma_n = \mathbb{Z}^n \rtimes_{-1} C_2$, where C_2 acts on $\mathbb{Z}^n$ by negation: $v \mapsto -v$.

We classify the proper actions of Γ_n on contractible n–manifolds.

The most powerful inspiration for our work is the remarkable rigidity theorem of Farrell and Jones concerning a discrete cocompact group of isometries of a simply connected nonpositively curved manifold (M, Γ). They classify the cocompact proper actions of such a Γ on a contractible manifold, if Γ is torsion-free.

The second major inspiration for our paper is the work of Cappell on UNil. If Γ as above has elements of order 2, then the nontrivial elements of UNil–groups coming from virtually cyclic subgroups of Γ can provide examples of cocompact Γ–manifolds (M', Γ) which are isovariantly homotopy equivalent to, but not homeomorphic to (M, Γ). So how do we classify such actions?

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The *Topological rigidity conjecture* stated below does this. We view it as a version of an old conjecture of Quinn, sharpened through the precision afforded by the work of Davis and Lück [23]. We then prove this conjecture for Γ_n using Bartels and Lück [5], and Connolly and Davis [15].

We can cast our results in terms of an action of a group $C_2 := \{1, \sigma\}$. We say an involution $\sigma\colon N \to N$ is H_1*–negative* if $\sigma_*(x) = -x$, for all $x \in H_1(N)$. We prove:

Theorem 1.1 *Let $\sigma\colon N \to N$ be an H_1–negative involution on a closed manifold homotopy equivalent to the n–torus T^n. Consider the C_2–manifold (N, C_2).*

1. *The fixed set N^{C_2} is discrete and consists of exactly 2^n points.*
2. *If $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, then (N^n, C_2) is equivariantly homeomorphic to the standard example, (T^n, C_2).*
3. *If $n \equiv 2, 3 \pmod 4$ and $n > 3$, there are infinitely many such C_2–manifolds, (N^n, C_2). All are isovariantly homotopy equivalent to (T^n, C_2), but no two are equivariantly homeomorphic. Each is smoothable, hence locally linear.*

By the *standard example* (T^n, C_2) above, we mean the involution $\sigma\colon T^n \to T^n$ given by $\sigma[x] = [-x]$ for all $[x] \in \mathbb{R}^n/\mathbb{Z}^n = T^n$. Recall that any n–manifold homotopy equivalent to the n–torus is homeomorphic to it; see Wall [53], Freedman and Quinn [27] and Anderson [1].

The construction of the exotic involutions mentioned in the theorem uses surgery theory, specifically Wall's realization [53, Theorems 5.8, 6.5]. Write $X := (T^n - (T^n)^{C_2})/C_2$, an open n–manifold. Define $\bar{X}$ as the obvious manifold compactification of X obtained by adding a copy of $\mathbb{RP}^{n-1}$ at each end of X. Note for all $n > 2$ that $\pi_1(\bar{X}) = \Gamma_n$ and that $\bar{X}$ is orientable if and only if n is even. Let $w_n\colon \Gamma_n \to \{\pm 1\}$ be the orientation character of $\bar{X}$. Then, for $n \geq 5$, an element $\theta \in L_{n+1}(\Gamma_n, w_n)$ determines a compact smooth manifold $\theta \cdot \bar{X}$, homotopy equivalent to $\bar{X}$ relative to the boundary. Passing to the two–fold cover and gluing in 2^n copies of D^n with the antipodal action, we get a smooth involution on the torus. All the exotic involutions in the above theorem arise in this way.

Observe that Γ_n is isomorphic to a rank n crystallographic group. This isometric action of Γ_n on $\mathbb{R}^n$ is given by $\mathbb{Z}^n$ acting by translation and C_2 acting by reflection through the origin. We let $(\mathbb{R}^n, \Gamma_n)$ denote this Γ_n–manifold.

We let $\mathcal{S}(\Gamma_n)$ be the set of equivariant homeomorphism classes of contractible n–dimensional manifolds equipped with a proper Γ_n–action. We compute $\mathcal{S}(\Gamma_n)$.

To parametrize the set $\mathcal{S}(\Gamma_n)$, we will need to use the unitary nilpotent groups of Cappell. For D_∞, these have been computed recently by Connolly and Koźniewski [20], Connolly and Davis [15], Connolly and Ranicki [21], and Banagl and Ranicki [2], yielding

$$\mathrm{UNil}_m(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \cong \begin{cases} 0 & \text{if } m \equiv 0 \pmod 4, \\ 0 & \text{if } m \equiv 1 \pmod 4, \\ (\mathbb{Z}/2\mathbb{Z})^\infty & \text{if } m \equiv 2 \pmod 4, \\ (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^\infty & \text{if } m \equiv 3 \pmod 4. \end{cases} \tag{1}$$

Let $\mathrm{mid}(\Gamma_n)$ be the set of maximal infinite dihedral subgroups of Γ_n. Let $(\mathrm{mid})(\Gamma_n)$ be a subset of $\mathrm{mid}(\Gamma_n)$ chosen so that it contains exactly one maximal infinite dihedral subgroup from each conjugacy class. Let D be a maximal infinite dihedral subgroup of Γ_n. For any integer n, with $\varepsilon = (-1)^n$, there is a composite map

$$\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \longrightarrow L_{n+1}(D, w_n) \longrightarrow L_{n+1}(\Gamma_n, w_n).$$

If n is odd, then there is an isomorphism $\mathrm{UNil}_{n-1}(\mathbb{Z};\mathbb{Z},\mathbb{Z}) \xrightarrow{\cong} \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^-,\mathbb{Z}^-)$.

Theorem 1.2 *Suppose $n \geq 4$. Write $\varepsilon := (-1)^n$. The Wall realization map induces a bijection of pointed sets, mapping the zero element to the basepoint $[\mathbb{R}^n, \Gamma_n]$:*

$$\partial_\oplus \colon \bigoplus_{D \in (\mathrm{mid})(\Gamma_n)} \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \xrightarrow{\approx} \mathcal{S}(\Gamma_n)$$

Consequently, $\mathcal{S}(\Gamma_n)$ consists of a single element if $n \equiv 0, 1 \pmod 4$, and $\mathcal{S}(\Gamma_n)$ is countably infinite if $n \equiv 2, 3 \pmod 4$.

We do not need to assume any conditions beyond continuity in order to obtain a full homeomorphism classification and to show all actions are smoothable. It turns out that Smith theory guarantees the fixed sets consist of isolated points (see Section 2). Also, local linearity is a consequence of our calculation (see Remark 4.2, which concludes that the forgetful map $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial\bar{X}) \to \mathcal{S}_{\mathrm{TOP}}(\partial\bar{X})$ is constant).

An action $\Gamma \times X \to X$ of a discrete group Γ on a locally compact Hausdorff space X is *proper* if $\{\gamma \in \Gamma \mid K \cap \gamma K \neq \varnothing\}$ is finite for each compact set $K \subset X$.

Note that, given $(M, \Gamma_n) \in \mathcal{S}(\Gamma_n)$, the quotient manifold $M^n/\mathbb{Z}^n$ is homotopy equivalent to, and hence homeomorphic to, the n–torus by the Borel conjecture; see Hsiang and Wall [29]. Therefore the universal cover M admits a homeomorphism to $\mathbb{R}^n$.

Outline of the argument

In Section 2, we show that any H_1–negative involution on an n–manifold homotopy equivalent to the n–torus has exactly 2^n fixed points. This allows one to deduce a correspondence between H_1–negative involutions on n–manifolds homotopy equivalent to the n–torus and contractible n–manifolds equipped with a proper Γ_n–action. Later in Section 2 we show that any compact C_2–manifold with finite fixed set has the C_2–homotopy type of a finite C_2–CW–complex. This allows one to conclude that any H_1–negative involution on a manifold homotopy equivalent to the n–torus is equivariantly homotopy equivalent to (T^n, C_2) and that any contractible n–manifold equipped with a proper Γ_n–action is equivariantly homotopy equivalent to $(\mathbb{R}^n, \Gamma_n)$.

For $n \geq 4$, the six structure sets we use are introduced in Section 3. These are

$$\begin{array}{ccc} \mathcal{S}(\Gamma_n), & \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n), & \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n), \\ \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X}), & \mathcal{S}_{\mathrm{TOP}}(T^n, C_2), & \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n, C_2). \end{array}$$

For example, the isovariant structure set $\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ is the set of equivalence classes of proper Γ_n–manifolds (M^n, Γ_n), together with an isovariant homotopy equivalence $M \to \mathbb{R}^n$. We show all six structure sets are isomorphic, and compute the fourth one to prove Theorem 1.2. The isomorphisms between the first and second, between the second and fifth, and the third and sixth structure sets are formal and are shown in Section 3. The isomorphism between the fifth and sixth structure set requires a detailed discussion of equivariance versus isovariance and is discussed in Appendix A. The isomorphism between the fourth and fifth structure set requires the use of end theory; see Lemma 3.1. Finally, the computation of the classical surgery-theoretic structure set $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X})$ uses the Farrell–Jones conjecture and is presented at the end of Section 4. This computation also uses the main result of Appendix B, which identifies the assembly map in surgery theory with a corresponding map in equivariant homology.

We prove Theorem 1.2 in Section 4 and then deduce Theorem 1.1 in Section 5.

The final bit of the paper, Section 6, is independent of [5] and gives examples of nonstandard structures on $(\mathbb{R}^n, \Gamma_n)$, hence of exotic H_1–negative involutions on tori. The intent is to show that Cappell's work, for straightforward reasons, gives obstructions to isovariant rigidity of a Γ–space when Γ has elements of order two. Shmuel Weinberger pointed out these counterexamples to simple isovariant rigidity some time ago. Since the argument was never published, we include it here.

1.2 Equivariant rigidity

This paper represents the start of a systematic attack on Quinn's ICM conjecture and the closely related questions of equivariant, isovariant and topological rigidity for a discrete group Γ. We take some time to formulate these questions precisely. Recall that two closed aspherical manifolds with the same fundamental group Γ are homotopy equivalent and that the Borel conjecture for Γ states that any such homotopy equivalence is homotopic to a homeomorphism.

Let Γ be a discrete group. A continuous function $f\colon A \to B$ between Γ–spaces is *equivariant*, or a Γ*–map*, if $f(\gamma x) = \gamma f(x)$ for all $x \in A$ and $\gamma \in \Gamma$; it is *isovariant* if furthermore $\gamma f(x) = f(x)$ implies $\gamma x = x$. A *model for* $E_{\mathrm{fin}}\Gamma$ is a Γ–space M Γ–homotopy equivalent to a Γ–CW–complex such that, for all subgroups H of Γ,

$$M^H \text{ is } \begin{cases} \text{contractible} & \text{if } H \text{ is finite,} \\ \text{empty} & \text{otherwise.} \end{cases}$$

Given any Γ–CW–complex X with finite isotopy groups, there is an equivariant map $X \to E_{\mathrm{fin}}\Gamma$, unique up to equivariant homotopy. It follows that any two models are Γ–homotopy equivalent. Furthermore, a model $E_{\mathrm{fin}}\Gamma$ exists for any group Γ.

A *cocompact manifold model for* $E_{\mathrm{fin}}\Gamma$ is a model M for $E_{\mathrm{fin}}\Gamma$ so that M/Γ is compact and so that M^F is a manifold for all finite subgroups F of Γ. A geometric example is given by a discrete cocompact group Γ of isometries of a simply connected complete nonpositively curved manifold M. *Equivariant (respectively, isovariant) rigidity holds for* Γ if any Γ–homotopy equivalence (respectively, Γ–isovariant homotopy equivalence) $M \to M'$ between cocompact manifold models for $E_{\mathrm{fin}}\Gamma$ is Γ–homotopic (respectively, Γ–isovariantly homotopic) to a homeomorphism.

With this terminology, our results can be restated as showing every proper Γ_n–action on a contractible manifold is a cocompact manifold model for $E_{\mathrm{fin}}\Gamma$, equivariant and isovariant rigidity for Γ_n holds when $n \equiv 0, 1 \pmod 4$ or $n = 2, 3$, and equivariant and isovariant rigidity fail for all other n. Previous results on equivariant and isovariant rigidity are found in Rosas [44], Connolly and Koźniewski [18; 19], Weinberger [54, Section 14.2], Prassidis and Spieler [38] and Moussong and Prassidis [37]. In particular, [19] gave the first examples of groups where isovariant rigidity fails; this proceeded via a version of Whitehead torsion. Proposition 1.1 below shows that the relevant Whitehead group vanishes for Γ_n. We give the first counterexamples to simple isovariant rigidity in print.

In this paper we restrict ourselves to the study of equivariant and isovariant rigidity of Γ_n, rather than for more general discrete groups Γ, for two reasons. First, by Theorem 1.1(1), the singular set is discrete, and that simplifies the local analysis immensely. Second, by Proposition 1.1 below, the group Γ_n is K–flat, so we avoid the subtleties of equivariant Whitehead torsion for topological manifolds. In forthcoming work [16], announced by the third author in [31], we will study equivariant rigidity of $E_{\mathrm{fin}}\Gamma$–manifolds with discrete singular set, without assuming the K–flat condition.

Remark 1.1 A key algebraic property of Γ_n is that it admits split epimorphisms

$$\varepsilon\colon \Gamma_n \longrightarrow \Gamma_1 = \mathbb{Z} \rtimes C_2 = C_2 * C_2$$

to the infinite dihedral group. The last equality results from noting that we have $\Gamma_1 = \langle (0,\sigma), (1,\sigma) \rangle = C_2 * C_2$. The existence of ε follows from the fact that every epimorphism $f\colon \mathbb{Z}^n \to \mathbb{Z}$ gives a split epimorphism $f \rtimes \mathrm{id}\colon \mathbb{Z}^n \rtimes C_2 \to \mathbb{Z} \rtimes C_2$. Thus Γ_n has an injective amalgamated product decomposition,

$$\Gamma_n = \varepsilon^{-1}(C_2 * 1) *_{\varepsilon^{-1}(1)} \varepsilon^{-1}(1 * C_2) \cong \Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1}.$$

Our analysis of $E_{\mathrm{fin}}\Gamma_n$ will have no issues with Whitehead torsion because of this:

Proposition 1.1 *The group Γ_n is K–flat, that is,* $\mathrm{Wh}(\Gamma_n \times \mathbb{Z}^k) = 0$ *for all $k \geq 0$.*

Proof We prove K–flatness of Γ_n by induction on n, as follows. First, note for $\Gamma_0 = C_2$ that $\mathrm{Wh}(C_2 \times \mathbb{Z}^k) = 0$, by using Rim's cartesian square of rings (see Milnor [35, Section 3]) and the vanishing of lower NK–groups of $\mathbb{Z}$ and $\mathbb{F}_2$; see Bass [6, Chapter XII].

Next, by Remark 1.1 and Waldhausen's sequence [51], we obtain

$$\mathrm{Wh}(\Gamma_{m-1} \times \mathbb{Z}^k)^{\oplus 2} \longrightarrow \mathrm{Wh}(\Gamma_m \times \mathbb{Z}^k) \xrightarrow{\partial} \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{m-1+k}]) = 0.$$

This sequence is exact, since the Nil term vanishes because the ring $\mathbb{Z}[\mathbb{Z}^{m-1+k}]$ is regular coherent. Therefore, by induction, we are done proving Γ_n is K–flat. □

1.3 The Topological rigidity conjecture

This section is motivated by the conjecture of F Quinn [39] at the 1986 ICM. It aims to say the same thing, but in a more precise way, by employing the language of Davis and Lück [23].

Our *Topological rigidity conjecture* concerns a discrete cocompact group Γ of isometries of a simply connected complete nonpositively curved manifold X^n (that is, a *Hadamard manifold*). It says, roughly, that any simple isovariant homotopy equivalence $f\colon M \to X$ should be isovariantly homotopic to a homeomorphism, except for the examples created by UNil–groups of virtually cyclic subgroups of Γ. But it does so by parametrizing the set of such f in terms of a homology group. The coefficient spectrum of this homology group, $\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}$, is an $\mathrm{Or}(\Gamma)$–spectrum in the sense of [23] (see Section 4). The homology is applied to a Γ–space with virtually cyclic isotropy. For these virtually cyclic subgroups of Γ, the nonzero homotopy groups of the spectrum are just the UNil of amalgamated products of finite groups.

To formulate the conjecture, one must restrict to isovariant homotopy equivalences since there is no reason to expect equivariant homotopy equivalences to be well behaved (see [54, Section 14.2] for some ill-behaved examples). We also restrict to *simple* isovariant homotopy equivalences, whose definition is indicated below, to separate out the roles of K–theory and L–theory. Furthermore, Theorem 1.1(2) shows that this conjecture cannot be extended to low dimensions.

Conjecture 1.1 *Let X^n be a Hadamard manifold of dimension $n > 3$. Let Γ be a discrete cocompact group of isometries of X. Assume the fixed set X^H has codimension greater than 2 in X^K whenever $K \subsetneq H$ are isotropy groups. There is a bijection*

$$H^{\Gamma}_{n+1}(E_{\mathrm{vc}}\Gamma; \underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xrightarrow{\approx} \mathcal{S}^{\mathrm{iso}}_{\mathrm{rel}}(X^n, \Gamma).$$

The elements of $\mathcal{S}^{\mathrm{iso}}_{\mathrm{rel}}(X, \Gamma)$ are equivalence classes of pairs (M, f), where M is a cocompact, locally flat topological Γ–manifold, and $f\colon M \to X$ is a simple Γ–isovariant homotopy equivalence that restricts to a homeomorphism on the singular set. *Locally flat* means if $M^H \subset M^K$ then M^H is a locally flat submanifold of M^K. Here, $f\colon M \to X$ and $f'\colon M' \to X$ are equivalent if there is a Γ–homeomorphism $h\colon M \to M'$ such that $f' \circ h$ is Γ–isovariantly homotopic to f. Shmuel Weinberger has been a long-time proponent of this "rel sing" structure set in a very similar conjecture (for example, see Cappell, Weinberger and Yan [14, Section 3]).

If Γ has no element of order 2, the conjecture implies each such f is isovariantly homotopic to a homeomorphism. The left side is defined in Davis, Quinn and Reich [24], using [23].

Quinn's conjecture should have included a vanishing hypothesis on equivariant Whitehead torsion, as first noticed in [18; 19]. We state this simpleness condition using his subsequent work. A Γ–isovariant homotopy equivalence $f\colon M \to X$ between locally flat, cocompact Γ–manifolds is *simple* if Quinn's stratified Whitehead torsion $\tau(\bar{f}\colon M/\Gamma \to X/\Gamma)$ vanishes. The element $\tau(\bar{f})$ is defined using [40, Corollary 1.6, Theorem 1.10(1)]. The geometric interpretation of $\tau(\bar{f}) = 0$ generalizes the interpretation for finite CW–complexes and is given by [40, Theorem 1.10(3)]: if f is simple, then there is a homotopically stratified space Z and a stratified cell-like map $z\colon Z \to M/\Gamma$ such that $\bar{f} \circ z\colon Z \to X/\Gamma$ is stratified homotopic to a cell-like map.

The Hadamard hypothesis is not, strictly speaking, necessary. We could have stated the conjecture when X is a locally flat, cocompact manifold model for $E_{\mathrm{fin}}\Gamma$ where each X^H has codimension greater than 2 in X^K. We stated it in the Hadamard case to minimize jargon and to remind the reader that verification of Conjecture 1.1 will require the Farrell–Jones conjecture in L–theory for Γ, and that, to date, the verification of the Farrell–Jones conjecture requires some geometric input. In the more general case, we could conjecture

$$H^{\Gamma}_{n+1}(X \longrightarrow \bullet; \underline{\boldsymbol{L}}) \xrightarrow{\approx} \mathcal{S}^{\mathrm{iso}}_{\mathrm{rel}}(X^n, \Gamma).$$

This alternative conjecture would have the advantage that its verification should be independent of the Farrell–Jones conjecture, but the disadvantage that the left-hand side is not particularly computable without the Farrell–Jones conjecture.

