THE MANIFOLD SET OF $\mathbb{RP}^n \# \mathbb{RP}^n$

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Let X be a finite simplicial complex. The *simple manifold set* $\mathcal{M}_{\mathrm{TOP}}(X)$ consists of the homeomorphism classes $[M]$ of closed topological manifolds M in the simple[1] homotopy type of X. Recall that the *simple structure set* $\mathcal{S}_{\mathrm{TOP}}(X)$ consists of the s-bordism classes $[M, h]$ of simple homotopy equivalences $h : M \to X$ where M is a closed topological manifold. Assuming the validity of the s-cobordism theorem, this equivalence relation becomes less esoteric: $[M, h] = [M', h']$ if and only if there exists a homeomorphism $\phi : M \to M'$ such that h is homotopic to $h' \circ \phi$. Write $\mathrm{hAut}(X)$ for the group of homotopy classes of simple homotopy equivalences $X \to X$. This has a canonical left-action on $\mathcal{S}_{\mathrm{TOP}}(X)$ given by post-composition. Therefore, the manifold set is the quotient of the structure set:

$$\mathcal{M}_{\mathrm{TOP}}(X) = \mathrm{hAut}(X) \backslash \mathcal{S}_{\mathrm{TOP}}(X).$$

Let X be a closed connected topological manifold of dimension n. Let Y be a connected codimension-one submanifold of X that is *separating:* $X - Y$ is disconnected. Assume Y is *incompressible in* X*:* $\pi_1 Y \to \pi_1 X$ is injective. So $X = X_1 \cup_Y X_2$ and there is an injective amalgamated product of fundamental groups: $G = G_1 *_H G_2$. A homotopy equivalence $h : M \to X$ from a closed manifold M is *split along* Y if h is transverse to Y and the restriction $h : h^{-1}Y \to Y$ is also a homotopy equivalence. Below we make use of the orientation character, a homomorphism $w : G \to \{\pm 1\}$.

The *simple split structure set* $\mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}(X; Y)$ consists of the split homotopy classes[2] of simple split homotopy equivalences $h : M \to X$. Note the forgetful map

$$\mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}(X; Y) \longrightarrow \mathcal{S}_{\mathrm{TOP}}(X).$$

Sylvain Cappell [5] defined the surgery obstruction to splitting a simple structure:

$$\mathcal{S}_{\mathrm{TOP}}(X) \longrightarrow \mathrm{UNil}^s_{n+1}(\mathbb{Z}[H]; \mathbb{Z}[G_1 - H]^{w_1}, \mathbb{Z}[G_2 - H]^{w_2}).$$

The target of his function is an algebraically defined abelian group, out of which he provides an algebraically defined monomorphism into Wall's group [4]:

$$\mathrm{UNil}^s_{n+1} \longrightarrow L^s_{n+1}(\mathbb{Z}[G]^w).$$

Thus we may consider the composite function

$$\mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}} \times \mathrm{UNil}^s_{n+1} \longrightarrow \mathcal{S}_{\mathrm{TOP}} \times L^s_{n+1} \longrightarrow \mathcal{S}_{\mathrm{TOP}},$$

where the last function is the right-action given by Wall realization ($n \geqslant 5$) [9]. Cappell's nilpotent normal cobordism construction [5] shows that this composite is surjective ($n \geqslant 6$). A relative form of it establishes injectivity, as announced in [2].

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[1] We only consider homotopy equivalences that are *simple:* they have zero Whitehead torsion.

[2] Using the h-cobordism theorem ($n \geqslant 4$), one can show that the connected sum is a bijection:

$$\# : \mathcal{S}_{\mathrm{TOP}}(X_1) \times \mathcal{S}_{\mathrm{TOP}}(X_2) \xrightarrow{\cong} \mathcal{S}^{\mathrm{split}}_{\mathrm{TOP}}(X_1 \# X_2; S^{n-1}).$$

Next, we describe the calculation of $\mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n)$. Stallings showed that the Whitehead group of $\mathbb{Z}/2 * \mathbb{Z}/2$ (and all its subgroups) is trivial [8]. Therefore, from above with $\varepsilon = (-1)^{n+1}$, we obtain a decomposition of the structure set ($n \geqslant 4$):

$$\mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n) \times \mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n) \times \mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \xrightarrow{\cong} \mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n). \tag{1}$$

López de Medrano calculated the structure set of real projective n-space [7]:

$$\mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n) \cong \begin{cases} (\mathbb{Z}/2)^{2k-1} & \text{if } n = 4k \\ (\mathbb{Z}/2)^{2k} & \text{if } n = 4k+1, 4k+2 \\ (\mathbb{Z}/2)^{2k} \oplus \mathbb{Z} & \text{if } n = 4k+3. \end{cases}$$

This is detected by normal invariants in $\mathbb{Z}/2$ along the submanifolds $\mathbb{RP}^i$ and, if $n = 4k+3$, the Browder–Livesay desuspension invariant in $\mathbb{Z}$. The above UNil-groups were computed partially by Cappell [3], later fully by Connolly–Davis [6]:

$$\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon) \cong \begin{cases} (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty & \text{if } n = 4k \\ (\mathbb{Z}/2)^\infty & \text{if } n = 4k+1 \\ 0 & \text{if } n = 4k+2, 4k+3. \end{cases}$$

For $n = 4k+1$, this is given by the Arf invariant for quadratic forms over the function field $\mathbb{F}_2(t)$. For $n = 4k$, this is given by a two-stage obstruction for quadratic linking forms. This completes the calculation of the structure set $\mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n)$.

Finally, we calculate the manifold set $\mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n)$. Cappell showed that the group $\mathrm{hAut}(\mathbb{RP}^n \# \mathbb{RP}^n)$ is generated by three diffeomorphisms $\gamma_1, \gamma_2, \gamma_3$ [3, p. 397]. Subsequently, Brookman–Davis–Khan [1] determined that the action of hAut on the left side of (1) induced by the bijection of (1) has quotient set

$$\mathrm{Sym}^2 \mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n) \times U_n \xrightarrow{\cong} \mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n).$$

Here, $\mathrm{Sym}^2 \mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n)$ denotes the unordered pairs in the manifold set. Also, U_n denotes the quotient set of $\mathrm{UNil}_{n+1}(\mathbb{Z};\mathbb{Z}^\varepsilon,\mathbb{Z}^\varepsilon)$ by the interchange of the two $\mathbb{Z}$-bimodules $\mathbb{Z}^\varepsilon$ (this switch map is induced by γ_1); this was computed in [1]. One easily shows that $\mathrm{hAut}(\mathbb{RP}^n)$ is generated by the 'reflection' β_n in $\mathbb{RP}^{n-1}$ so that $\mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n) = \mathcal{S}_{\mathrm{TOP}}(\mathbb{RP}^n)$ for $n = 4k, 4k+1, 4k+2$ and $\mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^{4k+3}) = (\mathbb{Z}/2)^{2k} \times \mathbb{Z}_{\geqslant 0}$. Thus we've described the manifold set $\mathcal{M}_{\mathrm{TOP}}(\mathbb{RP}^n \# \mathbb{RP}^n)$ ($n \geqslant 4$).

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