



Reduction of UNil for finite groups with normal abelian Sylow 2-subgroup

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ABSTRACT

Let F be a finite group with a Sylow 2-subgroup S that is normal and abelian. Using hyperelementary induction and cartesian squares, we prove that Cappell's unitary nilpotent groups $\text{UNil}_*(\mathbb{Z}[F]; \mathbb{Z}[F], \mathbb{Z}[F])$ have an induced isomorphism to the quotient of $\text{UNil}_*(\mathbb{Z}[S]; \mathbb{Z}[S], \mathbb{Z}[S])$ by the action of the group F/S . In particular, any finite group F of odd order has the same UNil-groups as the trivial group. The broader scope is the study of the L -theory of virtually cyclic groups, based on the Farrell–Jones isomorphism conjecture. We obtain partial information on these UNil when S is a finite abelian 2-group and when S is a special 2-group.

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1. Introduction

Our main theorem reduces the computation of UNil for finite groups with normal abelian Sylow 2-subgroup to the computation of UNil for its Sylow 2-subgroup. Throughout the paper, all multiplicative groups are equipped with trivial orientation character.

Theorem 1. *Suppose F is a finite group with a normal abelian Sylow 2-subgroup S . Then, for all $n \in \mathbb{Z}$, the following induced map from the group of coinvariants is an isomorphism:*

$$\text{incl}_* : \text{UNil}_n^{(-\infty)}(\mathbb{Z}[S]; \mathbb{Z}[S], \mathbb{Z}[S])_{F/S} \rightarrow \text{UNil}_n^{(-\infty)}(\mathbb{Z}[F]; \mathbb{Z}[F], \mathbb{Z}[F]).$$

Corollary 2. *Suppose F is a finite group of odd order. Then the map induced by the inclusion of the trivial subgroup is an isomorphism:*

$$\text{incl}_* : \text{UNil}_n^{(-\infty)}(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \rightarrow \text{UNil}_n^{(-\infty)}(\mathbb{Z}[F]; \mathbb{Z}[F], \mathbb{Z}[F]).$$

These are summands of the surgery groups $L_n^{(-\infty)}(\mathbb{Z}[\Gamma \times D_\infty])$ for $\Gamma = 1, F$. The target UNil has a complete set of invariants obtained first by the retraction $F \rightarrow 1$ to the source, and then from the two-stage obstruction theory of Connolly–Davis [11] (compare [10,3]).

Historically, these types of finite groups have been studied for classical L -theory.

Remark 3. Suppose F is a finite group with a normal abelian Sylow 2-subgroup. Then the classical L -groups $L_*^h(\mathbb{Z}[F])$ are computed by Bak [2] and Wall [35, Cor. 2.4.3]. The computation is extended to the colimit $L_*^{(-\infty)}(\mathbb{Z}[F])$ by Madsen–Rothenberg [22]. Therefore, by considering one such group F at a time, one can determine

$$L_*^{(-\infty)}(\mathbb{Z}[F \times D_\infty]) \quad \text{and} \quad L_*^{(-\infty)}(\mathbb{Z}[F][x]).$$

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1.1. Background

An involution $\bar{}$ on a ring R is an additive endomorphism that reverses products and whose square is the identity. For each $n \geq 0$, the surgery obstruction group $L_n(R)$ is defined as cobordism groups of n -dimensional quadratic Poincaré complexes over $(R, \bar{})$ [27]. Tensor product with the symmetric Poincaré complex over \mathbb{Z} of the complex projective plane $\mathbb{C}\mathbb{P}^2$ induces a periodicity isomorphism $L_n(R) \rightarrow L_{n+4}(R)$, and so the definition is extensible to all $n \in \mathbb{Z}$. These abelian groups fit into the surgery exact sequence [36], which can be used to compute the set $\mathcal{S}^s(X)/h\text{Aut}^s(X)$ of homeomorphism classes of manifolds in the simple homotopy type of a closed manifold X of dimension $n > 4$. Here, the ring $R = \mathbb{Z}[\pi_1(X)]$ has involution given by inversion of group elements.