The present paper proves Conjecture 1.1 for $\Gamma = \Gamma_n$, using the facts that the singular set in $\mathbb{R}^n$ is discrete and that all finite subgroups of Γ_n have vanishing lower K–theory. In fact our Theorem 1.2 is stronger than Conjecture 1.1 for Γ_n, and computes $\mathcal{S}(\Gamma_n)$ using Smith theory and other topological tools. The bijection in Conjecture 1.1 for $\Gamma = \Gamma_n$ is defined in Section 4 using maps of Cappell and Wall.

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2 Applications of Smith theory

Given a universal covering map $p\colon E \to B$ and an effective action of a group G on B, consider the group

$$\mathcal{D}(p, G) := \{h \in \mathrm{Homeo}(E) \mid p \circ h = g \circ p \text{ for some } g \in G\}.$$

There is an obvious exact sequence

$$1 \longrightarrow \mathcal{D}(p) \longrightarrow \mathcal{D}(p, G) \longrightarrow G \longrightarrow 1, \tag{2}$$

where $\mathcal{D}(p)$ is the deck transformation group of p (those which cover the identity on B).

For the quotient map $q\colon \mathbb{R}^n \to T^n$ and standard action $C_2 \curvearrowright T^n$, note $\mathcal{D}(q, C_2) = \Gamma_n$.

Theorem 2.1 *Let C_2 act on a manifold N^n homotopy equivalent to T^n so that $\sigma_*(\alpha) = -\alpha$ for all $\alpha \in H_1(N^n)$.*

(1) *The fixed set N^{C_2} consists of 2^n points and $\mathcal{D}(p, C_2) \cong \Gamma_n$, where p is the universal covering map of N^n. Moreover, if G is any nontrivial finite subgroup of $\mathcal{D}(p, C_2)$, then $\widetilde{N}^G$ consists of one point.*

(2) *Fix an isomorphism $\mathcal{D}(p, C_2) \cong \Gamma_n$. There is a Γ_n–homotopy equivalence of the universal covers, $\widetilde{J}\colon \widetilde{N}^n \to \mathbb{R}^n$. Any two such Γ_n–homotopy equivalences are Γ_n–homotopic. Furthermore $\widetilde{J}$ is the universal covering of a C_2–homotopy equivalence, $J\colon N^n \to T^n$.*

To prove this theorem we use lemmas concerning involutions on $\mathbb{R}^n$ and T^n. We state and prove these lemmas in their ultimate generality: actions of p–groups on contractible and aspherical manifolds.

Lemma 2.1 *Let G be a finite p–group.*

(1) *The fixed set of a G–action on a manifold is locally path-connected.*

(2) *The fixed set of a G–action on a contractible manifold is mod p acyclic and path-connected.*

(3) *If the fixed set of a G–action on a contractible manifold is compact, then the fixed set is a point.*

The proof of this lemma involves Smith theory. Our primary reference is Borel's *Seminar on transformation groups* [7]. Borel et al use Alexander–Spanier cohomology $\bar{H}^*(X; R)$ with coefficients in a commutative ring R. This is, in turn, isomorphic to Čech cohomology $\check{H}^*(X; R)$ for X paracompact Hausdorff, hence also for X metrizable; see Spanier [49, Corollary 6.8]. Of course, if X is a CW–complex then Alexander–Spanier and Čech and singular cohomology coincide; but fixed sets of actions are far from CW–complexes.

The proof of this lemma is inductive; any nontrivial p–group has a normal C_p subgroup, hence G/C_p acts on M^{C_p}. However the fixed set M^{C_p} is not necessarily a manifold, so we will have to work abstractly.

Proof of Lemma 2.1 (1) Consider the following four properties of a topological space: locally compact (lc), complete metrizable (cm), cohomology locally connected (clc_p) and cohomology finite-dimensional (cfd_p). We show that a manifold satisfies these four properties, that the fixed set of a C_p–action on a space which satisfies the four properties also satisfies the four properties, and that a space which satisfies the four properties is locally path-connected.

A space is *complete metrizable* if it admits a complete metric. A space X is *cohomology locally connected with respect to* $\mathbb{F}_p$ (written clc_p) if each neighborhood U of each $x \in X$ contains a neighborhood V of x so that the restriction map from U to V is zero on reduced Čech cohomology with coefficients in $\mathbb{F}_p$; see [7, I.1.3] or Bredon [8, II.17.1] for the definition. A locally compact space X is *cohomology finite-dimensional* (written cfd_p) if there is an integer $n \geq 0$ so that for all open sets U of X, the $(n+1)^{\mathrm{st}}$ Čech cohomology with compact supports vanishes: $\check{H}_c^{n+1}(U;\mathbb{F}_p) = 0$; see [7, I.1.2] or [8, II.16] for the definition.

Let M be topological n–manifold. Clearly it is locally compact. We claim that it is complete metrizable. Indeed by Urysohn's metrization theorem it admits the structure of a metric space (M,d). Since M is locally compact and the union of a countable number of compact sets, it admits a proper map $f\colon M \to \mathbb{R}$. Then a complete metric on M is given by $D(x,y) := d(x,y) + |f(x) - f(y)|$. A manifold is clc_p since it is locally contractible. A topological n–manifold is cfd_p: a reference is [8, Corollary 16.28]; it also follows from Poincaré duality.

Thus it suffices to show that the fixed set of a C_p–action on a locally compact, locally path-connected, cohomology finite-dimensional, complete metrizable space satisfies each of these properties. Note that the fixed set X^{C_p} is closed in X. It clearly follows that X^{C_p} is lc and cm, and it can be shown that any subspace of a cfd_p metrizable space is cfd_p (see [8, Theorem 16.8]). Finally, the fixed set of a C_p–action on a lc, cfd_p, clc_p space is clc_p (see [7, Proposition V.1.4], also [8, Exercise II.44].) This is the one point where Smith theory is used.

Thus we conclude by induction on $|G|$ that the fixed set M^G of a finite p–group acting on a manifold is clc_p and cm. By looking at Čech cohomology in degree zero, it is easy to see that a clc_p–space is locally connected (see [7, page 6] and [8, page 126].) A theorem of Moore, Menger and Mazurkiewicz (see Kuratowski [32, page 254]) shows that a locally connected complete metric space is locally path-connected.

(2) By a *mod p acyclic space*, we mean a Hausdorff space X which satisfies $\check{H}^*(X;\mathbb{F}_p) \cong \check{H}^*(\mathrm{pt};\mathbb{F}_p)$. A standard result from global Smith theory is that, if a finite p–group G acts on a cohomology finite-dimensional mod p acyclic space X, then its fixed set is also mod p acyclic; see for example [7, Corollary III.4.6]. In particular $\check{H}^0(M^G;\mathbb{F}_p) \cong \mathbb{F}_p$, so M^G is connected. By part (1), it is locally path-connected. Therefore M^G is path-connected.

(3) Since M is a topological manifold and is contractible, M is an orientable mod p cohomology manifold (see [7, Definition 3.3].) If G acts on an orientable mod p cohomology manifold M^n, then by Smith theory the fixed set M^G is an orientable mod p cohomology manifold [7, Theorem V.2.2] of dimension $d \leq n$. Recall that M^G is connected and mod p acyclic. If M^G is compact, then there is a fundamental cohomology class in dimension d, so $\check{H}^d(M^G;\mathbb{F}_p) \cong \mathbb{F}_p$ [7, Theorem I.4.3(1)]. But M^G is mod p acyclic so $d = 0$. Also, for a connected compact mod p homology manifold X of dimension d, $\check{H}^d(A;\mathbb{F}_p) = 0$ for any closed proper subset A of X by [7, Theorem I.4.3(1)]. Therefore M^G is a point. □

In the literature, we have not seen the application of Smith theory to path-connectivity. The above lemma seems to be the first occurrence. We use path-connectivity in the covering space arguments below.

Lemma 2.2 *Let G be a discrete group. Suppose that G acts effectively on a space M fixing a point $x \in M$. Let $p\colon (\widetilde{M}, \tilde{x}) \to (M, x)$ be the universal cover.*

(1) *There is a unique action $G \curvearrowright (\widetilde{M}, \tilde{x})$ so that $p\colon \widetilde{M} \to M$ is a G–map.*
(2) *The action $G \curvearrowright (M, x)$ gives a homomorphism $G \to \operatorname{Aut}(\pi_1(M, x))$ and the action $G \curvearrowright (\widetilde{M}, \tilde{x})$ gives a splitting of the short exact sequence*

$$1 \longrightarrow \mathcal{D}(p) \longrightarrow \mathcal{D}(p, G) \underset{\dashleftarrow}{\longrightarrow} G \longrightarrow 1.$$

The usual isomorphism $k\colon \pi_1(M, x) \to \mathcal{D}(p)$ is equivariant with respect to the G–action on the fundamental group and the G–action given on $\mathcal{D}(p)$ given by conjugation. Thus k sends $\pi_1(M, x)^G$ to

$$\mathcal{D}(p)^G = \{h \in \mathcal{D}(p) \mid ghg^{-1} = h \in \mathcal{D}(p, G) \text{ for all } g \in G\}.$$

(3) *The map $p^G\colon \widetilde{M}^G \to p(\widetilde{M}^G)$ is a regular $\mathcal{D}(p)^G$–cover.*
(4) *If M^G_x is the path component of M^G containing x, then $M^G_x \subset p(\widetilde{M}^G)$.*
(5) *If M is an aspherical manifold and G is a finite p–group, then $p(\widetilde{M}^G)$ is a connected component of M^G.*
(6) *If M is a compact, aspherical manifold and G is a finite p–group, then $p(\widetilde{M}^G)$ is compact.*

Proof (1) Any map $g\colon (M, x) \to (M, x)$ lifts to a unique map $\tilde{g}\colon (\widetilde{M}, \tilde{x}) \to (\widetilde{M}, \tilde{x})$ so that $p \circ \tilde{g} = g \circ p$. Uniqueness implies that $\widetilde{g \circ h} = \tilde{g} \circ \tilde{h}$. Define the action $G \curvearrowright \widetilde{M}$ by $gy := \tilde{g}(y)$.

(2) Clearly the G–action on (M,x) gives a G–action on the fundamental group and the G–action on $(\widetilde{M},\tilde{x})$ gives a splitting of the short exact sequence. The isomorphism k is specified by $k[\alpha](\tilde{x}) = \tilde{\alpha}(1)$ where $\tilde{\alpha}\colon I \to \widetilde{M}$ is the unique lift of $\alpha\colon I \to M$ so that $\tilde{\alpha}(0) = \tilde{x}$. One then checks that k is equivariant.

(3) Abstracting a bit, if $\pi\colon E \to B$ is a regular Γ–cover, if E_0 is a closed subset of E, and if Γ_0 is a subgroup of Γ so that $\Gamma_0 \curvearrowright E_0$, and if γE_0 and E_0 are disjoint whenever $\gamma \in \Gamma - \Gamma_0$, then $\pi_0\colon E_0 \to \pi(E_0)$ is a regular Γ_0–cover. To check this in the case at hand, let $y \in \widetilde{M}^G$. If $h \in \mathcal{D}(p)^G$, then for all $g \in G$, $ghy = hgy = hy$, so $hy \in \widetilde{M}^G$. Conversely, if $h \in \mathcal{D}(p) - \mathcal{D}(p)^G$, then for some $g \in G$, $hg \neq gh$, so $ghy \neq hgy = hy$ by the freeness of the $\mathcal{D}(p)$–action, so $hy \notin \widetilde{M}^G$.

(4) Let $y \in M^G_x$ and $\alpha\colon I \to M^G$ be a path from x to y. Let $\tilde{\alpha}\colon I \to \widetilde{M}$ be the unique lift of α with $\tilde{\alpha}(0) = \tilde{x}$. Then for any $g \in G$, $g\tilde{\alpha}$ is a lift starting at $g\tilde{x} = \tilde{x}$, so $g\tilde{\alpha} = \tilde{\alpha}$. Hence $\tilde{\alpha}(1) \in \widetilde{M}^G$ and since we have $p(\tilde{\alpha}(1)) = y$, we have arrived at our conclusion.

(5) By Lemma 2.1(2), $p(\widetilde{M}^G)$ is path-connected and so by part (4), $M^G_x = p(\widetilde{M}^G)$. On the other hand, by Lemma 2.1(1), M^G is locally path-connected so path components are components.

(6) The fixed set M^G is closed in M. Connected components are closed, so $p(\widetilde{M}^G)$ is closed in M^G. Hence if M is compact so is $p(\widetilde{M}^G)$. □

Proof of Theorem 2.1(1) Let σ be an involution on a manifold N^n homotopy equivalent to the n–torus such that $\sigma_*\alpha = -\alpha$ for all $\alpha \in H_1(N)$.

We first note that σ has a fixed point, because its Lefschetz number is $2^n \neq 0$. Indeed, $L(\sigma) = \sum_{k=0}^n (-1)^k \mathrm{Tr}(\sigma_*\colon H_k(N) \to H_k(N)) = \sum_{k=0}^n (-1)^k(-1)^k \binom{k}{n} = 2^n$. Thus by Lemma 2.2(2), $\mathcal{D}(p, C_2) \cong \Gamma_n$. Since $\mathcal{D}(p) \cong \mathbb{Z}^n$ is torsion-free, all nontrivial finite subgroups of Γ_n have order two and map isomorphically to C_2 under the map $\mathcal{D}(p, C_2) \to C_2$.

Let G by a nontrivial finite subgroup of $\mathcal{D}(p, C_2)$. We will show $\widetilde{N}^G$ is a point. By Lemma 2.1(2), $\widetilde{N}^G$ is nonempty. Since $\mathcal{D}(p, C_2)^G$ is the trivial subgroup, by Lemma 2.2(3), $\widetilde{N}^G$ is homeomorphic to $p(\widetilde{N}^G)$, which by Lemma 2.2(6) is compact. Hence by Lemma 2.1(3), $\widetilde{N}^G$ is a point.

Next we must show that $|N^{C_2}| = 2^n$. Each involution $s \in \mathcal{D}(p, C_2) = \Gamma_n$ determines a point, $p(\widetilde{N}^s)$ in N^{C_2}. Moreover, for involutions s and s', we have the following: $p(\widetilde{N}^s) = p(\widetilde{N}^{s'})$, if and only if for some deck transformation $t \in \mathcal{D}(p)$, $t(\widetilde{N}^s) = \widetilde{N}^{s'}$, if and only if $ts't^{-1} = s$ for some $t \in \mathcal{D}(p)$, if and only if s and s' are conjugate in Γ_n by an element $t \in \mathbb{Z}^n$, if and only if s and s' are conjugate in Γ_n (because

$\Gamma_n = \mathbb{Z}^n \cup s\mathbb{Z}^n$). Therefore this rule $s \mapsto p(\widetilde{N}^s)$ induces a bijection between the set of conjugacy classes of involutions in $\Gamma_n = \mathbb{Z}^n \rtimes C_2$ and N^{C_2}. Since each such conjugacy class is represented uniquely by an element of $\{(\varepsilon,\sigma) \mid \varepsilon \in \{0,1\}^n\}$, we obtain $|N^{C_2}| = 2^n$. □

Much of the above proof could be recast in a homological light. The exact sequence (2) is split, since $H^2(C_2;\mathcal{D}(p)) = 0$, so $\mathcal{D}(p, C_2) \cong \Gamma_n$. Then one shows, using the argument above, that for each nontrivial finite subgroup G of $\mathcal{D}(p, C_2)$, $\widetilde{N}^G$ is a point. It follows that fixed points of N^{C_2} are in one-to-one correspondence with conjugacy classes of splittings of (2) which is in turn in one-to-one correspondence with $H^1(C_2;\mathcal{D}(p)) \cong (C_2)^n$.

Jiang [30] used fixed point theory to show that if $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of a self-map $f\colon X \to X$ of a finite complex, then $p(\mathrm{Fix}\,\widetilde{f})$ is compact. This can be used to give an alternate proof that $p(\widetilde{N}^{C_2})$, and hence $\widetilde{N}^{C_2}$, is compact.

Before proving Theorem 2.1(2), we need the following useful fact.

Lemma 2.3 *Let (N^n, C_2) be compact C_2–manifold with finite fixed set. If $n \le 5$, assume $\pi_1(N_{\mathrm{free}}/C_2) \cong \Gamma_n$. Then (N, C_2) has the equivariant homotopy type of a finite C_2–CW–complex.*

Proof This is implicit, certainly in Quinn [40] but we will make it explicit here. First, the pair $(N/C_2, N^{C_2})$ is forward tame (tame in the sense of Quinn [40]) by [40, Propositions 2.6 and 3.6]. Second, the pair is reverse tame (tame in the sense of Siebenmann [47]) by [40, Proposition 2.14]. From this it follows from Siebenmann [47], when $n \ge 6$, that $N - N^{C_2}$ is the interior of a compact, free C_2 manifold, $\bar{N}$, with 2^n boundary components and N is homeomorphic to $\bar{N}/\sim$, where $\sim$ is the equivalence relation identifying each boundary component to a separate point. But $\bar{N}$ and each of its boundary components are homotopy equivalent to a finite free C_2 complex. It follows that $\bar{N}/\sim$ is homotopy equivalent to a finite C_2 complex with 2^n fixed points.

Alternatively, in any dimension, we can argue $(N - N^{C_2})/C_2$ and $\mathrm{Holink}(N/C_2, N^{C_2})$ are finitely dominated (see [40, Proposition 2.15]). Since the projective class groups vanish, $\widetilde{K}_0(\mathbb{Z}[C_2]) = 0 = \widetilde{K}_0(\mathbb{Z}[\Gamma_n])$, by Wall's finiteness theorem [52, Theorem F], each is homotopy equivalent to a finite CW–complex. So there is a finite C_2–CW pair (K, L), homotopy equivalent to the pair $(\mathrm{Cyl}(e_1), \mathrm{Holink}(N, N^{C_2}))$, where $e_1\colon \mathrm{Holink}(N, N^{C_2}) \to N - N^{C_2}$ is the evaluation map at time 1. This map passes to a C_2–homotopy equivalence of pairs:

$$(K \cup_L \mathrm{Cyl}(p), N^{C_2}) \longrightarrow (\mathrm{Cyl}(e_1) \cup_H \mathrm{Cyl}(e_0), N^{C_2}),$$

where $H = \text{Holink}(N, N^{C_2})$ and $p\colon L \to N^{C_2}$ is the composite C_2–map, $L \to H \xrightarrow{e_0} N^{C_2}$. But, as noted in Quinn [40], $\text{Cyl}(e_1) \cup_H \text{Cyl}(e_0)$ is C_2–homotopy equivalent to N and $K \cup_L \text{Cyl}(p)$ is a finite C_2–CW–complex. □

Proof of Theorem 2.1(2) For a discrete group Γ, recall that a Γ–space E is a *model for* $E_{\text{fin}}\Gamma$ if E^G is contractible for each finite subgroup $G \subset \Gamma$, E has the Γ–homotopy type of a Γ–CW–complex, and the Γ–space (E, Γ) is proper. Consider $\mathbb{R}^n$ with its Γ_n–action. The action is obviously proper. The only nontrivial finite subgroups have the form $\{1, \sigma\}$ for some involution σ; for this subgroup the fixed set is a single point. Finally, $\mathbb{R}^n$ admits the structure of a Γ_n–CW–complex. For it is the universal covering of (T^n, C_2), and (T^n, C_2) is the n–fold cartesian product of (S^1, C_2), (with the diagonal action), which is a C_2–CW–complex with exactly two (fixed) vertices.

Now let (N, C_2) be as in the hypothesis of Theorem 2.1. After the choice of an isomorphism $\mathcal{D}(p, C_2) \cong \Gamma_n$, $\widetilde{N}$ is a proper Γ_n–manifold, and has the Γ_n–homotopy type of a Γ_n–CW–complex by Lemma 2.3. For each finite subgroup G of Γ_n, $\widetilde{N}^G$ is contractible by Theorem 2.1(1). Therefore $\widetilde{N})$ is also Γ_n–universal, and there is a unique Γ_n–homotopy class of Γ_n–maps $g\colon \mathbb{R}^n \to \widetilde{N}$. By uniqueness, g and $\widetilde{J}$ are mutually Γ_n–homotopy inverse. Therefore $\widetilde{J}\colon \widetilde{N} \to \mathbb{R}^n$ and its quotient, $J\colon N \to T^n$ are C_2–equivariant homotopy equivalences. □

This paper focuses on actions of $C_2 = \{1, \sigma\}$ on a torus N for which $\sigma_*(x) = -x$ for all $x \in H_1(N)$. But the following lemma (with Theorem 2.1) shows that this is *equivalent* to saying that the torus has at least one isolated fixed point:

Lemma 2.4 *Suppose $C_2 = \{1, \sigma\}$ acts on a torus N and there is at least one fixed point which is isolated in N^{C_2}. Then $\sigma_*(x) = -x$ for all $x \in H_1(N)$.*

Proof Lift the involution to an involution $\widetilde{\sigma}\colon \widetilde{N} \to \widetilde{N}$ with an isolated fixed point $\widetilde{x} \in \widetilde{N}$. The group $G = \{1, \widetilde{\sigma}\}$ fixes only $\widetilde{x}$ since the fixed set is mod 2 acyclic. But the centralizer, $Z_{\mathcal{D}(p)}(G)$, acts freely on this fixed set $\{\widetilde{x}\}$, since the action is proper. So if $t \in \mathcal{D}(p)$, and $t\widetilde{\sigma} = \widetilde{\sigma}t$, then $t = 1$. Therefore, if $x \in H_1(N) \cong \mathcal{D}(p)$, and $\sigma_*(x) = x$, then $x = 0$. This implies that $\sigma_*(x) = -x$ for all $x \in H_1(N)$. □

3 Equivariant and isovariant structures

Definition 3.1 The *equivariant structure set* $\mathcal{S}_{\text{TOP}}(\mathbb{R}^n, \Gamma_n)$, is the set of equivalence classes of pairs $((M, \Gamma_n), f)$, where (M, Γ_n) is a manifold with a cocompact proper Γ_n–action, and $f\colon M \to \mathbb{R}^n$ is a Γ_n–equivariant homotopy equivalence. Often we write such pairs as (M, f). Two such pairs (M, f) and (M', f') are equivalent if there is an equivariant homeomorphism $h\colon M \to M'$ and an equivariant homotopy H from f to $f' \circ h$. One defines the *isovariant structure sets* $\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$ and $\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n, C_2)$ similarly, except one requires that f is an isovariant homotopy equivalence and that H is an isovariant homotopy.

Remark 3.1 We make no requirements on the fixed sets of subgroups in M and N above, because we have seen (see Section 2) these fixed sets are discrete. A result of Quinn [40, Propositions 2.6, 3.6] then guarantees there are no local pathologies in such manifolds.

The universal covering of a C_2–homotopy equivalence, $f\colon N^n \to T^n$ is a Γ_n–homotopy equivalence $\tilde{f}\colon \tilde{N}^n \to \mathbb{R}^n$. Moreover $\tilde{f}$ is isovariant if f is isovariant. This gives obvious bijections

$$u\colon \mathcal{S}_{\mathrm{TOP}}(T^n, C_2) \xrightarrow{\approx} \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n), \tag{3}$$

$$u^{\mathrm{iso}}\colon \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n, C_2) \xrightarrow{\approx} \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n). \tag{4}$$

Consider the closed C_2–manifold $T := T^n$. Write $X := T_{\mathrm{free}}/C_2$, an open n–manifold. Define $\bar{X}$ as the obvious manifold compactification of X obtained by adding a copy of $\mathbb{RP}^{n-1}$ at each end of X. Label the boundary components as

$$\partial \bar{X} = \bigsqcup_{i=1}^{2^n} \partial_i \bar{X}.$$

Definition 3.2 For $n \geq 5$, the *structure set* $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X})$ is the set of equivalence classes of triples $(\bar{Y}, \bar{h}, \partial \bar{h})$, where $\bar{Y}$ is a compact n–manifold and $(\bar{h}, \partial \bar{h})\colon (\bar{Y}, \partial \bar{Y}) \to (\bar{X}, \partial \bar{X})$ is a homotopy equivalence of pairs. Such triples $(\bar{Y}^0, \bar{h}^0, \partial \bar{h}^0)$, $(\bar{Y}^1, \bar{h}^1, \partial \bar{h}^1)$ are equivalent if there is a homeomorphism $\varphi\colon \bar{Y}^0 \to \bar{Y}^1$ such that $(\bar{h}^1, \partial \bar{h}^1) \circ \varphi$ is homotopic to $(\bar{h}^0, \partial \bar{h}^0)$. Compare [53, Section 10]; we used $\mathrm{Wh}(\Gamma_n) = 0$.

For the four-dimensional case, some modifications are required.

Definition 3.3 For $n = 4$, the *structure set* $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X})$ is the set of equivalence classes of triples $(\bar{Y}, \bar{h}, \partial \bar{h})$, where $\bar{Y}$ is a compact topological 4–manifold and $(\bar{h}, \partial \bar{h})\colon (\bar{Y}, \partial \bar{Y}) \to (\bar{X}, \partial \bar{Y})$ is a $\mathbb{Z}[\Gamma_4]$–homology equivalence of pairs. Such triples $(\bar{Y}^0, \bar{h}^0, \partial \bar{h}^0)$ and $(\bar{Y}^1, \bar{h}^1, \partial \bar{h}^1)$ are equivalent if there is a $\mathbb{Z}[\Gamma_4]$–homology h–bordism $(\bar{W}; \bar{Y}^0, \bar{Y}^1) \to \bar{X} \times (I; 0, 1)$ between them. Compare with [27, Section 11.3].

For our application, we give a more explicit form of this general notion.

Remark 3.2 Since $\mathrm{Wh}(\Gamma_4)=0$ and Γ_4 is "good" in the sense of [27], we can simplify Definition 3.3. First, for a representative $(\overline{Y},\overline{h},\partial\overline{h})$, by the manifold-theoretic plus-construction rel boundary [27, Section 11.1], we can assume that $\overline{h}\colon \partial\overline{Y}\to\partial\overline{X}$ is a homotopy equivalence. Second, two triples $(\overline{Y}^0,\overline{h}^0,\partial\overline{h}^0)$ and $(\overline{Y}^1,\overline{h}^1,\partial\overline{h}^1)$ are equivalent, by plus-construction on $\overline{W}$, if and only if there are

- a nonorientable closed 3–manifold $P=\bigsqcup_{i=1}^{16}P_i$,
- $\mathbb{Z}[\Gamma_4]$–homology h–cobordisms $(\overline{Z}^j;\partial\overline{Y}^j,P)$ for both $j=0,1$,
- $\mathbb{Z}[\Gamma_4]$–homology equivalences $g^j\colon \overline{Z}^j\to\partial\overline{X}$ extending $\partial\overline{h}^j\colon \partial\overline{Y}^j\to\partial\overline{X}$,
- a homeomorphism $\varphi\colon \overline{Y}^0\cup_{\partial\overline{Y}^0}\overline{Z}^0\to\overline{Y}^1\cup_{\partial\overline{Y}^1}\overline{Z}^1$ relative to P,

such that $(\overline{h}^1\cup g^1)\circ\varphi$ is homotopic to $(\overline{h}^0\cup g^0)$ relative to P; see [27, Section 11.1].

Lemma 3.1 *Suppose $n\geq 4$. There is a bijection*

$$\Phi\colon \mathcal{S}_{\mathrm{TOP}}(\overline{X},\partial\overline{X})\xrightarrow{\approx}\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n,C_2).$$

Proof First, suppose $n\geq 5$. Let $(\overline{h},\partial\overline{h})\colon(\overline{Y},\partial\overline{Y})\to(\overline{X},\partial\overline{X})$ be a homotopy equivalence of pairs, where $\overline{Y}$ is a compact n–dimensional topological manifold. So $[\overline{h},\partial\overline{h}]$ is an element of $\mathcal{S}_{\mathrm{TOP}}(\overline{X},\partial\overline{X})$. Denote $\widehat{X}$ and $\widehat{Y}$ as the corresponding double cover of $\overline{X}$ and $\overline{Y}$. Passage to double covers induces a C_2–equivariant homotopy equivalence $(\widehat{h},\partial\widehat{h})\colon(\widehat{Y},\partial\widehat{Y})\to(\widehat{X},\partial\widehat{X})$. Each component $\partial_i\widehat{Y}$ of $\partial\widehat{Y}$ is homotopy equivalent to, and therefore homeomorphic to the sphere S^{n-1}; see Smale [48] and [27]. Hence the cone $c(\partial_i\widehat{Y})$ is homeomorphic to the disc D^n. So $N:=\widehat{Y}\cup\bigsqcup_i c(\partial_i\widehat{Y})$ is a topological C_2–manifold. Thus, by coning off each map $\partial_i\widehat{h}$, we obtain a function

$$\Phi\colon \mathcal{S}_{\mathrm{TOP}}(\overline{X},\partial\overline{X})\longrightarrow\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T,C_2)\quad(\overline{h},\partial\overline{h})\longmapsto\widehat{h}\cup\bigsqcup_i c(\partial_i\widehat{h}).$$

Now we show that Φ is a bijection by exhibiting its inverse. Let $f\colon N\to T^n$ be a C_2–isovariant homotopy equivalence. This induces a proper homotopy equivalence

$$f_{\mathrm{free}}/C_2\colon N_{\mathrm{free}}/C_2\longrightarrow T_{\mathrm{free}}/C_2.$$

Then, since all the ends of T_{free}/C_2 are tame, the ends of N_{free}/C_2 are also tame. Note $\mathrm{Wh}(C_2)=\widetilde{K}_0(\mathbb{Z}[C_2])=0$. Then, by a theorem of Siebenmann [47] (or of Freedman if $n=5$, see [27]), we can fit a unique boundary, $\partial\overline{N}$ onto $(N-N^{C_2})/C_2$, thereby creating a compact manifold $\overline{N}$, unique up to homeomorphism. So we can extend f_{free}/C_2 to $\partial\overline{N}$. (Here a small proper equivariant homotopy of f_{free}/C_2 may be needed before the extension.) This construction, $[N,f]\mapsto[\overline{N},\overline{f}]$ is clearly inverse to Φ. We conclude that Φ is both surjective and injective.

Finally, it remains to consider Φ for $n=4$. Let $(\bar{h},\partial\bar{h})\colon (\bar{Y},\partial\bar{Y}) \to (\bar{X},\partial\bar{X})$ be a map of pairs such that $\bar{h}\colon \bar{Y} \to \bar{X}$ is a homotopy equivalence of 4–manifolds and each $\partial_i\bar{h}\colon \partial_i\bar{Y} \to \partial_i\bar{X}$ is a $\mathbb{Z}[C_2]$–homology equivalence of 3–manifolds. Recall the notation $\widehat{(\cdot)}$ for the double cover of $\overline{(\cdot)}$, used above for $n \geq 5$. For each i, by [27, Proposition 11.1C], there is a compact contractible 4–manifold $c^*(\partial_i\widehat{Y})$ with a C_2–action such that its C_2–equivariant boundary is the homology 3–sphere $\partial_i\widehat{Y}$ and it has a single fixed point. It is unique up to C_2–homeomorphism. Using that isolated fixed point, one can construct a C_2–isovariant homotopy equivalence

$$c^*(\partial_i\widehat{h})\colon c^*(\partial_i\widehat{Y}) \longrightarrow c(\partial_i\widehat{X}).$$

Suppose $(\bar{Y}^0,\bar{h}^0,\partial\bar{h}^0)$ is equivalent to $(\bar{Y}^1,\bar{h}^1,\partial\bar{h}^1)$, in the sense of Definition 3.3. In the setting of Remark 3.2, there is a C_2–homeomorphism $\widehat{Y}^0 \cup \widehat{Z}^0 \to \widehat{Y}^1 \cup \widehat{Z}^1$. For each $j=0,1$, by uniqueness in [27, Proposition 11.1C], there are C_2–homeomorphisms

$$c^*(\partial_i\widehat{Y}^j) \longrightarrow \widehat{Z}_i^j \cup c^*(\widehat{P}_i).$$

Using the identity map on each $c^*(\widehat{P}_i)$, these produce a C_2–homeomorphism

$$N^{0*} := \widehat{Y}^0 \cup \bigsqcup_i c^*(\partial_i\widehat{Y}^0) \longrightarrow N^{1*} := \widehat{Y}^1 \cup \bigsqcup_i c^*(\partial_i\widehat{Y}^1).$$

Thus we may define Φ on equivalence classes similarly to the high-dimensional case, except we use the "homotopy cones" c^* instead of the "honest cones" c. The argument for showing that Φ is a bijection as in the high-dimensional case, except for surjectivity we invoke the weak end theorem [27, Theorem 11.9B] and for injectivity we invoke the classification of weak collars [27, Theorem 11.9C(3)]. □

Lemma 3.2 *Suppose $n \geq 4$. Consider the above-defined structure sets.*

(1) *The following forgetful maps are bijections:*

$$\Psi\colon \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n,\Gamma_n) \xrightarrow{\approx} \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n,\Gamma_n)$$

$$\psi\colon \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n,C_2) \xrightarrow{\approx} \mathcal{S}_{\mathrm{TOP}}(T^n,C_2)$$

(2) *The following forgetful map is a bijection:*

$$\chi\colon \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n,\Gamma_n) \xrightarrow{\approx} \mathcal{S}(\Gamma_n)$$

Proof For part (1), in view of the bijections u and u^{iso}, it suffices to prove that ψ is a bijection. This is achieved by Theorem A.1, which we provide in Appendix A.

For part (2), by Theorem 2.1(2), it is immediate that χ is injective. We must prove χ is surjective. Let $[M,\Gamma_n]$ be in $\mathcal{S}(\Gamma_n)$. Since M is a contractible n–manifold and the restricted action $\mathbb{Z}^n \curvearrowright M$ is free, the quotient space $M/\mathbb{Z}^n$ is an n–manifold and an Eilenberg–Mac Lane space of form $K(\mathbb{Z}^n, 1) \simeq T^n$. The action $\Gamma_n \curvearrowright M$ descends to $C_2 \curvearrowright M/\mathbb{Z}^n$ with $(\alpha \mapsto -\alpha)$ on H_1. So, by Theorem 2.1(2), there is a Γ_n–homotopy equivalence $\widetilde{J}\colon M \to \mathbb{R}^n$, yielding an element $[(M, \Gamma_n), \widetilde{J}] \in \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$. Thus χ is surjective. □

Proposition 3.1 *Suppose $n \geq 4$. The following map α is a bijection:*

$$\alpha := \chi \circ u \circ \psi \circ \Phi\colon \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X}) \xrightarrow{\approx} \mathcal{S}(\Gamma_n), \quad [\bar{X}, \mathrm{id}] \longmapsto [\mathbb{R}^n, \Gamma_n]$$

Proof This follows immediately from (3) and Lemmas 3.1 and 3.2. □

4 Calculation of the isovariant structure set

Our ultimate goal here is to prove Theorem 1.2. We also establish the Topological rigidity conjecture (of Section 1.3) for the crystallographic groups Γ_n.

Throughout this section, we assume $n \geq 4$ and shall use the shorthand $\Gamma := \Gamma_n$. For each family $\mathcal{F}$ of subgroups of Γ, we write $E_{\mathcal{F}}\Gamma$ for the classifying space for Γ–CW–complexes whose isotropy groups are in $\mathcal{F}$. We use the families $\mathtt{fin}$, $\mathtt{vc}$, and $\mathtt{all}$, consisting of finite, virtually cyclic and all subgroups respectively. For the remainder of this section, since the subgroups of $\Gamma = \Gamma_n$ have trivial reduced lower K–theory, for ease of reading, we shall simply write $\underline{\boldsymbol{L}}$ for the $\mathrm{Or}(\Gamma)$–spectrum $\underline{\boldsymbol{L}}^h$.

Recall the Wall realization map [53, Theorems 5.8, 6.5], relative to the boundary:

$$\partial^{\mathrm{Wall}}\colon L^h_{n+1}(\Gamma, w_n) \longrightarrow \mathcal{S}_{\mathrm{TOP}}(\bar{X}) \longrightarrow \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X})$$

Using Cappell's map [12], define a composite homomorphism

$$\beta\colon \bigoplus_{D \in (\mathrm{mid})(\Gamma)} \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon}) \longrightarrow \bigoplus_{D \in (\mathrm{mid})(\Gamma)} L^h_{n+1}(D, w_n) \longrightarrow L^h_{n+1}(\Gamma, w_n).$$

Now we can define the desired basepoint-preserving function

$$\partial_{\oplus} := \alpha \circ \partial^{\mathrm{Wall}} \circ \beta\colon \bigoplus_{D \in (\mathrm{mid})(\Gamma)} \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon}) \longrightarrow \mathcal{S}(\Gamma).$$

It remains to show that $\partial_{\oplus}$ is a bijection of sets. This will span several lemmas.

4.1 Algebraic structure groups and equivariant homology

For cofibrant pairs (A, B) of topological spaces, A Ranicki [41] defined the algebraic structure groups $\mathcal{S}_*^h$ as the homotopy groups of the homotopy cofiber of an assembly map $\alpha\langle 1\rangle$, so that there is a long exact sequence

$$\cdots \to H_*(A, B; \boldsymbol{L}\langle 1\rangle) \xrightarrow{\alpha\langle 1\rangle} L_*^h(A, B) \to \mathcal{S}_*^h(A, B) \to H_{*-1}(A, B; \boldsymbol{L}\langle 1\rangle) \xrightarrow{\alpha\langle 1\rangle} \cdots .$$

Here $\boldsymbol{L}\langle 1\rangle$ is the 1–connective cover of the 4–periodic surgery spectrum $\boldsymbol{L}$ algebraically defined in [41]. (The homotopy invariant functor $\mathcal{S}_*^h$ is a desuspended chain-complex analogue of the geometric structure groups $\mathcal{S}_*^h$ originally defined by F Quinn.) When a map $i\colon B \to A$ is understood, we shall write (A, B) for the cofibrant pair $(\mathrm{Cyl}(i), B)$. The relative L–groups $L_*^h(A, B) = L_*^h(i\colon B \to A)$ were defined algebraically by Ranicki [42], following C T C Wall [53].

For computational purposes, we employ the nonconnective version $\mathcal{S}_*^{\mathrm{per},h}$ of $\mathcal{S}_*^h$. It is the homotopy groups of a homotopy cofiber of an assembly map α:

$$\cdots \to H_*(A, B; \boldsymbol{L}) \xrightarrow{\alpha} L_*^h(A, B) \to \mathcal{S}_*^{\mathrm{per},h}(A, B) \to H_{*-1}(A, B; \boldsymbol{L}) \xrightarrow{\alpha} \cdots$$

Remark 4.1 Suppose $B = \varnothing$ and that A is the quotient of a free Γ–action on a space $\widetilde{A}$ each of whose components is simply connected. Write $\Pi_0(\widetilde{A})$ for the Γ–set of components of $\widetilde{A}$; there is a canonical Γ–map $\widetilde{A} \to \Pi_0(\widetilde{A})$. By Theorem B.1, the Quinn–Ranicki assembly map can be naturally identified with the Davis–Lück assembly map, at the spectrum level. Then the cofibers of these assembly maps agree in a functorial manner. Specifically, Appendix B constructs an isomorphism in $\mathrm{Ho}\,\mathrm{sc}(\Gamma, 1)\mathcal{CW}\text{–}\mathfrak{Spectra}$, whose value on $\widetilde{A}$ after the application of homotopy groups gives an isomorphism

$$H_*^\Gamma(\Pi_0(\widetilde{A}), \widetilde{A}; \underline{\boldsymbol{L}}) \xrightarrow{\cong} \mathcal{S}_*^{\mathrm{per},h}(A). \tag{5}$$

Write $\mathbb{R}^n_{\mathrm{free}} := \{x \in \mathbb{R}^n \mid \Gamma_x = 1\}$ for those points with trivial isotropy group. Observe that $\mathbb{R}^n_{\mathrm{free}}$ equivariantly deformation retracts to the universal cover of $\bar{X}$. There is a canonical Γ–map from $\mathbb{R}^n$ to its singleton $\{\mathbb{R}^n\}$ with trivial Γ–action.