Let $\mathcal{B}_-, \mathcal{B}_+$ be (R, R) -bimodules with involution $\widehat{}$ satisfying $\widehat{r \cdot b \cdot s} = \widehat{s} \cdot \widehat{b} \cdot \widehat{r}$. The splitting obstruction group $\text{UNil}_{2k}^h(R; \mathcal{B}_-, \mathcal{B}_+)$ is defined as the Witt group of interlocking quadratic forms over \mathcal{B}_\pm whose adjoints compose to a nilpotent endomorphism. Product with the circle S^1 defines the groups $\text{UNil}_{2k-1}^h(R; \mathcal{B}_-, \mathcal{B}_+)$ [7]. These abelian groups satisfy 4-periodicity and are 2-primary. In the case of group rings of an amalgamated free product or HNN-extension G , there is a split monomorphism $\text{UNil}_n^h \rightarrow L_n^h(\mathbb{Z}[G])$ and UNil_n^h has exponent 8 [15]. If the manifold X has a π_1 -injective, two-sided submanifold X_0 , we obtain such a decomposition of fundamental groups. A homotopy equivalence $h : M \rightarrow X$ is **splittable** along X_0 if h is homotopic to a map h' transverse to X_0 and $(h')^{-1}(X_0) \rightarrow X_0$ is a homotopy equivalence. If X_0 has dimension > 4 , Sylvain Cappell’s nilpotent normal cobordism construction [8] provides a bijection sending a homotopy structure $[h] \in \mathcal{S}(X)$ with vanishing Whitehead torsion in Nil_0 to a normally cobordant split solution $[h'] \in \mathcal{S}^{\text{split}}(X; X_0)$ and a splitting obstruction $[\text{split}(h)] \in \text{UNil}_{n+1}^h$ [6]. The element $\text{split}(h)$ vanishes if and only if h is splittable along X_0 .

Now, the source group of Corollary 2 is computed [3, Theorem D] [11, Theorem 1.10].

Theorem 4 (Banagl, Connolly, Davis, Ranicki). *Let $n \in \mathbb{Z}$. Then*

$$\text{UNil}_n^h(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{4} \\ 0 & \text{if } n \equiv 1 \pmod{4} \\ \mathbb{Z}\langle x \rangle / 2 & \text{if } n \equiv 2 \pmod{4} \\ \mathbb{Z}\langle x \rangle / 4 \times \bigoplus_3 \mathbb{Z}\langle x \rangle / 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The next corollary is immediate from the above discussion, Corollary 2, and the semiperiodicity isomorphism $\text{UNil}_n^h(R; R^-, R^-) = \text{UNil}_{n+2}^h(R; R, R)$ [7].

Corollary 5. *Let Z be a closed, connected, oriented n -manifold with finite fundamental group of odd order. For any $n + m > 5$, define the codimension one submanifold*

$$X_0 := Z \times S^{m-1} \text{ of } X := Z \times (\mathbb{R}P^m \# \mathbb{R}P^m).$$

Here, the sphere S^{m-1} defines the connected sum of the real projective spaces.

Suppose m is odd and $n + m \equiv 0, 3 \pmod{4}$, or m is even and $n + m \equiv 1, 2 \pmod{4}$. Then any simple homotopy equivalence $h : M \rightarrow X$ is splittable along X_0 , where M is a closed manifold. Otherwise, there are a countably infinite number of counterexamples within any given cobordism class of M . \square

1.2. Decorations and correspondences

First, we describe a correspondence in algebraic K -theory. Let R be a ring, and let $i \in \mathbb{Z}$. Using the ring map $\text{aug}_0 : R[x] \rightarrow R$, Hyman Bass [4] defined a decomposition

$$\widetilde{K}_i(R[x]) = \widetilde{K}_i(R) \times NK_i(R).$$

Specializing to the case of an amalgamated product $R[D_\infty] = R[C_2] *_R R[C_2]$ of rings, Friedhelm Waldhausen [33] determined a decomposition

$$K_i(R[D_\infty]) = \frac{K_i(R[C_2]) \times K_i(R[C_2])}{K_i(R)} \times \widetilde{\text{Nil}}_{i-1}(R; R, R).$$

These two K -theory Nil-groups agree by the Davis–Khan–Ranicki correspondence.