Lemma 4.1 *There is a commutative diagram with long exact rows:*

$$\begin{array}{ccccccc}
H_*^\Gamma(\mathbb{R}^n, \mathbb{R}^n_{\mathrm{free}}; \underline{\boldsymbol{L}}) & \longrightarrow & H_*^\Gamma(\{\mathbb{R}^n\}, \mathbb{R}^n_{\mathrm{free}}; \underline{\boldsymbol{L}}) & \longrightarrow & H_*^\Gamma(\{\mathbb{R}^n\}, \mathbb{R}^n; \underline{\boldsymbol{L}}) & \longrightarrow & H_{*-1}^\Gamma(\mathbb{R}^n, \mathbb{R}^n_{\mathrm{free}}; \underline{\boldsymbol{L}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{S}_*^{\mathrm{per},h}(\partial\bar{X}) & \longrightarrow & \mathcal{S}_*^{\mathrm{per},h}(\bar{X}) & \longrightarrow & \mathcal{S}_*^{\mathrm{per},h}(\bar{X}, \partial\bar{X}) & \longrightarrow & \mathcal{S}_{*-1}^{\mathrm{per},h}(\partial\bar{X})
\end{array}$$

Furthermore, the vertical maps are isomorphisms of abelian groups.

Proof The top row is the long exact sequence of the triple $(\{\mathbb{R}^n\}, \mathbb{R}^n, \mathbb{R}^n_{\mathrm{free}})$ in Γ–equivariant $\boldsymbol{L}$–homology [23]. The bottom row is the long exact sequence of the pair $(\bar{X}, \partial\bar{X})$ in algebraic structure groups; see Ranicki [43]. The inner left vertical map is induced on homotopy groups from (5) for $\widetilde{A} = \mathbb{R}^n_{\mathrm{free}}$. Write $D(\mathbb{R}^n_{\mathrm{sing}})$ for the Γ–subset of points in $\mathbb{R}^n$ of distance less than or equal to 0.2 from $\mathbb{R}^n_{\mathrm{sing}}$. Define $S(\mathbb{R}^n_{\mathrm{sing}}) := \partial D(\mathbb{R}^n_{\mathrm{sing}})$. The outer vertical maps are induced on homotopy groups from (5) for $\widetilde{A} = S(\mathbb{R}^n_{\mathrm{sing}})$, precomposed with the inverse of the induced excision map

$$H^{\Gamma}_*(D(\mathbb{R}^n_{\mathrm{sing}}), S(\mathbb{R}^n_{\mathrm{sing}}); \underline{\boldsymbol{L}}) \longrightarrow H^{\Gamma}_*(\mathbb{R}^n, \mathbb{R}^n_{\mathrm{free}}; \underline{\boldsymbol{L}}).$$

By functoriality of (5), the left square's diagram of spectra homotopy-commutes. In particular, the left square itself commutes. Then the inner right map is induced from a well-defined homotopy class of map of spectra. So the middle square and right square are defined and commute. Therefore, since (5) implies the outer maps and inner left map are isomorphisms, by the Five Lemma, we conclude that the inner right map is an isomorphism also. □

Remark 4.2 By the Isomorphism conjecture [5] and Bartels' splitting theorem [3] on the top row, we conclude the connecting homomorphism $\mathcal{S}^{\mathrm{per},h}_*(\bar{X}, \partial\bar{X}) \to \mathcal{S}^{\mathrm{per},h}_{*-1}(\partial\bar{X})$ is zero. Therefore, using Ranicki's natural bijection (see Remark 4.3 below), the forgetful map $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial\bar{X}) \to \mathcal{S}_{\mathrm{TOP}}(\partial\bar{X})$ is constant.

We now calculate $H^{\Gamma}_*(E_{\mathrm{vc}}\Gamma, E_{\mathrm{fin}}\Gamma; \underline{\boldsymbol{L}})$ by using a specific model for the spaces involved. Models of $E_{\mathrm{vc}}G$ for crystallographic groups G are due to Connolly, Fehrman and Hartglass [17]. For any group G, Lück and Weiermann [33] built models of $E_{\mathrm{vc}}G$ from $E_{\mathrm{fin}}G$. However, the following lemma is shown directly for our Γ.

Let C be an infinite cyclic subgroup of Γ. Let $\mathcal{P}_C$ denote the collection of all affine lines $\ell \subset \mathbb{R}^n$ which are stabilized by C. Endow $\mathcal{P}_C$ with the affine structure and topology of a copy of $\mathbb{R}^{n-1}$. Since $\mathcal{P}_C$ is a partition of $\mathbb{R}^n$, there is a continuous quotient map $\pi_C \colon \mathbb{R}^n \to \mathcal{P}_C$. Since C is normal in Γ, the Γ–action on $\mathbb{R}^n$ extends to a Γ–action on the mapping cylinder, $\mathrm{Cyl}(\pi_C)$.

Let $\mathrm{mic}(\Gamma)$ denote the collection of maximal infinite cyclic subgroups of Γ.

Lemma 4.2 *A model E for the classifying space $E_{\mathrm{vc}}\Gamma$ (which classifies Γ–CW–complexes with virtually cyclic isotropy) is the union along $\mathbb{R}^n$ of mapping cylinders:*

$$E := \bigcup_{C \in \mathrm{mic}(\Gamma)} \mathrm{Cyl}(\pi_C \colon \mathbb{R}^n \longrightarrow \mathcal{P}_C)$$

Proof If H is a finite nontrivial subgroup of Γ, then E^H is a tree with one edge in $\mathrm{Cyl}(\pi)^C$ for each $C \in \mathrm{mic}(\Gamma)$. So E^H is contractible. If H is infinite cyclic or infinite dihedral, there is just one $C \in \mathrm{mic}(\Gamma)$ for which $\mathrm{Cyl}(\pi_C)^H$ is nonempty. For this C, observe that $\mathrm{Cyl}(\pi_C)^H$ is a single point when H is dihedral and is all of $\mathrm{Cyl}(\pi_C)$ when H is cyclic. Also $E = E^{\{1\}}$ is contractible, and E^H is empty if H is not virtually cyclic.

Finally, we must prove that E has the structure of a Γ–CW–complex. We begin by assuming K is a Γ–CW structure on $\mathbb{R}^n$ which is *convex*. By this we mean each closed cell is convex, and its boundary is a subcomplex. It suffices to show how to extend K to a Γ–CW structure over each mapping cylinder, $\mathrm{Cyl}(\pi_C)$ in E.

So fix C and parametrize $\mathrm{Cyl}(\pi_C)$ as

$$\mathrm{Cyl}(\pi_C) = \mathbb{R}^n \times [-1,1] \cup_{\pi_C} \mathcal{P}_C, \quad \text{where } (x,1) = \pi_C(x) \text{ for all } x \in \mathbb{R}^n.$$

There are convex Γ–CW structures, L on $\mathcal{P}_C$, and $\widehat{L}$ on $\mathbb{R}^n$, so that each j–cell f of L has the form $\pi_C(\widehat{f})$ for some $(j+1)$–cell $\widehat{f}$ of $\widehat{L}$. This endows $\mathbb{R}^n \times [0,1] \cup_{\pi_C} \mathcal{P}_C$ with the structure of a Γ–CW–complex, K_+ so that $\mathbb{R}^n \times 0$ is the complex $\widehat{L}$. Now $\widehat{L}$ and K have a common subdivision K', since each is convex. There is then a CW structure K_- on $\mathbb{R}^n \times [-1,0]$ in which K, K' and $\widehat{L}$ are identified with $\mathbb{R}^n \times \{-1\}, \mathbb{R}^n \times \{-\frac{1}{2}\}$ and $\mathbb{R}^n \times \{0\}$ respectively as subcomplexes. (Also, $e \times [-1,-\frac{1}{2}]$ and $f \times [-\frac{1}{2},0]$ are cells of K_- if e and f are cells of K and $\widehat{L}$ respectively.) Then $K_+ \cup K_-$ is the required Γ–CW structure on $\mathrm{Cyl}(\pi_C)$. □

Each infinite dihedral subgroup D of Γ contains a unique maximal infinite cyclic subgroup C. Moreover, D has a unique invariant line, $\ell_D \subset \mathbb{R}^n$. The image of ℓ_D in $\mathcal{P}_C$ is a single point, which we denote by the singleton $\{\ell_D\} = \pi_C(\ell_D)$.

Lemma 4.3 *The inclusion-induced map is an isomorphism of abelian groups:*

$$\bigoplus_{D \in (\mathrm{mid})(\Gamma)} H_*^D(\{\ell_D\}, \ell_D; \underline{\boldsymbol{L}}) \longrightarrow H_*^\Gamma(E, \mathbb{R}^n; \underline{\boldsymbol{L}})$$

Proof Lemma 4.1 of [24] allows one to translate between maps induced by Γ–maps of classifying spaces for actions with isotropy in a family to maps induced by maps of $\mathrm{Or}(\Gamma)$–spectra. There is a homotopy cofiber sequence of $\mathrm{Or}(\Gamma)$–spectra,

$$\boldsymbol{L}_{\mathrm{fin}} \longrightarrow \boldsymbol{L} \longrightarrow \boldsymbol{L}/\boldsymbol{L}_{\mathrm{fin}}.$$

By [24, Lemma 4.1(ii)], the following absolute homology group vanishes:

$$H_*^\Gamma(\mathbb{R}^n; \underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) = 0$$

Also, by [24, Lemma 4.1(iii)], the following relative homology group vanishes:

$$H_*^\Gamma(E,\mathbb{R}^n;\underline{\boldsymbol{L}}_{\mathrm{fin}}) = 0$$

So we obtain a composite isomorphism, informally first observed by Quinn:

$$H_*^\Gamma(E,\mathbb{R}^n;\underline{\boldsymbol{L}}) \xrightarrow{\cong} H_*^\Gamma(E,\mathbb{R}^n;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xleftarrow{\cong} H_*^\Gamma(E;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \tag{6}$$

Now, since $N_\Gamma(C) = \Gamma$, by Lemma 4.2 and excision, we obtain

$$\bigoplus_{C\in\mathrm{mic}(\Gamma)} H_*^\Gamma(\mathcal{P}_C;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xrightarrow{\cong} H_*^\Gamma(E;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}). \tag{7}$$

Fix $C\in\mathrm{mic}(\Gamma)$. Observe that the action of the group Γ/C on the parallel pencil $\mathcal{P}_C$ has a discrete singular set

$$\mathrm{sing}\mathcal{P}_C := \{\ell_D \in \mathcal{P}_C \mid C\subset D \text{ for some } D\in\mathrm{mid}(\Gamma)\}.$$

Let U be a Γ–tubular neighborhood of $\mathrm{sing}\mathcal{P}_C$ in $\mathcal{P}_C$. Write $V := \mathcal{P}_C - \mathrm{sing}\mathcal{P}_C$. Recall, by a theorem of J Shaneson [46], that the following assembly map is a homotopy equivalence:

$$A_C\colon S^1_+\wedge \boldsymbol{L}(1) \xrightarrow{\simeq} \boldsymbol{L}(C)$$

That is, the spectrum $(\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}})(\Gamma/C)$ is contractible. So, since V has isotropy C,

$$H_*^\Gamma(U\cap V;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) = 0 = H_*^\Gamma(V;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}).$$

Since $\mathrm{sing}\mathcal{P}_C$ is discrete, there is a Γ–homotopy equivalence

$$\coprod_{\substack{D\supset C\\ D\in(\mathrm{mid})(\Gamma)}} \Gamma\times_D\{\ell_D\} \xrightarrow{\cong} \mathrm{sing}\mathcal{P}_C \xrightarrow{\simeq} U.$$

So the homotopy and excision axioms of equivariant homology imply

$$\bigoplus_{\substack{D\supset C\\ D\in(\mathrm{mid})(\Gamma)}} H_*^\Gamma(\Gamma\times_D\{\ell_D\};\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xrightarrow{\approx} H_*^\Gamma(U;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xrightarrow{\cong} H_*^\Gamma(\mathcal{P}_C;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}).$$

Thus (7) and the induction axiom of equivariant homology imply

$$\bigoplus_{D\in(\mathrm{mid})(\Gamma)} H_*^D(\{\ell_D\};\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xrightarrow{\cong} H_*^\Gamma(E;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}).$$

Finally, since ℓ_D is a model for $E_{\mathrm{fin}}D$, by [24, Lemma 4.1(ii)] again, we obtain

$$H_*^D(\{\ell_D\},\ell_D;\underline{\boldsymbol{L}}) \xrightarrow{\cong} H_*^D(\{\ell_D\},\ell_D;\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \xleftarrow{\cong} H_*^D(\{\ell_D\};\underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}). \qquad \square$$

Lemma 4.4 *Recall $\varepsilon = (-1)^n$. Let D be an infinite dihedral subgroup of Γ. Then the following composite map is an isomorphism of abelian groups:*

$$\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \longrightarrow L_{n+1}(D,w_n) = H^D_{n+1}(\{\ell_D\};\underline{\boldsymbol{L}}) \longrightarrow H^D_{n+1}(\{\ell_D\},\ell_D;\underline{\boldsymbol{L}})$$

Proof Denote the map under consideration by ϕ. Consider the three maps

$$L_{n+1}(C_2,\varepsilon)\oplus L_{n+1}(C_2,\varepsilon) \xrightarrow{i} H^D_{n+1}(\ell_D;\underline{\boldsymbol{L}}) \xrightarrow{j} L_{n+1}(D,w_n),$$
$$L_n(1) \xrightarrow{k} L_n(C_2,\varepsilon)\oplus L_n(C_2,\varepsilon).$$

Observe that ϕ has a factorization given by the commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Cok}(i) & \xrightarrow{\partial^{\mathrm{alg}}} & \mathrm{Ker}(k) \\
 & & \downarrow{\scriptstyle j_*} & & \\
\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) & \xrightarrow{c} & \mathrm{Cok}(j\circ i) & \xrightarrow{\partial^{\mathrm{top}}} & \mathrm{Ker}(k) \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \kappa} & & \\
H^D_{n+1}(\{\ell_D\},\ell_D;\underline{\boldsymbol{L}}) & \xleftarrow{\cong} & \mathrm{Cok}(j). & &
\end{array}$$

Here, the map ∂^{alg} is a monomorphism induced from the connecting map in the Mayer–Vietoris sequence for D–equivariant $\boldsymbol{L}$–homology. The map ∂^{top} is induced from the connecting map in Cappell's exact sequence in L–theory. By [12, Theorem 2], the map c is injective. By [12, Theorem 5(ii)], the middle row is exact. By Bartels' Theorem [3], the bottom horizontal map is an isomorphism. For general group-theoretic reasons, the middle column is exact and κ is surjective.

Note, by the calculation in [53, Theorem 13A.1], that $\mathrm{Ker}(k)=0$ for all n. Then c is surjective and $\mathrm{Cok}(i)=0$. Therefore κ, hence ϕ, is an isomorphism. □

Remark 4.3 By Proposition 1.1 and the s–cobordism theorem, there is a bijection

$$\mathcal{S}_{\mathrm{TOP}}(\bar{X},\partial\bar{X}) \xrightarrow{\approx} \mathcal{S}^h_{\mathrm{TOP}}(\bar{X},\partial\bar{X}).$$

For any compact manifold W of dimension $n\geq 6$, Ranicki gives a bijection from the geometric structure set to the algebraic structure group [43, Theorem 18.5]:

$$\mathcal{S}^h_{\mathrm{TOP}}(W,\partial W) \xrightarrow{\approx} \mathcal{S}^h_{n+1}(W,\partial W)$$

In the case $(W,\partial W)=(\bar{X},\partial\bar{X})$ this bijection is valid for $n=5$ since all the fundamental groups are good in the sense of Freedman and Quinn [27], and it is valid for $n=4$ since we use homology equivalences on the 3–dimensional boundary.

The Atiyah–Hirzebruch spectral sequence shows that $H_{n+1}(W, \partial W; \boldsymbol{L}/\boldsymbol{L}\langle 1\rangle) = 0$ for a compact n–manifold W with boundary. A diagram chase using this fact together with the definitions of the algebraic structure groups give a monomorphism

$$\mathcal{S}^h_{n+1}(W, \partial W) \rightarrowtail \mathcal{S}^{\mathrm{per},h}_{n+1}(W, \partial W).$$

Recall the definitions of ∂^{Wall} and β from the introduction of Section 4.

Proposition 4.1 *The following composite function is a bijection of pointed sets:*

$$\partial^{\mathrm{Wall}} \circ \beta \colon \bigoplus_{D \in (\mathrm{mid})(\Gamma)} \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon}) \longrightarrow \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X})$$

Proof Consider the commutative diagram

$$\begin{array}{ccccccc}
\bigoplus \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^{\varepsilon}, \mathbb{Z}^{\varepsilon}) & \longrightarrow & \bigoplus L^h_{n+1}(D, w_n) & \longrightarrow & L^h_{n+1}(\Gamma, w_n) & \longrightarrow & \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X}) \\
\downarrow \cong & \swarrow & & & \downarrow & & \downarrow \cong \\
\bigoplus H^D_{n+1}(\{\ell_D\}, \ell_D; \underline{\boldsymbol{L}}) & \xrightarrow{\cong} & H^{\Gamma}_{n+1}(E, \mathbb{R}^n; \underline{\boldsymbol{L}}) & \xrightarrow{\cong} & H^{\Gamma}_{n+1}(\{\mathbb{R}^n\}, \mathbb{R}^n; \underline{\boldsymbol{L}}) & \xrightarrow{\cong} & \mathcal{S}^{\mathrm{per},h}_{n+1}(\bar{X}, \partial \bar{X}).
\end{array}$$

The composition of the three maps in the top row is $\partial^{\mathrm{Wall}} \circ \beta$. By Lemma 4.4, the leftmost vertical map is an isomorphism. By Lemma 4.3, the leftmost map of the bottom row is an isomorphism. By Lemma 4.2 and the Farrell–Jones conjecture [5], the middle map of the bottom row is an isomorphism. By Lemma 4.1, the rightmost map of the bottom row is an isomorphism.

In particular, the composite from the upper left of the diagram to the lower right of the diagram must be surjective. Hence the right vertical map is surjective. By Remark 4.3, the rightmost vertical map is injective. Hence the right vertical map is bijective. Therefore, the three top horizontal maps compose to a bijection. □

Observe that the map $\mathcal{S}^h_{n+1}(\bar{X}, \partial \bar{X}) \longrightarrow \mathcal{S}^{\mathrm{per},h}_{n+1}(\bar{X}, \partial \bar{X})$ from the nonconnective structure group to the connective structure group is an isomorphism in our case, since the rightmost vertical map in the diagram above is an isomorphism.

Proof of Theorem 1.2 This follows from Propositions 3.1 and 4.1, and (1). □

4.2 Verification of the Topological rigidity conjecture

Lastly, we show that our family of crystallographic examples satisfies our conjecture.

Proof of Conjecture 1.1 for $(X^n, \Gamma) = (\mathbb{R}^n, \Gamma_n)$ Consider the commutative diagram

$$\begin{array}{ccccccc} H^\Gamma_{n+1}(\mathbb{R}^n; \underline{\boldsymbol{L}}) & \xrightarrow{A^{\mathrm{vc}}_{\mathrm{fin}}} & H^\Gamma_{n+1}(E; \underline{\boldsymbol{L}}) & \xrightarrow{A^{\mathrm{all}}_{\mathrm{vc}}} & L^h_{n+1}(\Gamma, w_n) & \xrightarrow{\partial^{\mathrm{Wall}}} & \mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial \bar{X}) \\ & & \downarrow & & \downarrow & & \downarrow \approx \\ & & H^\Gamma_{n+1}(E, \mathbb{R}^n; \underline{\boldsymbol{L}}) & \xrightarrow{\cong} & H^\Gamma_{n+1}(\{\mathbb{R}^n\}, \mathbb{R}^n; \underline{\boldsymbol{L}}) & \xrightarrow{\cong} & \mathcal{S}^{\mathrm{per},h}_{n+1}(\bar{X}, \partial \bar{X}). \end{array}$$

The three bijections hold as in the earlier diagram. Define a precursor map $\widehat{\partial}$ by

$$\widehat{\partial} := u^{\mathrm{iso}} \circ \Phi \circ \partial^{\mathrm{Wall}} \circ A^{\mathrm{all}}_{\mathrm{vc}} \colon H^\Gamma_{n+1}(E; \underline{\boldsymbol{L}}) \longrightarrow \mathcal{S}^{\mathrm{iso}}_{\mathrm{rel}}(\mathbb{R}^n, \Gamma).$$

By a theorem of A Bartels [3], there is a short exact sequence

$$0 \longrightarrow H^\Gamma_{n+1}(\mathbb{R}^n; \underline{\boldsymbol{L}}) \xrightarrow{A^{\mathrm{vc}}_{\mathrm{fin}}} H^\Gamma_{n+1}(E; \underline{\boldsymbol{L}}) \longrightarrow H^\Gamma_{n+1}(E, \mathbb{R}^n; \underline{\boldsymbol{L}}) \longrightarrow 0.$$

Then $\widehat{\partial}$ induces a map ∂ from $\mathrm{Cok}(A^{\mathrm{vc}}_{\mathrm{fin}})$. Using the identification (6), we obtain

$$\partial \colon H^\Gamma_{n+1}(E_{\mathrm{vc}}\Gamma; \underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}) \longrightarrow \mathcal{S}^{\mathrm{iso}}_{\mathrm{rel}}(\mathbb{R}^n, \Gamma).$$

Therefore, since u^{iso} and Φ are bijections, this desired map ∂ is also a bijection. □

5 Classification of involutions on tori

The goal of this section is to prove Theorem 1.1.

Proof of Theorem 1.1(1) This is immediate from Theorem 2.1(1). □

Proof of Theorem 1.1(2) The case $n = 0$ is trivial: $T^0 = \mathbb{R}^0/\mathbb{Z}^0 = \mathrm{pt}$.

Assume $n = 1$. Set $D^1_\pm := \{z = x + iy \in S^1 \subset \mathbb{C} \mid \pm y \geq 0\}$.

Write $a, b \in N$ for the fixed points of σ. Let $f \colon D^1_+ \to N$ be a homeomorphism of D^1_+ onto either arc in N with endpoints a and b. Extend f to a continuous map $f \colon S^1 \to N$ by setting

$$f(z) = \sigma f(\bar{z}) \quad \text{for all } z \in D^1_-.$$

Then $f \colon (S^1, C_2) \to (N, C_2)$ is an equivariant homeomorphism.

Assume $n = 2, 3$. There is a homeomorphism $f \colon N \to T^n$ (by work of Perelman [1] for $n = 3$). We want to show that each fixed point $x \in N^{C_2}$ has an invariant neighborhood D such that (D, C_2) is homeomorphic to (D^n, C_2), the orthogonal action fixing only 0.

To see this, note the involution σ of (N,x) lifts to an involution of the universal cover $(\widetilde{N},\widetilde{x})$ (for any point $\widetilde{x}$ over x) whose one point compactification provides an involution $\widetilde{\sigma}$ with two fixed points on S^n. If this involution is standard, this yields arbitrarily small standard disk neighborhoods of $\widetilde{x}$ and the required invariant standard disk neighborhood (D,C_2) of x in N.

But this involution on S^n *is* standard. For, when $n=2$ this was proved by Kérékjartò, Brouwer and Eilenberg (see Constantin and Kolev [22]); when $n=3$ it was proved by Hirsch and Smale, and Livesay; see Rubinstein [45].

Around each fixed point remove the interior of such an invariant standard disk, to obtain a compact manifold with a free involution, (N_0^n,σ_0) whose boundary consists of 2^n copies of S^{n-1} with the antipodal involution. This manifold with free involution is smooth. This is by Moise [36, Theorem 9.1] and Whitehead [56] if $n=3$, and by the classification of surfaces in $n=2$. Gluing back the 2^n standard disks, we conclude N is smooth, and σ is smooth.

If $n=3$, a theorem of Meeks and Scott [34] then proves there is a flat, invariant Riemannian metric on (N,C_2). So we may assume $N=T^3$ and C_2 acts by isometries, and the origin is an isolated fixed point. The group of all isometries fixing the origin is $O(3)\cap GL_3(\mathbb{Z})$. Only $-I$ acts with the origin as an isolated fixed point. This is the standard involution on T^3. This proves the theorem when $n=3$.

If $n=2$, we see by the Euler characteristic that N/C_2 must be S^2, and (N,C_2) must be the two–fold cover, branched at four points of S^2. This, again, is the standard involution on T^2. This proves the theorem when $n=2$.

Assume $n\geq 4$ and $n\equiv 0,1 \pmod 4$. By Theorem 2.1(2), there is a C_2–homotopy equivalence $J\colon N\to T^n$. Recall, from Section 3, the bijection $u\colon \mathcal{S}_{\mathrm{TOP}}(T^n,C_2)\to \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n,\Gamma_n)$. By Lemma 3.2(2), there is a bijection $\chi\colon \mathcal{S}_{\mathrm{TOP}}(\mathbb{R}^n,\Gamma_n)\to\mathcal{S}(\Gamma_n)$. By Theorem 1.2, $\mathcal{S}(\Gamma_n)$ is a singleton. Thus $\mathcal{S}_{\mathrm{TOP}}(T^n,C_2)$ is also. Therefore J is C_2–homotopic to a homeomorphism. □

The proof of Theorem 1.1(3) will take a little preliminary work. Let $\mathcal{T}_n$ denote the set of equivariant homeomorphism classes of H_1–negative C_2–manifolds (N^n,C_2) for which N^n has the homotopy type of T^n. We must prove that $\mathcal{T}_n$ is infinite in the case that $n\equiv 2,3 \pmod 4$ and $n\geq 6$. Write $\mathrm{hAut}(T^n,C_2)$ for the group of C_2–homotopy classes of C_2–homotopy equivalences, $f\colon (T^n,C_2)\to(T^n,C_2)$. Note, by Theorem 2.1(2), that

$$\mathcal{T}_n\approx \mathcal{S}_{\mathrm{TOP}}(T^n,C_2)/\,\mathrm{hAut}(T^n,C_2). \tag{8}$$

We begin by constructing a homomorphism $\mathrm{Aut}(\Gamma_n)\to\mathrm{hAut}(T^n,C_2)$.

Recall $\Gamma_n = \mathbb{Z}^n \rtimes_{-1} C_2$. For increased clarity below, we shall write A_n for the subgroup of translations in Γ_n and write $\sigma_0 \in \Gamma_n$ for the reflection through 0:

$$A_n := \{(x,1) \in \mathbb{Z}^n \rtimes_{-1} C_2 \mid x \in \mathbb{Z}^n\}, \quad \sigma_0 := (0,\sigma) \in \mathbb{Z}^n \rtimes_{-1} C_2$$

Also, below we shall use the identification

$$k\colon A_n \longrightarrow H_1(T^n), \quad (x,1) \longmapsto ([0,1] \to T^n; \theta \mapsto \theta x + \mathbb{Z}^n).$$

For each automorphism $a\colon \Gamma_n \to \Gamma_n$ choose an a–equivariant continuous map

$$\text{(9)} \quad \widetilde{J}_a\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \text{that is,} \quad \widetilde{J}_a(\gamma \cdot v) = a(\gamma)\cdot \widetilde{J}_a(v), \text{ for all } (\gamma, v) \in \Gamma_n \times \mathbb{R}^n.$$

Note $\widetilde{J}_a$ is unique up to a–equivariant homotopy. Since $a(A_n) = A_n$, we see $\widetilde{J}_a$ descends to a map, $J_a\colon (T^n, C_2) \to (T^n, C_2)$. So $[J_a] \in \mathrm{hAut}(T^n, C_2)$. From (9) we see that for all $a, b \in \mathrm{Aut}(\Gamma_n)$, $\widetilde{J}_{ab}$ and $\widetilde{J}_a\widetilde{J}_b$ are ab–equivariantly homotopic.

For each $x \in \Gamma_n$, we write $c(x)$ for the automorphism

$$c(x)\colon \Gamma_n \longrightarrow \Gamma_n, \quad \gamma \longmapsto x\gamma x^{-1}.$$

If $t \in A_n$ is any translation, a valid choice for $\widetilde{J}_{c(t)}$ is

$$\widetilde{J}_{c(t)}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad v \longmapsto t\cdot v,$$

since (9) holds for this choice. So this construction specifies a homomorphism

$$J\colon \mathrm{Aut}(\Gamma_n)/c(A_n) \longrightarrow \mathrm{hAut}(T^n, C_2), \quad [a] \longmapsto [J_a].$$

Write $\mathrm{Aut}(\Gamma_n)_{\sigma_0} := \{a \in \mathrm{Aut}(\Gamma_n) \mid a(\sigma_0) = \sigma_0\}$. For $a \in \mathrm{Aut}(\Gamma_n)_{\sigma_0}$, let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the unique linear isomorphism satisfying: $T(t\cdot 0) = a(t)\cdot 0$, for all $t \in A_n$. Then $T(\gamma\cdot x) = a(\gamma)\cdot x$ for all $x \in \mathbb{R}^n$, and therefore a valid choice for $\widetilde{J}_a$ is: $\widetilde{J}_a = T$.

Proposition 5.1 *The map J is an isomorphism.*

Proof Let $[a] \in \mathrm{Ker}(J)$. We show $[a] = 1$. Using the isomorphism $k\colon A_n \cong H_1(T^n)$ and the fact that $(J_a)_* = \mathrm{id}\colon H_1(T^n) \to H_1(T^n)$, note that $a(t) = t$ for all $t \in A_n$. Also J_a fixes the discrete set $(T^n)^{C_2}$, since $[J_a] = 1$. Therefore $\widetilde{J}_a(0) \in \mathbb{Z}^n$. Replacing $a \in [a]$ with $a\cdot c(t)$ for a suitable $t \in A_n$, we conclude for our new a that $\widetilde{J}_a(0) = 0$. So $a(\sigma_0) = \sigma_0$. But $\Gamma_n = \langle A_n, \sigma_0\rangle$. So $a = \mathrm{id}_{\Gamma_n}$ and J is injective.

Now we show J is surjective. Let $[f] \in \mathrm{hAut}(T^n, C_2)$. Here $f\colon T^n \to T^n$ is a C_2–map. Let $a'' \in \mathrm{Aut}(\Gamma_n)_{\sigma_0}$ satisfy: $a''(t) = k^{-1} f_*^{-1}(k(t)) \in A_n$ for all $t \in A_n$. Then $(J_{a''} f)_* = \mathrm{id}_{H_1(T^n)}$. Let $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ be a lift of f. Note $0 \in \mathbb{R}^n_{\mathrm{sing}}$, and so $p := \widetilde{J}_{a''}\widetilde{f}(0) \in \mathbb{R}^n_{\mathrm{sing}}$. There is an involution $\sigma_p \in \Gamma_n - A_n$ fixing p.

Let res: $\mathrm{Aut}(\Gamma_n) \to \mathrm{Aut}(A_n)$ be the restriction homomorphism. Observe Ker(res) acts transitively on $\Gamma_n - A_n$, since $\Gamma_n = \langle A_n, \sigma \rangle$ for any $\sigma \in \Gamma_n - A_n$, and each such σ is an involution. Therefore there exists $a' \in \mathrm{Ker(res)}$ such that $a'(\sigma_p) = \sigma_0$. So $0 = \widetilde{J}_{a'}(p) = \widetilde{J}_{a'} \widetilde{J}_{a''} \widetilde{f}(0)$. Note $\widetilde{J}_{a'a''} \widetilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is A_n equivariant. But $J_{a'a''} f$ is C_2–equivariant, so $\widetilde{J}_{a'a''} \widetilde{f}$ is σ_0 equivariant. Therefore $\widetilde{J}_{a'a''} \widetilde{f}$ is Γ_n–equivariant and so $\widetilde{J}_{a'a''} \widetilde{f}$ is Γ_n–homotopic to $\mathrm{id}_{\mathbb{R}^n}$. Therefore $[f] = [J_a]$, where $a = (a'a'')^{-1}$. So J is surjective. □

Proof of Theorem 1.1(3) Assume $n \equiv 2, 3 \pmod 4$ and $n \geq 6$. For any group G, we are going to abbreviate

$$H^G := H^G_{n+1}(E_{\mathrm{vc}} G; \underline{\boldsymbol{L}}/\underline{\boldsymbol{L}}_{\mathrm{fin}}).$$

From Proposition 5.1 and Section 4.2 and (8), we see that $H^{\Gamma_n}/\,\mathrm{Aut}(\Gamma_n) \approx \mathcal{T}_n$. So we must prove that this set $H^{\Gamma_n}/\,\mathrm{Aut}(\Gamma_n)$ is infinite. The proof is based on the fact that, for any maximal infinite dihedral subgroup D, we have $H^D \cong \mathrm{UNil}_{n+1}(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ by Lemma 4.4, and so H^D is an infinite set by (1). We will produce an injective map from an infinite set,

$$H^D/\,\mathrm{Aut}(D) \longrightarrow H^{\Gamma_n}/\,\mathrm{Aut}(\Gamma_n),$$

and this will show that $\mathcal{T}_n$ is an infinite set.

Take $D \in (\mathrm{mid})(\Gamma_n)$, a maximal dihedral subgroup. The inclusion $i_D\colon D \to \Gamma_n$ induces a map, $i_{D*}\colon H^D \to H^{\Gamma_n}$. It is easy to see that each automorphism of D extends to an automorphism of Γ_n. For any $a \in \mathrm{Aut}(\Gamma_n)$, we have

$$a_* \circ i_{D*} = i_{a(D)*} \circ (a|_D)_*\colon H^D \longrightarrow H^{\Gamma_n},$$

where $a|_D\colon D \to a(D)$ denotes the restriction of a.

By Lemma 4.3, we have that the induced map $i_{D*}\colon H^D \to H^\Gamma_n$ is a monomorphism, and $i_{D*}(H^D) \cap i_{a(D)*}(H^D) = 0$ if $a(D)$ is not conjugate to D. But if $a(D) = c_\gamma(D)$ for some $\gamma \in \Gamma_n$, then the following diagram commutes:

$$\begin{array}{ccc} H^D & \xrightarrow{i_{D*}} & H^{\Gamma_n} \\ \downarrow{\scriptstyle (c_{\gamma^{-1}} a|_D)_*} & & \downarrow{\scriptstyle (c_{\gamma^{-1}})_* a_*} \\ H^D & \xrightarrow{i_{D*}} & H^{\Gamma_n} \end{array} \tag{10}$$

By a theorem of Taylor [50], for any $\gamma \in D$, the map $c_{\gamma*}\colon H^D \to H^D$ is just multiplication by $(-1)^k$, if $w(\gamma) = (-1)^k$. Now if $n \equiv 3 \pmod 4$, then $w(\gamma) = 1$ for all γ, and if $n \equiv 2 \pmod 4$, then $2 \cdot H^D = 0$, so, in all cases $c_{\gamma*} = \mathrm{id}$. For the same reason, if $\gamma \in \Gamma_n$, then $c_{\gamma*} = \mathrm{id}\colon H^{\Gamma_n} \to H^{\Gamma_n}$.

This together with (10) proves, first, that the induced map

$$(i_D)_*\colon H^D/\operatorname{Aut}(D) \to H^{\Gamma_n}/\operatorname{Aut}(\Gamma_n)$$

is injective, and second that $\operatorname{Inn}(D)$ acts trivially on the infinite group H^D. But $\operatorname{Aut}(D)/\operatorname{Inn}(D) \cong C_2$, so $H^D/\operatorname{Aut}(D)$ is an infinite set. Therefore $H^{\Gamma_n}/\operatorname{Aut}(\Gamma_n)$ and $\mathcal{T}_n$ are also infinite sets, as required. □

6 A nontrivial element of $\mathcal{S}(\Gamma_n)$

In this section, which is independent of the rest of the paper, we give a classical argument for the existence of nontrivial elements of $\mathcal{S}(\Gamma_n)$ for some n. Indeed the argument for the case $n \equiv 2 \pmod 4$, could have been written in 1976. It was in fact pointed out by Weinberger to the first author many years ago.

Theorem 6.1 *Suppose $n \equiv 2$ or $3 \pmod 4$ and $n \geq 6$. There exists a cocompact action of the group Γ_n on a manifold M^n such that (M^n, Γ_n) is simply isovariantly homotopy equivalent to $(\mathbb{R}^n, \Gamma_n)$ but is not equivariantly homeomorphic to $(\mathbb{R}^n, \Gamma_n)$.*

The proof depends only on an idea of Farrell [25] and on Cappell's splitting theorem [11, Theorem 6; 13]. It does not depend on [5].

Let $w_n\colon \Gamma_n = \mathbb{Z}^n \rtimes C_2 \to \{\pm 1\}$ be the homomorphism such that $\ker(w_n) = \mathbb{Z}^n$ if n is odd, and $w_n(\Gamma_n) = \{1\}$ if n is even. By Remark 1.1, there is a group isomorphism $\Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1} \to \Gamma_n$. By Cappell [12], this decomposition defines a split monomorphism

$$\rho\colon \mathrm{UNil}^h_{n+1}(R; \mathcal{B}, \mathcal{B}') \to L^h_{n+1}(\mathbb{Z}[\Gamma_n], w_n),$$
$$R = \mathbb{Z}[\mathbb{Z}^{n-1}], \quad \mathcal{B} = \mathcal{B}' = \mathbb{Z}[\Gamma_{n-1} - \mathbb{Z}^{n-1}].$$

Here R is a ring with involution given by $\bar{a} = a^{-1}$ for all $a \in \mathbb{Z}^{n-1} \subset \Gamma_{n-1}$. Also $\mathcal{B}$ and $\mathcal{B}'$ are R–bimodules with involution $\bar{b} = (-1)^n b^{-1}$ for all $b \in \Gamma_{n-1} - \mathbb{Z}^{n-1}$.