Theorem 6 ([12]). *Let R be a ring. Then, for all $i \in \mathbb{Z}$, there is a natural isomorphism*

$$\widetilde{\text{Nil}}_i(R; R, R) \longrightarrow NK_{i+1}(R).$$

Next, we discuss the appropriate K -theory decorations in L -theory [28, Section 17].

Definition 7. Let R be a ring with involution. For a given $i \leq 1$, let $S \subseteq \widetilde{K}_i(R)$ and $T \subseteq NK_i(R)$ be $*$ -invariant subgroups. For each $n \in \mathbb{Z}$, define the **intermediate lower, lower, and ultimate nilpotent L -groups** by

$$\begin{aligned} NL_n^T(R) &:= \text{Ker}(\text{aug}_0 : L_n^{S \times T}(R[x]) \longrightarrow L_n^S(R)) \\ NL_n^{(i)}(R) &:= NL_n^{NK_i(R)}(R) \\ NL_n^{(-\infty)}(R) &:= \text{Ker}(\text{aug}_0 : L_n^{(-\infty)}(R[x]) \longrightarrow L_n^{(-\infty)}(R)). \end{aligned}$$

For typographical convenience, we shall abbreviate $NL_n(R) := NL_n^{(-\infty)}(R)$.

Definition 8. Define the **simple, free, and projective nilpotent L -groups** by

$$\begin{aligned} NL_n^s(R) &:= NL_n^{0 \subseteq NK_1(R)}(R) \\ NL_n^h(R) &:= NL_n^{(1)}(R) = NL_n^{0 \subseteq NK_0(R)}(R) \\ NL_n^p(R) &:= NL_n^{(0)}(R). \end{aligned}$$

There are natural decompositions

$$\begin{aligned} L_n^{S \times T}(R[x]) &= L_n^S(R) \oplus NL_n^T(R) \\ L_n^{(i)}(R[x]) &= L_n^{(i)}(R) \oplus NL_n^{(i)}(R) \\ L_n^{(-\infty)}(R[x]) &= L_n^{(-\infty)}(R) \oplus NL_n^{(-\infty)}(R). \end{aligned}$$

Using pairs of finitely generated free R -modules with additional unitary structure, Sylvain Cappell [7] defined the UNil-groups

$$\text{UNil}_n^h(R; R, R) = \text{UNil}_n^{\widetilde{\text{Nil}}_0(R; R, R)}(R; R, R)$$

and showed that they fit into a decomposition

$$L_n^h(R[D_\infty]) = \frac{L_n^h(R[C_2]) \times L_n^h(R[C_2])}{L_n^h(R)} \times \text{UNil}_n^h(R; R, R).$$

Following Cappell, we use a Shaneson-type sequence to define lower decorations.

Definition 9. Write $\text{UNil}_n^{(1)}(R; R, R) := \text{UNil}_n^h(R, R, R)$. For every $i \leq 0$, define

$$\text{UNil}_n^{(i)}(R; R, R) := \frac{\text{UNil}_n^{(i+1)}(R[C_\infty]; R[C_\infty], R[C_\infty])}{\text{UNil}_n^{(i+1)}(R; R, R)}.$$

There is a forget-decoration map which fits into a Rothenberg-type sequence

$$\dots \longrightarrow \text{UNil}_n^{(i+1)} \longrightarrow \text{UNil}_n^{(i)} \longrightarrow \widehat{H}^n(C_2; \widetilde{\text{Nil}}_{i-1}) \xrightarrow{\partial} \text{UNil}_{n-1}^{(i+1)} \longrightarrow \dots$$

We define the **projective and ultimate unitary nilpotent L -groups** by

$$\begin{aligned} \text{UNil}_n^p(R; R, R) &:= \text{UNil}_n^{(0)}(R; R, R) \\ \text{UNil}_n^{(-\infty)}(R; R, R) &:= \text{colim}_{i \leq 1} \text{UNil}_n^{(i)}(R; R, R). \end{aligned}$$

The earlier analogue to [12] in L -theory was the Connolly–Ranicki correspondence.