Lemma 6.1 (Cappell) *The action of the abelian group $L^h_{n+1}(\mathbb{Z}[\Gamma_n], w_n)$ on the set $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial\bar{X})$ restricts to a free action of $\mathrm{UNil}^h_{n+1}(R; \mathcal{B}, \mathcal{B}')$ on $\mathcal{S}_{\mathrm{TOP}}(\bar{X}, \partial\bar{X})$.*

Proof Cappell's splitting theorem (see [11, Theorem 6; 13]) applies only to a *closed* manifold X with $\Gamma_n = \pi_1(X)$, if X admits a splitting $X = X_1 \cup_Y X_2$ consistent with the decomposition $\Gamma_n \cong \Gamma_{n-1} *_{\mathbb{Z}^{n-1}} \Gamma_{n-1}$. As stated, it does not apply to $\bar{X}$, since $\partial\bar{X}$ is nonempty. But $\bar{X}$ does split along a *closed submanifold*

$$\bar{X} = X_1 \cup_Y X_2, \quad \text{where} \quad Y := \{[t_1, \ldots, t_n] \in T^n \mid t_1 = \pm\tfrac{1}{4}\}/C_2.$$

Here X_1 (and X_2) is defined similarly in $\bar{X}$ but with $t_1 \in [-\frac{1}{4}, \frac{1}{4}]$ (respectively, $t_1 \in [\frac{1}{4}, \frac{3}{4}]$). The fundamental groups of X_1, X_2, Y are the groups appearing in Remark 1.1 with $f(a_1, \dots, a_n) = a_1$, So, by Theorem 6.2, we are done. Again, the key point is $\partial Y = \varnothing$. □

Lemma 6.2 *The group* $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon)$ *is a summand of* $\mathrm{UNil}_n(R; \mathcal{B}, \mathcal{B}')$ *if* $\varepsilon = (-1)^n$*. Furthermore,* $\mathrm{UNil}_{4k+2}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ *and* $\mathrm{UNil}_{4k+3}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ *are nonzero.*

Proof The first claim is immediate from the split epimorphism

$$\varepsilon\colon \Gamma_n \to \Gamma_1 = C_2 * C_2$$

of Remark 1.1, which induces a split epimorphism

$$\mathrm{UNil}_n(R; \mathcal{B}, \mathcal{B}') \to \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^\varepsilon, \mathbb{Z}^\varepsilon).$$

The second claim has been known for many years; see Cappell [10] or [20]. But for the reader's convenience, here is a very easy proof. Farrell (see [25]) extended Cappell's homomorphism ρ, mentioned above, to a homomorphism

$$\rho'\colon \mathrm{UNil}_{2k}(R; R, R) \to L_{2k}(R)$$

for any ring with involution R. But the nonzero element of $L_2(\mathbb{Z})$ is the class of the rank-two (-1)–quadratic form with Arf invariant 1. This element is $\rho'([\zeta])$, where $[\zeta] \in \mathrm{UNil}_2(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ is the class of the unilform, $\zeta = (P, \lambda, \mu, P', \lambda', \mu')$, where

$$P = \mathbb{Z}e, \quad P' = \mathbb{Z}f, \quad \lambda = 0, \quad \lambda' = 0, \quad \mu(e) = \mu'(f) = 1 \pmod 2.$$

Finally a quick proof that $\mathrm{UNil}_{4k+3}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ has an element of order 4 can be found in [15, Corollary 1.9]. It uses almost no machinery. □

Proof of Theorem 6.1 By Lemmas 6.2, 6.1 and 3.1, there is an element $[M, f] \neq [\mathbb{R}^n, \mathrm{id}]$ in $\mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n)$. By Lemma 3.2, we conclude that (M, Γ_n) is not equivariantly homeomorphic to $(\mathbb{R}^n, \Gamma_n)$. □

6.1 Free action of UNil on the structure set of a pair

Our purpose here is to show that our Lemma 6.1 is a formal consequence of the L–theoretic exact sequence of Cappell, appearing in [12; 13].

Let X be a compact, connected topological manifold of dimension $n \geq 6$. Let Y a connected, separating, codimension-one submanifold of X without boundary (that is, $\partial Y = \varnothing$). Assume the induced map $\pi_1(Y) \to \pi_1(X)$ of groups is injective. Furthermore, assume the induced map $\pi_1(\partial X) \to \pi_1(X)$ of groupoids is injective.

Write $X = X_1 \cup_Y X_2$ for the induced decomposition of manifolds. Write $G = G_1 *_F G_2$ for the induced injective amalgam of fundamental groups, where $F := \pi_1(Y)$. Finally, write $H := \pi_1(\partial X)$ as the fundamental groupoid of the boundary.

For simplicity of notation, we shall suppress all the orientation characters. Furthermore, to avoid K–theoretic difficulties, *we assume throughout this subsection* that the projective class group for the codimension-one submanifold Y vanishes,

$$\widetilde{K}_0(\mathbb{Z}[F]) = 0.$$

Theorem 6.2 *On the structure set $\mathcal{S}^h_{\mathrm{TOP}}(X, \partial X)$ of the pair, Wall's action of the group $L^h_{n+1}(G)$ restricts to a free action of Cappell's subgroup,*

$$\mathrm{UNil}^h_{n+1} := \mathrm{UNil}^h_{n+1}(\mathbb{Z}[F]; \mathbb{Z}[G_1 - F], \mathbb{Z}[G_2 - F]).$$

Thus we slightly generalize the case of $\partial X = \varnothing$ of Cappell [11, Theorem 2]. Our proof relies only on his algebraic results [12, Theorems 2 and 5; 13].

Theorem 6.3 (Cappell) *There is a homomorphism*

$$\iota\colon \mathrm{UNil}^h_* \longrightarrow L^h_*(G)$$

whose composite with a map of Wall (see [53, Theorem 9.6]) is an isomorphism

$$\mathrm{UNil}^h_* \xrightarrow{\iota} L^h_*(G) \longrightarrow L^h_* \begin{pmatrix} F & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{pmatrix}.$$

Furthermore, there is an exact sequence

$$\cdots \to \mathrm{UNil}^h_{*+1} \oplus L^h_*(F) \to L^h_*(G_1) \oplus L^h_*(G_2) \to L^h_*(G) \xrightarrow{\binom{s}{\partial}} \mathrm{UNil}^h_* \oplus L^h_{*-1}(F) \to \cdots$$

such that $s \circ \iota = \mathrm{id}$. In particular, ι is split injective with a preferred left inverse. □

Write $\mathtt{all}$ as the family of all subgroups of G. Write $\mathtt{fac}$ as the family of subgroups of G conjugate into either G_1 or G_2.

Theorem 6.3 implies the following composite map is an isomorphism:

$$\phi\colon \mathrm{UNil}^h_* \xrightarrow{\iota} L^h_*(G) = H^G_*(E_{\mathtt{all}}G; \underline{\mathbf{L}}^h) \xrightarrow{h} H^G_*(E_{\mathtt{all}}G, E_{\mathtt{fac}}G; \underline{\mathbf{L}}^h)$$

This implies a relative version; recall $H = \pi_1(\partial X)$. Consider the homomorphism

$$j\colon L^h_*(G) \longrightarrow L^h_*(G, H).$$

Write the fundamental groupoid $H = H_1 \sqcup \cdots \sqcup H_m$ as the disjoint union of its vertex groups H_i. The associated G–set is defined by

$$G/H := G/H_1 \sqcup \cdots \sqcup G/H_m.$$

Observe, since $Y \cap \partial X = \varnothing$, for each i that $H_i \subset G_1$ or $H_i \subset G_2$; hence $H_i \in \mathfrak{fac}$. Therefore there is a canonical G–map $G/H \to E_{\mathfrak{fac}}G$.

Corollary 6.1 *There is a split short exact sequence*

$$0 \longrightarrow H_*^G(E_{\mathfrak{fac}}G, G/H; \underline{\boldsymbol{L}}^h) \xrightarrow{A_{\mathfrak{fac}}} L_*^h(G, H) \xrightarrow{s'} \mathrm{UNil}_*^h \longrightarrow 0.$$

The preferred right inverse for s' is the composite $j \circ \iota$.

Proof There is a long exact sequence of the triple,

$$\cdots \to H_*^G(E_{\mathfrak{fac}}G, G/H; \underline{\boldsymbol{L}}^h) \to H_*^G(E_{\mathfrak{all}}G, G/H; \underline{\boldsymbol{L}}^h) \xrightarrow{k} H_*^G(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G; \underline{\boldsymbol{L}}^h) \to \cdots .$$

So, by the above discussion, we may define a homomorphism

$$s'\colon L_*^h(G, H) = H_*^G(E_{\mathfrak{all}}G, G/H; \underline{\boldsymbol{L}}^h) \xrightarrow{k} H_*(E_{\mathfrak{all}}G, E_{\mathfrak{fac}}G; \underline{\boldsymbol{L}}^h) \xrightarrow{\phi^{-1}} \mathrm{UNil}_*^h .$$

That is, $s' := \phi^{-1} \circ k$. Note $h = k \circ j$. Recall $\phi = h \circ \iota$. Then

$$s' \circ (j \circ \iota) = (\phi^{-1} \circ k) \circ (j \circ \iota) = \phi^{-1} \circ (h \circ \iota) = \phi^{-1} \circ \phi = \mathrm{id}.$$

Therefore, k has right inverse $j \circ \iota \circ \phi^{-1}$, and the above exact sequence splits. □

Now we are ready to prove the main theorem of this subsection.

Proof of Theorem 6.2 Ranicki defined algebraic structure groups $\mathcal{S}_*^h(X, \partial X)$, a homomorphism $L_*^h(G, H) \to \mathcal{S}_*^h(X, \partial X)$, and a pointed bijection

$$\mathcal{S}_{\mathrm{TOP}}^h(X, \partial X) \xrightarrow{\approx} \mathcal{S}_{n+1}^h(X, \partial X)$$

such that it is equivariant with respect to the actions of $L_{n+1}^h(G, H)$ (see [43]). Also observe, from Remark 4.3, that there is a monomorphism

$$\mathcal{S}_{n+1}^h(X, \partial X) \rightarrowtail \mathcal{S}_{n+1}^{\mathrm{per},h}(X, \partial X).$$

Write W as the composite homomorphism $L^h_{n+1}(G,H) \to \mathcal{S}^{\mathrm{per},h}_{n+1}(X,\partial X)$, which is compatible with Wall's action of $L^h_{n+1}(G,H)$ on the structure set $\mathcal{S}^h_{\mathrm{TOP}}(X,\partial X)$. Thus it suffices to show that the following composite is a monomorphism:

$$\mathrm{UNil}^h_{n+1} \xrightarrow{\iota} L^h_{n+1}(G) \xrightarrow{j} L^h_{n+1}(G,H) \xrightarrow{W} \mathcal{S}^{\mathrm{per},h}_{n+1}(X,\partial X)$$

By definition of the algebraic structure groups, there is an exact sequence

$$H_{n+1}(X,\partial X;\boldsymbol{L}) \xrightarrow{A} L^h_{n+1}(G,H) \xrightarrow{W} \mathcal{S}^{\mathrm{per},h}_{n+1}(X,\partial X).$$

Also, using Theorem B.1, there is a commutative diagram of assembly maps:

$$\begin{array}{ccc} H^G_{n+1}(\widetilde{X}, G\times_H \widetilde{\partial X}; \underline{\boldsymbol{L}}^h) & \longrightarrow & H^G_{n+1}(E_{\mathfrak{fac}}G, G/H; \underline{\boldsymbol{L}}^h) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle A_{\mathfrak{fac}}} \\ H_{n+1}(X,\partial X;\boldsymbol{L}) & \xrightarrow{\quad A \quad} & L^h_{n+1}(G,H) \end{array}$$

Then, by Corollary 6.1, note

$$\mathrm{Ker}(W) = \mathrm{Im}(A) \subseteq \mathrm{Im}(A_{\mathfrak{fac}}) = \mathrm{Ker}(s') \quad \text{and} \quad \mathrm{Im}(j\circ\iota)\cap \mathrm{Ker}(s') = 0.$$

So $W\circ j\circ\iota$ is a monomorphism. Therefore UNil^h_{n+1} acts freely on $\mathcal{S}^h_{\mathrm{TOP}}(X,\partial X)$. □

Appendix A: From equivariance to isovariance

We want to prove that the forgetful map $\psi\colon \mathcal{S}^{\mathrm{iso}}_{\mathrm{TOP}}(T^n,C_2) \to \mathcal{S}_{\mathrm{TOP}}(T^n,C_2)$ is bijective when $n\ge 4$. It seems best to approach this from a general study of isovariance. It shall be immediate from Theorem A.1 below that ψ is bijective. The assumption of a discrete singular set in our Theorem A.1 is key.

Let G be a finite group. For any G–spaces X and Y, write $[X,Y]_G$ and $[X,Y]^{\mathrm{iso}}_G$ for the set of G–equivariant and G–isovariant homotopy classes of maps, respectively.

Let X be a G–space with a fixed point p. The *homotopy link* of p in X, denoted t^pX, and the *homotopy tangent space* of X at p, denoted t_pX, are defined by

$$t^pX := \mathrm{Holink}(X,p) = \{\sigma\colon [0,1]\to X \mid \sigma^{-1}(p)=\{0\}\},$$
$$t_pX := t^pX \cup \{\sigma_p\}.$$

Here σ_p denotes the constant path at p, and t_pX has the compact-open topology. This is the metric topology of uniform convergence if X is a metric space.

There is a G–subspace $X_{(p)} \subset X$ and isovariant evaluation map e_1 defined by

$$X_{(p)} := (X - X^G) \cup \{p\}, \quad e_1\colon t_p X_{(p)} \longrightarrow X, \quad \sigma \longmapsto \sigma(1).$$

Let U be a neighborhood of p in X. If U is homeomorphic to $\mathbb{R}^n$, then

(11) $\qquad e_1$ restricts to a homotopy equivalence $t^p U \simeq \mathbb{R}^n - \{0\}$.

If U is G–invariant and p is an isolated fixed point, then the inclusion $\iota\colon t_p U \to t_p X$ is an isovariant homotopy equivalence.

An action of a group G on a set X is *semifree* if the action is free away from the fixed set, that is, the action of G on $X - X^G$ is a free action.

Lemma A.1 *Let X and Y be metric spaces on which G acts semifreely and isometrically. Assume q is an isolated fixed point of Y. The rule $f \mapsto f|_{X - X^G}$ gives a bijection between isovariant and equivariant homotopy classes,*

$$[X, t_q Y]_G^{\mathrm{iso}} \cong [X - X^G, t^q Y]_G.$$

Proof Let $f\colon X - X^G \to t^q Y$ be a G–map. We first show f is G–homotopic to an extendible map, f'. (Here f' is *extendible*, if $\lim_{x \to x_0} f'(x) = \sigma_q$ for any $x_0 \in X^G$.) This will prove that the restriction map is a surjection. We assume X and Y have metrics, d_X and d_Y, bounded by 1. Write d_t for the induced metric on $t_q Y$.

For $x \in X - X^G$, set $\|x\| := d_X(x, X^G)$. For $\sigma \in t^q Y$, set $\|\sigma\| := d_t(\sigma, \sigma_q)$. If, for all $x \in X - X^G$, $\|f(x)\| \le \|x\|$, then f is obviously extendible. In $(X - X^G) \times I$, the subset $(X - X^G) \times 0$ is disjoint from the closed subset

$$B := \{(x,t) \in (X - X^G) \times I \mid d_Y(f(x)(t), q) \ge \|x\|\}.$$

Consider the continuous map

$$\phi\colon X - X^G \longrightarrow (0,1], \quad x \longmapsto d_\times((x,0), B),$$

where $d_\times$ denotes the product metric on $(X - X^G) \times I$. Observe

$$\phi(x) = d_\times((x,0),(x,\phi(x))).$$

If $0 \le t < \phi(x)$ then $(x,t) \notin B$. Therefore $d_Y(f(x)(t), q) \le \|x\|$ for all $t \in [0, \phi(x)]$.

Define $f'\colon X - X^G \to t^q Y$ as the map whose adjoint is

$$\mathcal{A}f' := \mathcal{A} \circ \Phi\colon (X - X^G) \times I \longrightarrow Y, \quad \text{where} \quad \Phi(x,t) := (x, t \cdot \phi(x)).$$

By construction, $\|f'(x)\| \leq \|x\|$ for all $x \in X - X^G$. So f' is extendible. But a G–homotopy, $f_s, 0 \leq s \leq 1$, from f to f' is defined by

$$\mathcal{A} f_s := \mathcal{A} f \circ \Phi_s \colon (X - X^G) \times I \longrightarrow Y, \quad \text{where} \quad \Phi_s(x,t) := (x, t \cdot (s\phi(x) + (1-s))).$$

Note that if f is extendible, then each f_s is extendible too.

The same simple argument shows that if $f\colon (X - X^G) \times I \to t^q Y$, is a homotopy between two extendible G–homotopy equivalences, then f' supplies an extendible homotopy. One merely changes the definition of B to

$$B := \{(x,t) \in (X - X^G) \times I \mid d_Y(f(x,s)(t), q) \geq \|x\| \text{ for some } s \in [0,1]\}.$$

This proves that $[X - X^G, t^q Y]_G \cong [X, t_q Y]_G^{\text{iso}}$, as required. □

Because our argument employs a somewhat unusual form of Poincaré duality, namely (13), we spend some space introducing it here.

Lemma A.2 *Let (X, A) be a compact Hausdorff G–pair satisfying this condition:*

(12) *There exists a G–homotopy $h\colon X \times I \to X$ and there exists a G–neighborhood N of A so that*

$$\forall x \in X, h(x,0) = x, \quad \forall a \in A, h(a,t) = a, \quad \forall n \in N, h(n,1) \in A.$$

Assume that $X - A$ is an m–dimensional manifold with boundary, and that the action of G on $X - A$ is free.

Then for any left $\mathbb{Z}G$–module B, there is an isomorphism

$$H_G^*(X - A, \partial(X - A); B) \cong H^G_{m-*}(X, A; B). \tag{13}$$

Notation For any G–pair (X, A), the groups $H_*^G(X, A; B)$ and $H_G^*(X, A; B)$ denote the homology of the complexes $C(X, A) \otimes_{\mathbb{Z}G} B$ and $\mathrm{Hom}_{\mathbb{Z}G}(C(X, A), B)$.

Proof of Lemma A.2 Let h and N be as in (12). Let $f\colon X \to X$ be the function $x \mapsto h(x,1)$. Note $f(X) = A$. Consider the collection

$$\mathcal{U} := \{G\text{–open } U \text{ in } X \mid A \subset U \subset N\}.$$

For each $U \in \mathcal{U}$, write $f_U\colon C(X,U) \to C(X,A)$ for the chain map induced by f and $i_U\colon C(X,A) \to C(X,U)$ for the chain map induced by the inclusion $A \subset U$.

Since $X - A$ is a noncompact m–manifold with boundary, Poincaré duality takes the form of an isomorphism

$$\mathcal{P}\colon H_G^*(X - A, \partial(X - A); B) \longrightarrow H^{G,lf}_{m-*}(X - A; B),$$

where the right-hand group is the homology of the complex $C^{lf}(X-A)\otimes_{\mathbb{Z}G} B$. We identify the complex $C^{lf}(X-A)$ of locally finite chains with the inverse limit

$$C^{lf}(X-A) = \lim_{U\in\mathcal{U}} C(X,U).$$

The systems of maps given by f_U and i_U then define maps

$$C(X,A) \xrightarrow{i_{\mathcal{U}}} \lim_{U\in\mathcal{U}} C(X,U) = C^{lf}(X-A) \xrightarrow{f_{\mathcal{U}}} C(X,A).$$

We will prove below that $i_{\mathcal{U}}$ and $f_{\mathcal{U}}$ are chain homotopy inverse over $\mathbb{Z}G$. Therefore the required isomorphism is the composite

$$(i_{\mathcal{U}}\otimes \mathrm{id}_B)_*^{-1}\circ \mathcal{P}\colon H^*_G(X-A,\partial(X-A);B) \longrightarrow H^G_{m-*}(X,A;B).$$

We now show $i_{\mathcal{U}}$ and $f_{\mathcal{U}}$ are chain homotopy inverse to one another. First note that the map $f_{\mathcal{U}}\circ i_{\mathcal{U}}\colon C(X,A)\to C(X,A)$ equals the chain map $f_\#$ induced by f. So the homotopy h induces a $\mathbb{Z}G$–chain homotopy: $\mathrm{id}_{C(X,A)} \simeq_G (f_{\mathcal{U}}\circ i_{\mathcal{U}})$.