Theorem 10 ([10]). Let R be a ring with involution. Then, for all $n \in \mathbb{Z}$, there is a natural isomorphism

$$r^h : \text{UNil}_n^h(R; R, R) \longrightarrow NL_n^h(R)$$

which descends to natural isomorphisms

$$\begin{aligned} r^p : \text{UNil}_n^p(R; R, R) &\longrightarrow NL_n^p(R) \\ r^{(-\infty)} : \text{UNil}_n^{(-\infty)}(R; R, R) &\longrightarrow NL_n^{(-\infty)}(R). \end{aligned}$$

1.3. Tools

For the benefit of the reader, we list the technical tools for this paper.

- (i) The abelian groups $\text{UNil}_*(R; R, R)$ vanish if 2 is a unit in R (Theorem 12).
- (ii) Hyperelementary induction for the functor $NL_n(\mathbb{Z}[-])$ on finite groups is concentrated at the prime $p = 2$ (Theorem 20).
- (iii) A cartesian square (a pullback-pushout diagram) of rings with involution induces a Mayer–Vietoris exact sequence of NL -groups (Theorem 17).
- (iv) Chain bundles are used implicitly (Proposition 25) to prove that $NL_*(\mathbb{Z}[\zeta_d]) = 0$ for all $d > 1$ odd. Again, the emphasis turns out to be at the prime 2, and we obtain that $NL_*(\mathbb{Z}[\zeta_e])$ is a basic building block for all $e \geq 1$ a power of 2.

1.4. Motivation

The Farrell–Jones isomorphism conjecture in L -theory [16] states for any discrete group Γ that $L_*(\mathbb{Z}[\Gamma])$ is determined by $L_*(\mathbb{Z}[V])$ of all virtually cyclic subgroups V of Γ together with certain homological information. That is, the blocked assembly map is conjectured to be an isomorphism:

$$H_n(B_{vc}(\Gamma); \mathbb{L}_\bullet^{(-\infty)}(\mathbb{Z}[-])) \longrightarrow L_n^{(-\infty)}(\mathbb{Z}[\Gamma]).$$

For computation of the source, there are spectral sequences (Atiyah–Hirzebruch [26] and p -chain Davis–Lück [13]) which converge to the cosheaf \mathbb{L} -homology of the classifying space $B_{vc}(\Gamma)$ for Γ -actions with virtually cyclic isotropy. A group V is **virtually cyclic** if it contains a cyclic subgroup of finite index. Equivalently:

- (I) V is a finite group, or
- (II) V is a group extension $1 \rightarrow F \rightarrow V \rightarrow C_\infty \rightarrow 1$ for some finite group F , or
- (III) V is a group extension $1 \rightarrow F \rightarrow V \rightarrow D_\infty \rightarrow 1$ for some finite group F .

The L -theory of type I, with various decorations, is determined classically by Wall and others [35,22]. The L -theory of type II is determined by a combination of the L -theory of type I and the monodromy map $(1 - \alpha_*)$ in the Cappell–Ranicki–Shaneson–Wall sequence [36, Section 12B] [30]. The groups V of type III admit a decomposition

$$V = V_- *_F V_+$$

as an injective amalgam with

$$[V_\pm : F] = 2.$$

Sylvain Cappell developed a Mayer–Vietoris type sequence [6] that determines the groups $L_*^h(\mathbb{Z}[V])$ as an extension of the L -groups of the type I groups F, V_-, V_+ and of his splitting obstruction groups

$$\text{UNil}_*^h(\mathbb{Z}[F]; \mathbb{Z}[V_- \setminus F], \mathbb{Z}[V_+ \setminus F]).$$

The recent computations [9–11,3] of these UNil -groups for $F = 1$ provide a starting point for our determination of the L -groups $L_*(\mathbb{Z}[V])$ for certain classes of type III virtually cyclic groups V .

1.5. Outline of proof

The main Theorem 1 and its Corollary 2 are proven at the end of Section 5. The method is to apply 2-hyperelementary induction (Section 3) and then to use Mayer–Vietoris sequences for cartesian squares (Section 5). Odd order information is removed by vanishing theorems (Section 2) and by homological analysis of cyclotomic number rings (Section 4).