Finally, we show that there exists a $\mathbb{Z}G$–chain homotopy $\mathrm{id}_{C^{lf}(X-A)} \simeq_G (i_{\mathcal{U}}\circ f_{\mathcal{U}})$. Choose a sequence $(n\mapsto O_n)$ of open, G–invariant neighborhoods of A in X such that $\mathrm{Cl}_X(O_{n+1})\subset O_n\subset N$ and $A=\bigcap_{n=1}^\infty O_n$. Define an order preserving function, $T\colon \mathcal{U}\to\mathcal{U}$ by setting $T(U)=O_n$ where n is the first integer for which $h(\mathrm{Cl}_X(O_n)\times I)\subset U$. Write $j_U\colon C(X,T(U))\to C(X,U)$ for the map induced by the inclusion $T(U)\subset U$. Then for each U, the homotopy h induces a chain homotopy

$$h_U\colon j_U \simeq_G (i_U\circ f_{T(U)})\colon C(X,T(U)) \longrightarrow C(X,U).$$

But note that

$$\lim_{U\in\mathcal{U}} j_U = \mathrm{id}_{C^{lf}(X-A)},\quad \lim_{U\in\mathcal{U}} i_U = i_{\mathcal{U}},\quad \lim_{U\in\mathcal{U}} f_{T(U)} = f_{\mathcal{U}}.$$

Therefore $\lim_{U\in\mathcal{U}} h_U$ provides the desired chain homotopy. □

Recall that a map $f\colon (X,A)\to(Y,B)$ is *strict* if $f^{-1}(B)=A$. A homotopy equivalence between the pairs (X,A) and (Y,B) is *strict* if there are corresponding homotopies $(X\times I, A\times I)\to(Y,B)$ and $(Y\times I,B\times I)\to(X,A)$ that are strict.

Definition A.1 Let X be a G–space, and let A a closed G–subset of X. We call a pair (U,h) a *tamed neighborhood* if it consists of a G–invariant neighborhood U of A in X and a strict G–map $h\colon (U\times I, U\times\{0\}\cup A\times I)\to (X,A)$ that restricts to the inclusion on $U\times\{1\}$ and the projection on $A\times I$. We say A is *tame in X* if A has a tamed neighborhood (U,h) in X. It follows that (X,A) satisfies (12).

Remark A.1 Note that if $A = \{p\}$ is an isolated fixed point with tamed neighborhood (U,h), then $U^G = \{p\}$ and the adjoint of $h\colon U \times I \to X$ is an isovariant map $\lambda_p\colon U \to t_p X$. Furthermore, it follows from Quinn [40, Proposition 3.6 or 2.6] that if X is assumed to be a G–manifold and V is any neighborhood of p, then there exists a tamed neighborhood (U,h) of p with $U \subset V$.

Lemma A.3 *Let U^m be a compact semifree G–manifold with G–collared boundary. Assume U^G is tame in U. Let V^n be a G–manifold, and q an isolated fixed point of V. Let $f\colon U \to t_q V$ be a G–map such that $f|_{\partial U}$ is isovariant. If (U,U^G) is 1–connected and $m \le n+1$, then f is G–homotopic* rel ∂U *to an isovariant map.*

Proof There exists a G–map $F\colon U - U^G \to t^q V$ extending the G–map

$$f|_{\partial(U-U^G)}\colon \partial(U-U^G) \to t^q V,$$

since, using Lemma A.2, the obstructions lie in the groups

$$H^i_G(U-U^G, \partial(U-U^G); \pi_{i-1}(t^q V)) \cong H^G_{m-i}(U, U^G; \pi_{i-1}(t^q V)).$$

If $i < n$ then the coefficient group is zero, by (11). If $i \ge n$ then the homology group is zero, because (U,U^G) is a 1–connected pair and $m-i \le m-n \le 1$. Therefore, by Lemma A.1, there is an isovariant map $f'\colon U \to t_q V$ such that $f'|_{\partial U}$ is isovariantly homotopic to $f|_{\partial U}$. But since ∂U has a G–collar in U, the isovariant G–homotopy extension property applies, and we can choose f' so that $f'|_{\partial U} = f|_{\partial U}$. Finally since $t_q V$ is G–contractible, the maps $f, f'\colon U \to t_q V$ are G–homotopic. □

Let X be a G–manifold with boundary. Recall the *singular set* of X is

$$X_{\text{sing}} := \{x \in X \mid gx = x \text{ for some } g \neq 1 \in G\}.$$

A neighborhood U of X_{sing} in X is a *k–neighborhood* if it is a G–invariant codimension zero submanifold with bicollared frontier in X, such that the pair (U, X_{sing}) is k–connected. We only use this concept for $k = 0, 1$.

Theorem A.1 *Let X^n and Y^n be compact G–manifolds without boundary. Assume X_{sing} and Y_{sing} are finite sets, and assume $n \ge 4$. Let $f\colon X \to Y$ be a G–map such that the restriction $f_{\text{sing}}\colon X_{\text{sing}} \to Y_{\text{sing}}$ is bijective.*

(1) *If f is 1–connected, then f is G–homotopic to an isovariant map.*

(2) *Suppose $f = f_0$ is isovariant and 2–connected. If f_0 is G–homotopic to an isovariant map $f_1\colon X \to Y$, then f_0 is G–isovariantly homotopic to f_1.*

Proof of Theorem A.1(1) We argue in two major steps.

Step 1 We find 0–neighborhoods U, V of $X_{\mathrm{sing}}, Y_{\mathrm{sing}}$ such that $U = f^{-1}(V)$.

It follows from Siebenmann's thesis [47] that each neighborhood of X_{sing} contains a 0–neighborhood if $n = 4$, and a 1–neighborhood if $n \geq 5$. By Remark A.1, we can choose a 0–neighborhood V of Y_{sing} in Y, so small that for each $q \in Y_{\mathrm{sing}}$, the component V_q containing q, admits the structure of a tamed neighborhood of q, say (V_q, h_q). By [27] applied to $(f - f_{\mathrm{sing}})/G$, we may assume f is transverse to ∂V. Then $N := f^{-1}(\partial V)$ is a bicollared codimension one G–submanifold of X. It is the boundary and frontier of $U := f^{-1}(V)$, a G–neighborhood of $f^{-1}(Y_{\mathrm{sing}})$.

Our desire is that U be a 0–neighborhood. We plan to accomplish this by handle exchanges along N realized through a homotopy of f.

Define the closures $X_0 := \mathrm{Cl}_X(X - U)$ and $Y_0 := \mathrm{Cl}_Y(Y - V)$. Note X_0 is a manifold with boundary $N = \partial X_0$. Also note $X = U \cup_N X_0$ and $Y = V \cup_{\partial V} Y_0$.

We now recall the aforementioned notion of *handle exchange* along N.

Suppose we can find a map, $i\colon (D^k, \partial D^k) \times \{0\} \to (X_0, N)$ (or alternatively, a map, $i\colon (D^k, \partial D^k) \times \{0\} \to (U, N)$), together with an extension of $f \circ i$ to a map

$$j\colon (D^k, \partial D^k) \times (I, \{1\}) \longrightarrow (Y_0, \partial Y_0) \quad \text{(or to } (V, \partial V)\text{)}.$$

If $k < n/2$, we can, after a homotopy, thicken i to an equivariant embedding and an equivariant extension still called i; also, we can thicken j to a continuous G–map, still called j:

$$i\colon G \times (D^k, \partial D^k) \times D^{n-k} \longrightarrow (X_0, N) \quad \text{(or to } (U - U^G, N)\text{)},$$
$$j\colon G \times (D^k, \partial D^k) \times D^{n-k} \times (I, \{1\}) \longrightarrow (Y_0, N) \quad \text{(or to } (V - V^G, \partial V)\text{)}.$$

Now deform f by a G–homotopy, stationary off $i(G \times \mathrm{Int}(D^k \times D^{m-k}))$, to a map f' so that f' is still transverse to N, but

$$f'^{-1}(V) = U \cup i(G \times D^k \times \tfrac{1}{2} D^{m-k})$$
$$\text{(or } f'^{-1}(Y_0) = X_0 \cup i(G \times D^k \times \tfrac{1}{2} D^{m-k})\text{)}.$$

Note this homotopy is rel ∂X. If $q \in Y_{\mathrm{sing}}$, set $U_q := f^{-1}(V_q)$ and $N_q := N \cap U_q$.

Assume $k = 1$. This handle exchange process decreases the number of components of N provided that i is chosen so that $\mathrm{Im}(i)/G$ meets two components of N/G. After finitely many such handle exchanges then, we arrive at a map f' for which N_q is connected for each $q \in Y_{\mathrm{sing}}$. Therefore U_q is connected too. So U is a 0–neighborhood of X^G.

Step 2 In this step we find a G–homotopy from f to a G–isovariant map.

From Step 1, we have $f^{-1}(Y_{\mathrm{sing}}) \subset U = f^{-1}(V)$. Now, we need only show how to deform $f|_U$ rel ∂U equivariantly to a G–isovariant map, $f'\colon U \to Y$.

But if $q \in Y_{\mathrm{sing}}$ and $g \in G$ and $gq \neq q$, then we have $U_q \cap U_{gq} = \varnothing$. Therefore, if we choose one point q from each G–orbit in Y_{sing} it is sufficient to show how to deform $f|_{U_q}$ rel ∂U_q to a G_q–isovariant map $f'_q\colon U_q \to Y$. Here, $G_q := \{g \in G \mid gq = q\}$ denotes the isotropy group of q.

Fix $q \in Y_{\mathrm{sing}}$. Let $Y_{(q)} = (Y - Y^G) \cup \{q\}$. Recall $f|_{U_q} = e_1 \circ \lambda_q \circ f|_{U_q}$, where $\lambda_q\colon V_q \to t_q Y_{(q)}$, and $e_1\colon t_q Y_{(q)} \to Y$ is equivariant. By Lemma A.3, there is a G_q–isovariant map $F_q\colon U_q \to t_q Y_{(q)}$ for which $(\lambda_q \circ f|_{U_q}) \simeq_{G_q} F_q$ rel ∂U_q. Define $f'_q := e_1 \circ F_q$. Then f'_q is G_q–isovariant and $f|_{U_q} \simeq_{G_q} f'_q$, as required. □

Proof of Theorem A.1(2) The argument is entirely similar to that for the proof of Theorem A.1(1). Realize the homotopy from f_0 to f_1 by a G–map

$$(F, f_0 \sqcup f_1)\colon X \times (I, \partial I) \longrightarrow Y \times (I, \partial I).$$

Step 1 We find 1–neighborhoods U, V of $X_{\mathrm{sing}} \times I$, $Y_{\mathrm{sing}} \times I$ so that $F^{-1}(V) = U$.

Choose a tamed neighborhood (W, h) of Y_{sing} in Y. Let W_q be the component of W containing q, for each $q \in Y_{\mathrm{sing}}$. Since $\dim(Y \times I) \geq 5$, by Siebenmann's thesis [47] again, we can choose a 1–neighborhood V of $Y_{\mathrm{sing}} \times I$ in $W \times I$, such that $V \cap (Y \times \partial I)$ is a 0–neighborhood of $Y_{\mathrm{sing}} \times \partial I$ in $Y \times \partial I$.

Let $\partial_0 V$ be the frontier of V in $Y \times I$. Then $\partial_0 V$ is a codimension 0–submanifold of ∂V, and $\partial\partial_0 V = \partial_0 V \cap (Y \times \partial I)$.

Make F transverse to $\partial_0 V$ after a homotopy which is isovariant on $X \times \partial I$. Let $U = F^{-1}(V)$ and let $U_0 = f^{-1}(\partial_0 V)$, the frontier of U in $X \times I$. Then $\partial_0 U$ is a manifold with boundary and $\partial\partial_0 U \subset X \times \partial I$. Also $(\partial_0 U, \partial\partial_0 U)$ is bicollared in $(X \times I, \partial(X \times I))$.

Proceed as in the proof of Theorem A.1(1) to make U a 0–neighborhood of $X_{\mathrm{sing}} \times I$. As before, we let U_q be the component of U containing $(F^G)^{-1}(q \times I)$, and let $\partial_0 U_q = \partial_0 U \cap U_q$.

We plan to make $\partial_0 U_q$ simply connected for each q. We repeat the "innermost circles" argument of Browder [9] doing handle exchanges along $\partial_0 U_q$ using 2–handles to kill off the finitely many generators of each $\pi_1(\partial_0 U_q)$. We do one q at a time, choosing one q from each G orbit of Y_{sing}. In the end we get a new $F\colon X \times I \to Y \times I$ with its new U for which $\partial_0 U_q$ is 1–connected for each $q \in Y_{\mathrm{sing}}$. This implies that

$\pi_1(X \times I) = \pi_1(U_q) * \pi_1(X \times I - \operatorname{Int}(U_q))$, and $\mathrm{incl}_*\colon \pi_1(U_q) \to \pi_1(X \times I)$ is injective. But V_q is simply connected and therefore U_q is simply connected by the diagram

$$\begin{array}{ccc} \pi_1(U_q) & \xrightarrow{(f|_{U_q})_*} & \pi_1(V_q) = \{1\} \\ \Big\downarrow{\scriptstyle \mathrm{incl}_*} & & \Big\downarrow{\scriptstyle \mathrm{incl}_*} \\ \pi_1(X \times I) & \xrightarrow[\cong]{F_*} & \pi_1(Y \times I). \end{array}$$

Therefore U is a 1–neighborhood of $X_{\mathrm{sing}} \times I$ in $X \times I$, and $U = F^{-1}(V)$.

Step 2 In this step we show f_0 is isovariantly homotopic to f_1.

Let $H = p_1 \circ F\colon X \times I \to Y$. Note H is a G–homotopy from f_0 to f_1, with

$$H^{-1}(Y^G) \subset U \subset H^{-1}(W), \quad H(U_q) \subset W_q \quad \text{for all } q \in Y^G.$$

As in the proof of Theorem A.1(1), for each $q \in Y^G$, $\lambda_q \circ H|_{U_q}\colon U_q \to t_q Y_{(q)}$ is homotopic rel ∂U_q to an isovariant map, by Lemma A.3. Therefore F is homotopic rel $X \times I - \operatorname{Int}(U)$, to an isovariant map $H'\colon X \times I \to Y$, which serves as the required isovariant homotopy from f_0 to f_1. □

Observe that we did note use any end theorems in the proof above.

Appendix B: Quinn–Ranicki = Davis–Lück in the case of free actions

In this section we identify the Quinn–Ranicki assembly map with a map in equivariant homology in the case of a group acting freely on a CW–complex, where all components are simply connected. (The connected components may be permuted by the group action.) For a connected CW–complex this follows from the characterization of assembly maps due to Weiss and Williams [55], but in our case we must use the equivariant characterization of assembly maps given in [23, Section 6]. Hambleton and Pedersen generalized the work of Weiss and Williams in a different direction. Unfortunately, Davis and Lück [23] did not connect the map in equivariant homology with the Farrell–Jones conjecture. This was remedied by Hambleton and Pedersen in [28, Corollary 10.2], which identified the stratified assembly map used in the original formulation of the Farrell–Jones conjecture [26] with the map in equivariant homology induced by $E_{\mathrm{vc}}G \to E_{\mathrm{all}}G$. Thus their work [28] applied to the contractible case, while ours applies to the free case.

Let $\operatorname{Ho}\mathsf{Spectra}$ be the *homotopy category*, given by formally inverting weak homotopy equivalences. There is a *localization functor* $\operatorname{Ho}\colon \mathsf{Spectra} \to \operatorname{Ho}\mathsf{Spectra}$ sending weak equivalences to isomorphisms, and this functor is initial with respect to all such functors from $\mathsf{Spectra}$. The functor Ho is a bijection on objects. Homotopy groups $\pi_i\colon \mathsf{Spectra} \to \mathsf{Ab}$ factor through the functor Ho. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$–spectrum* is a functor from $\mathcal{C}$ to $\mathsf{Spectra}$, a *map of $\mathcal{C}$–spectra* is a natural transformation, and a *weak equivalence of $\mathcal{C}$–spectra* is a map of $\mathcal{C}$–spectra $\boldsymbol{E} \to \boldsymbol{F}$ which induces a weak homotopy equivalence of spectra $\boldsymbol{E}(c) \to \boldsymbol{F}(c)$ for all objects c in $\mathcal{C}$. There is a localization functor $\operatorname{Ho}\colon \mathcal{C}\text{–}\mathsf{Spectra} \to \operatorname{Ho}\mathcal{C}\text{–}\mathsf{Spectra}$. A key property is that if $\boldsymbol{E}$ and $\boldsymbol{F}$ are $\mathcal{C}$–spectra which become isomorphic in $\operatorname{Ho}\mathcal{C}\text{–}\mathsf{Spectra}$, then there is a $\mathcal{C}$–spectrum $\boldsymbol{G}$ and weak equivalences $\boldsymbol{E} \leftarrow \boldsymbol{G} \to \boldsymbol{F}$.

For a groupoid $\mathcal{G}$, let $\boldsymbol{L}(\mathcal{G})$ be the corresponding L–spectrum, as in [23, Section 2]. This is a functor from the category of groupoids to the category of spectra which satisfies the additional property that an equivalence $F\colon \mathcal{G} \to \mathcal{G}'$ of groupoids induces a weak equivalence $\boldsymbol{L}(F)\colon \boldsymbol{L}(\mathcal{G}) \to \boldsymbol{L}(\mathcal{G}')$ of spectra.

Ranicki, motivated by the earlier geometric work of Quinn, defined the *assembly map* [43, Chapter 14], a natural transformation of functors from Top to $\mathsf{Spectra}$,

$$\boldsymbol{A}(Z)\colon \boldsymbol{H}(Z;\boldsymbol{L}(1)) \longrightarrow \boldsymbol{L}(\Pi_1 Z),$$

where $\Pi_1 Z$ is the fundamental groupoid of Z. When Z is a point, the assembly map is a weak equivalence. The *algebraic structure spectrum* $\mathcal{S}^{\mathrm{per}}(Z)$ is defined to be the homotopy cofiber of $\boldsymbol{A}(Z)$. Its homotopy groups $\mathcal{S}^{\mathrm{per}}_*(Z) := \pi_*\mathcal{S}^{\mathrm{per}}(Z)$ are the algebraic structure groups used in Section 4; one can do this for pairs also.

Fix a group G. Consider the orbit category $\mathrm{Or}(G)$ and the $\mathrm{Or}(G)$–spectrum

$$\underline{\boldsymbol{L}}\colon \mathrm{Or}(G) \longrightarrow \mathsf{Spectra}, \quad G/H \longmapsto \boldsymbol{L}(\overline{G/H}),$$

where $\overline{G/H}$ is the groupoid associated to the G–set G/H. For a G–CW–complex X, consider the spectrum

$$\boldsymbol{H}^G(X;\underline{\boldsymbol{L}}) := \mathrm{map}_G(-,X)_+ \wedge_{\mathrm{Or}(G)} \underline{\boldsymbol{L}}(-).$$

Then, by definition, $H^G_*(X;\underline{\boldsymbol{L}}) = \pi_*\boldsymbol{H}^G(X;\underline{\boldsymbol{L}})$.

Write $G\mathcal{CW}$ for the category whose objects are G–CW–complexes and whose morphisms are cellular G–maps. (Actually, for set-theoretic reasons we need to restrict our G–CW–complexes to a fixed universe; for our purposes it will suffice to assume that the underlying space of each CW–complex is embedded in $\mathbb{R}^\infty$.) A *$(G,\mathcal{F})$–CW–complex* is a G–CW–complex with isotropy in a family $\mathcal{F}$. Let $(G,\mathcal{F})\mathcal{CW}$ be the full subcategory of $G\mathcal{CW}$ whose objects are $(G,\mathcal{F})$–CW–complexes. Let $\mathrm{Or}(G,\mathcal{F})$

be the full subcategory of $(G,\mathcal{F})\mathcal{CW}$ whose objects are the discrete G–spaces G/H with $H \in \mathcal{F}$. The symbol 1 will denote both the trivial group and the family of subgroups of G consisting of the trivial group. Let $\mathrm{sc}(G,1)\mathcal{CW}$ be the full subcategory of $(G,1)\mathcal{CW}$ whose objects are free G–CW–complexes all of whose components are simply connected.

Let X be a free G–CW–complex. Let $\Pi_0 X$ be the G–set of path components of X. Here is the main theorem of this appendix.

Theorem B.1 *There is a commutative diagram in* $\mathrm{Ho}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$*:*

$$\begin{array}{ccc} \boldsymbol{H}(X/G;\boldsymbol{L}(1)) & \longrightarrow & \boldsymbol{L}(\Pi_1(X/G)) \\ \downarrow & & \downarrow \\ \boldsymbol{H}^G(X;\underline{\boldsymbol{L}}) & \longrightarrow & \boldsymbol{H}^G(\Pi_0 X;\underline{\boldsymbol{L}}) \end{array}$$

(1) *The top map is the assembly map $\boldsymbol{A}(X/G)$ and is a map of $(G,1)\mathcal{CW}$–$\mathsf{Spectra}$.*

(2) *The bottom map is induced by the G–map $X \to \Pi_0 X$ and is a map of $(G,1)\mathcal{CW}$–$\mathsf{Spectra}$.*

(3) *The right map is the composite of the formal inverse of the weak equivalence of $(G,1)\mathcal{CW}$–spectra $\boldsymbol{L}(\Pi_1(EG\times_G X)) \to \boldsymbol{L}(\Pi_1(X/G))$ and the map of $(G,1)\mathcal{CW}$–spectra $\boldsymbol{L}(\Pi_1(EG\times_G X)) \to \boldsymbol{H}^G(\Pi_0 X;\underline{\boldsymbol{L}})$ defined in Lemma B.1. This map is a weak equivalence when restricted to* $\mathrm{sc}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$*.*

(4) *The left map is an isomorphism in* $\mathrm{Ho}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$*.*

The proof of the theorem is quite formal and applies more generally. What is needed is a functor from groupoids to spectra which takes equivalences of groupoids to weak equivalences of spectra and an assembly map which is a weak equivalence when X is a point. So, for example, our theorem applies equally well to K–theory. See Remark B.2 below for the modifications necessary for the L–theory nonorientable case.

Let G be a discrete group. Let S be a G–set. Define the *action groupoid* $\overline{S}$ as the category whose object set is S, and whose morphisms from s to t are triples (t,g,s) such that $t = gs$, and whose composition law is $(t,g,s)\circ(s,f,r) = (t,gf,r)$. Define a functor

$$\boldsymbol{L}^G\colon (G,1)\mathcal{CW} \longrightarrow \mathsf{Spectra}, \qquad X \longmapsto \boldsymbol{L}(\Pi_1(EG\times_G X)).$$

The next lemma relates $\boldsymbol{L}^G$ to the above functor $\underline{\boldsymbol{L}}\colon \mathrm{Or}(G) \to \mathsf{Spectra}$.

Lemma B.1 *Let G be a discrete group.*

(1) *For a discrete G–set S, there is a homeomorphism of spectra*

$$\boldsymbol{H}^G(S;\underline{\boldsymbol{L}}) \cong \boldsymbol{L}(\bar{S}),$$

natural in S.

(2) *For a free G–CW–complex X, there is a map of groupoids*

$$\Phi(X)\colon \Pi_1(EG \times_G X) \longrightarrow \overline{\Pi_0 X},$$

which is an equivalence of groupoids when all the components of X are simply connected. Furthermore, Φ is natural in X; that is, $\Phi(-)$ is a map of $(G,1)\mathcal{CW}$–groupoids.

(3) *There is a map of $(G,1)\mathcal{CW}$–spectra*

$$\Lambda(X)\colon \boldsymbol{L}^G(X) \longrightarrow \boldsymbol{H}^G(\Pi_0 X;\underline{\boldsymbol{L}}),$$

whose restriction to $\mathrm{sc}(G,1)\mathcal{CW}$ is a weak equivalence.

Proof (1) The homeomorphism is given by

$$\boldsymbol{H}^G(S;\underline{\boldsymbol{L}}) \longrightarrow \boldsymbol{L}(\bar{S}),$$
$$[(f,x) \in \mathrm{map}_G(G/K,S)_+ \wedge L(\overline{G/K})_n] \longmapsto L(\bar{f})_n(x) \in L(\bar{S})_n.$$

If S is an orbit G/K, then the inverse is given by $x \in L(\overline{G/K})_n \mapsto [(\mathrm{id},x) \in \mathrm{map}_G(G/K,G/K)_+ \wedge L(\overline{G/K})_n]$. The case of a general G–set follows since both $\boldsymbol{H}^G(-;\underline{\boldsymbol{L}})$ and $\boldsymbol{L}(-)$ convert disjoint unions to one-point unions of spectra.

(2) We first need some notation. For a subset A of a topological space Y, let $\Pi_1(Y,A)$ be the full subcategory of the fundamental groupoid $\Pi_1 Y$ whose objects are points in A. If $\Pi_0 A \to \Pi_0 Y$ is onto, then there is an equivalence of groupoids $\Pi_1 Y \to \Pi_1(Y,A)$ whose definition depends on a choice of a path from y to a point in A for every $y \in Y$.

Let $p\colon EG \times X \to EG \times_G X$ be the quotient map. We will define $\Phi(X)$ as a composite

$$\Pi_1(EG \times_G X) \xrightarrow{\Theta(X)} \Pi_1(EG \times_G X, p(\{e_0\} \times X)) \xrightarrow{\Psi(X)} \overline{\Pi_0 X}.$$

We first define $\Theta(X)$ by making choices in the universal space EG. Choose a point $e_0 \in EG$. For each $e \in EG$, choose a path $\sigma_e\colon I \to EG$ from e to e_0, choosing the paths so that, for all $g \in G$ and $t \in I$, $g(\sigma_e(t)) = \sigma_{ge}(t)$. This can be accomplished by choosing a set-theoretic section $s\colon BG \to EG$ of the covering map, and defining the remaining σ_e by equivariance. Then for $p(e,x) \in EG \times_G X$, define the path

$\theta_{p(e,x)}(t) := p(\sigma_e(t), x)$. This path is independent of the choice of representative of $p(e,x)$. These paths give the equivalence of groupoids $\Theta(X)$, natural in X.

We now define $\Psi(X)$ using the fact that p is a covering map. On objects, define $\Psi(X)(p(e_0,x)) := C(x) \in \Pi_0 X$, where $C(x)$ is the path component of x in X. For a morphism represented by a path $\alpha\colon I \to EG \times_G X$ with $\alpha(0) = p(e_0,x)$ and $\alpha(1) = p(e_0,y)$, let $\widetilde{\alpha}\colon I \to EG \times X$ be the lift of α starting at (e_0,x). Then $\widetilde{\alpha}(1) = (ge_0, gy)$ some $g \in G$. Then define $\Psi(X)[\alpha] := (C(x), g, C(y))$. We leave the geometric details of verifying that this is a functor to the reader, but note that we follow that convention that a path α determines a morphism from $\alpha(1)$ to $\alpha(0)$ in the fundamental groupoid.

Suppose all the components of X are simply connected. We now show that $\Psi(X)$ is an equivalence of groupoids. Choose a base point for each component of X. Define a functor $\overline{\Pi_0 X} \to \Pi_1(EG \times_G X, p(\{e_0\} \times X))$ on objects by sending $C(x)$ to $p(e_0,x)$, and on morphisms by sending $(C(x), g, C(y))$ to $[p \circ \widetilde{\alpha}]$, where $\widetilde{\alpha}\colon I \to EG \times X$ is a path from (e_0,x) to (ge_0, gy). This $\widetilde{\alpha}$ is unique up to homotopy rel endpoints since X is simply connected. This ends the proof of (2).

(3) Define $\Lambda(X)$ as the composite of $\boldsymbol{L}(\Phi(X))$ and the isomorphism from (1). □

Remark B.1 We next recast the axiomatic approach of [23, Section 6]. Our terminology is self-consistent but does not precisely match that of [23]; in particular we drop the adverb "weakly". A functor $\boldsymbol{E}\colon (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is *homotopy invariant* if any homotopy equivalence induces a weak equivalence of spectra. A functor $\boldsymbol{E}\colon (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is *excisive* if $\boldsymbol{E}(-)$ and $\boldsymbol{H}^G(-;\boldsymbol{E}|_{\mathrm{Or}(G,\mathcal{F})})$ are isomorphic objects in $\mathrm{Ho}(G,\mathcal{F})\mathcal{CW}$–$\mathsf{Spectra}$. This is equivalent to the notion of weakly $\mathcal{F}$–excisive given in [23].[1] By [23, Theorem 6.3(2)], a map $\boldsymbol{T}\colon \boldsymbol{E} \to \boldsymbol{F}$ of excisive $(G,\mathcal{F})$–CW-spectra is a weak equivalence if and only if $\boldsymbol{T}(G/H)\colon \boldsymbol{E}(G/H) \to \boldsymbol{F}(G/H)$ is a weak equivalence of spectra for all $H \in \mathcal{F}$. An *excisive approximation* of a homotopy invariant functor $\boldsymbol{E}\colon (G,\mathcal{F})\mathcal{CW} \to \mathsf{Spectra}$ is a map $\boldsymbol{T}\colon \boldsymbol{E}' \to \boldsymbol{E}$ of $(G,\mathcal{F})$–CW-spectra such that $\boldsymbol{E}'$ is excisive and $\boldsymbol{T}(G/H)$ is a weak equivalence for all orbits G/H with $H \in \mathcal{F}$.

We next assert existence and uniqueness of excisive approximations. Theorem 6.3(2) of [23] constructs a specific excisive approximation $\boldsymbol{E}^{\%} \to \boldsymbol{E}$ which is functorial

[1] Indeed [23, Theorem 6.3(1,3)] implies that if $\boldsymbol{E}$ is weakly $\mathcal{F}$–excisive in the sense of [23], then there is a $(G,\mathcal{F})$–spectrum $\boldsymbol{E}^{\%}$ and weak equivalences $\boldsymbol{E} \leftarrow \boldsymbol{E}^{\%} \to \boldsymbol{H}^G(-;\boldsymbol{E}|_{\mathrm{Or}(G,\mathcal{F})})$. Conversely, if $\boldsymbol{E}(-)$ and $\boldsymbol{H}^G(-;\boldsymbol{E}|_{\mathrm{Or}(G,\mathcal{F})})$ are isomorphic objects in $\mathrm{Ho}(G,\mathcal{F})\mathcal{CW}$–$\mathsf{Spectra}$, then there are weak equivalences $\boldsymbol{E} \leftarrow \boldsymbol{F} \to \boldsymbol{H}^G(-;\boldsymbol{E}|_{\mathrm{Or}(G,\mathcal{F})})$ for some $(G,\mathcal{F})\mathcal{CW}$–spectrum $\boldsymbol{F}$. But [23, Theorem 6.3(1)] shows that $\boldsymbol{H}^G(-;\boldsymbol{E}|_{\mathrm{Or}(G,\mathcal{F})})$ is weakly $\mathcal{F}$–excisive, and hence so is any weakly equivalent $(G,\mathcal{F})\mathcal{CW}$–spectrum.

in $\boldsymbol{E}$. Excisive approximations are unique in the sense that, given any two excisive approximations $\boldsymbol{T}'\colon \boldsymbol{E}' \to \boldsymbol{E}$ and $\boldsymbol{T}''\colon \boldsymbol{E}'' \to \boldsymbol{E}$, there is an isomorphism $\boldsymbol{S}\colon \boldsymbol{E}' \to \boldsymbol{E}''$ in $\mathrm{Ho}(G,\mathcal{F})\mathcal{CW}$–$\mathsf{Spectra}$ such that $\boldsymbol{T}' = \boldsymbol{T}'' \circ \boldsymbol{S}$. Indeed, to verify that $\boldsymbol{S}$ exists, it suffices to compare any excisive approximation $\boldsymbol{T}'\colon \boldsymbol{E}' \to \boldsymbol{E}$ is equivalent to the functorial excisive approximation $\boldsymbol{T}^{\%}\colon \boldsymbol{E}^{\%} \to \boldsymbol{E}$. Consider the commutative diagram in $(G,\mathcal{F})\mathcal{CW}$–$\mathsf{Spectra}$:

$$\begin{array}{ccc} \boldsymbol{E}'^{\%} & \longrightarrow & \boldsymbol{E}' \\ \big\downarrow & & \big\downarrow \\ \boldsymbol{E}^{\%} & \longrightarrow & \boldsymbol{E} \end{array}$$

Since the left and top maps are both weak equivalences, we obtain an isomorphism in the homotopy category $\boldsymbol{S} := (\boldsymbol{E}' \leftarrow \boldsymbol{E}'^{\%} \to \boldsymbol{E}^{\%})$ with $\boldsymbol{T}' = \boldsymbol{T}^{\%} \circ \boldsymbol{S}$, as desired.

Proof of Theorem B.1 The theorem will be proven by concatenating three commutative squares. The first is a commutative diagram in $\mathrm{Ho}(1,1)\mathcal{CW}$–$\mathsf{Spectra}$, which we will apply below in the case $Z = X/G$:

$$\begin{array}{ccc} \boldsymbol{H}(Z;\boldsymbol{L}(1)) & \longrightarrow & \boldsymbol{L}(\Pi_1 Z) \\ \big\downarrow & & \big\downarrow \\ (\boldsymbol{L}^1)^{\%}(Z) & \longrightarrow & \boldsymbol{L}^1(Z) \end{array}$$

The right map is the identity. The top map is the assembly map $\boldsymbol{A}(Z)$. The left map exists in the homotopy category (see end of Remark B.1) and is an isomorphism since both horizontal maps are $(1,1)$–excisive approximations of $\boldsymbol{L}(\Pi_1 Z) = \boldsymbol{L}^1(Z)$.

Next comes a commutative diagram in $\mathrm{Ho}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$:

$$\begin{array}{ccc} (\boldsymbol{L}^1)^{\%}(X/G) & \longrightarrow & \boldsymbol{L}^1(X/G) \\ \big\uparrow & & \big\uparrow \\ (\boldsymbol{L}^G)^{\%}(X) & \longrightarrow & \boldsymbol{L}^G(X) \end{array}$$

The right map is induced by the homotopy equivalence $EG \times_G X \to X/G$, inducing an equivalence of fundamental groupoids, hence a weak equivalence of spectra. The left map exists and is an isomorphism, since the top map and the composite of the bottom and right maps are $(G,1)$–excisive approximations of $\boldsymbol{L}^G(X)$.

Our final commutative diagram is in $\mathrm{Ho}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$:

$$\begin{array}{ccc} (\boldsymbol{L}^G)^{\%}(X) & \longrightarrow & \boldsymbol{L}^G(X) \\ \downarrow & & \downarrow \\ \boldsymbol{H}^G(X;\underline{\boldsymbol{L}}) & \longrightarrow & \boldsymbol{H}^G(\Pi_0 X;\underline{\boldsymbol{L}}) \end{array}$$

The top map is the functorial excisive approximation of $\boldsymbol{L}^G(X)$. The bottom map is induced by the G–map $X \to \Pi_0 X$ and is an excisive approximation of $\boldsymbol{H}^G(\Pi_0(-);\underline{\boldsymbol{L}})$. The right map is defined in Lemma B.1(3) and is an isomorphism when restricted to $\mathrm{Ho}\,\mathrm{sc}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$. Functoriality gives a map

$$(\boldsymbol{L}^G)^{\%}(X) \to \boldsymbol{H}^G(\Pi_0(-);\underline{\boldsymbol{L}})^{\%}(X),$$

and the bottom map is an excisive approximation implies $\boldsymbol{H}^G(\Pi_0(-);\underline{\boldsymbol{L}})^{\%}(X) \to \boldsymbol{H}^G(X;\underline{\boldsymbol{L}})$; define the left map as the composite. Since the left map is a map of excisive functors and is a homotopy equivalence when $X = G/1$, the left map is an isomorphism in $\mathrm{Ho}(G,1)\mathcal{CW}$–$\mathsf{Spectra}$. □

Remark B.2 We next indicate the modifications needed for the statement and proof of Theorem B.1 in the nonorientable case. A *groupoid with orientation character* $\mathcal{G}^w$ is a groupoid $\mathcal{G}$ together with a functor $w\colon \mathcal{G} \to \{\pm 1\}$, where $\{\pm 1\}$ is the category with a single object and two morphisms $\{+1,-1\}$ where $-1 \circ -1 = +1$. A map of groupoids with orientation character is a map of groupoids which preserves the orientation character. Let GWOC denote the category of groupoids with orientation character. There is an L–theory functor $\boldsymbol{L}\colon \mathsf{GWOC} \to \mathsf{Spectra}$. (The definition in [23] can be easily modified to cover this case; see also [4].) Two maps $F_0, F_1\colon \mathcal{G}^w \to \mathcal{G}'^{w'}$ of groupoids with orientation character are *equivalent* if there is a natural transformation which is orientation preserving in the sense that $w'(F_0(x) \to F_1(x)) = +1$ for all objects x of $\mathcal{G}$. A map $F\colon \mathcal{G}^w \to \mathcal{G}'^{w'}$ is an *equivalence of groupoids with orientation characters* if there is a map $F'\colon \mathcal{G}'^{w'} \to \mathcal{G}^w$ so that both composites $F \circ F'$ and $F' \circ F$ are equivalent to the respective identity. An equivalence of groupoids with orientation characters gives an weak equivalence of L–spectra.

Now suppose G is a group with orientation character $w\colon G \to \{\pm 1\}$. Following forthcoming work of Davis and Lindenstrauss, we discuss two related groupoids with orientation character. First, if S is a G–set, give the action groupoid $\overline{S}$ the orientation character $(t,g,s) \mapsto w(g)$. This gives a functor $\mathrm{Or}(G) \to \mathsf{GWOC}$ defined on objects by $G/H \mapsto (\overline{G/H})^w$ and hence a functor

$$\underline{\boldsymbol{L}}\colon \mathrm{Or}(G) \longrightarrow \mathsf{Spectra}, \quad G/H \longmapsto \boldsymbol{L}((\overline{G/H})^w).$$

Suppose $\phi\colon \widehat{Y} \to Y$ is a double cover. Define the *fundamental groupoid with orientation character* $\Pi_1^w(Y)$, as follows. The objects are the points of $\widehat{Y}$. A morphism from $\hat{y}$ to $\hat{y}'$ is a path α from $w(\hat{y}')$ to $w(\hat{y})$. A morphism is assigned $+1$ if the unique lift of α starting at $\hat{y}'$ ends at $\hat{y}$; otherwise assign the morphism -1.

Recall G is a group with orientation character w. Given a free G–CW–complex X, let $w\colon EG \times_{\mathrm{Ker}(w)} X \to EG \times_G X$ be the corresponding double cover. Thus, for a fixed (G, w), there is a functor $\boldsymbol{L}^G$ defined by

$$\boldsymbol{L}^G\colon (G,1)\mathcal{CW} \longrightarrow \mathsf{Spectra}, \quad X \longmapsto \boldsymbol{L}(\Pi_1^w(EG \times_G X)).$$

Then, after modifying $\boldsymbol{L}$, $\boldsymbol{L}^G$ and $\overline{S}$ as indicated above, the statement and proof of Lemma B.1 remain valid. The same is true for Theorem B.1 after accounting for Ranicki's version of the assembly map in the nonorientable case [43, Appendix A].

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