2. Localization, completion, and excision for NL

In this section, we document a known vanishing result for $\text{UNil} = NL$ (Theorem 12) and apply it to show that localization and completion properties are concentrated at the prime 2 (Theorem 16). Also, we state the Mayer–Vietoris sequence for a well-known cartesian square that allows us to pass from the group ring of a group G to the quotient G/K by a finite subgroup K (Theorem 17).

Definition 11. Let $j \in \mathbb{Z}$ and write $\epsilon := (-1)^j$. Define the **Tate cohomology group**

$$\widehat{H}^j(A) := \widehat{H}^j(C_2; A) = \frac{\{a \in A \mid a = \epsilon \bar{a}\}}{\{b + \epsilon \bar{b} \mid b \in A\}},$$

where the group C_2 acts on the ring A by its involution.

We start with an elementary vanishing theorem (cf. Karoubi [23, Theorem 7]). For other kinds of rings with involution, it turns out that the vanishing of Tate cohomology implies the vanishing of the nilpotent L -group (Proposition 25).

Theorem 12. Suppose 2 is a unit in a ring A with involution. Let $j, n \in \mathbb{Z}$. Then

$$\widehat{H}^j(A) = 0 \quad \text{and} \quad NL_n(A) = 0.$$

The proof requires two algebraic lemmas.

Lemma 13. For any $r \in \mathbb{N}$, the rational number $\binom{-1/2}{r}$ lies in $\mathbb{Z}[\frac{1}{2}]$.

Proof. This follows immediately from the observation

$$\binom{-1/2}{r} = \binom{2r}{r} \left(-\frac{1}{4}\right)^r,$$

proven inductively, and from the fact that $\binom{2r}{r} \in \mathbb{Z}$. \square

Lemma 14. Suppose ρ is a nilpotent element of a ring S that contains 2 as a unit. Then there exists a unit V of S commuting with ρ such that $V^2 = (1 + \rho)^{-1}$.

Proof. Observe the nilpotence of $\rho \in S$ implies that $1 + \rho \in S^\times$, with inverse

$$(1 + \rho)^{-1} = \sum_{r=0}^{\infty} (-1)^r \rho^r.$$

By Lemma 13 and the nilpotence of ρ , we can define an element $V \in S$ that commutes with ρ :

$$V := \sum_{r=0}^{\infty} \binom{-1/2}{r} \rho^r.$$

Then the binomial theorem implies

$$V^2 = (1 + \rho)^{-1}.$$

In particular, we obtain $V \in S^\times$. \square

Proof of Theorem 12(1). Write $\epsilon := (-1)^j$. Suppose $a \in A$ is ϵ -symmetric. Define

$$b := \frac{1}{2}a \in A.$$

Then note $a = \epsilon \bar{a}$ implies

$$a = \frac{1}{2}(a + a) = \frac{1}{2}(a + \epsilon \bar{a}) = b + \epsilon \bar{b}.$$

Therefore every ϵ -symmetric element of A is also ϵ -even. Hence $\widehat{H}^j(A) = 0$. \square

Proof of Theorem 12(2). First consider the case $n = 2k$ and write $\epsilon := (-1)^k$. Recall that Higman linearization [31, Lemma 4.2] involves stabilization by hyperbolic planes and zero-torsion isomorphisms. Then we can represent

$$\vartheta = [P[x], f_0 + xf_1] \in NL_{2k}^h(A)$$

by a “linear” nonsingular ϵ -quadratic form over $A[x]$ with null-augmentation to A . Here, P is a finitely generated free left R -module, and $f_0, f_1 : P \rightarrow P^* = \text{Hom}_R(P, R)^t$ are left R -module morphisms. Recall, since 2 is a unit in $A[x]$, that the ϵ -symmetrization map is an isomorphism [27, Proposition 1.4.3]:

$$L_{2k}^h(A[x]) \xrightarrow{1+T_\epsilon} L_h^{2k}(A[x]).$$

In fact, the quadratic refinement of a symmetric form is recovered, uniquely up to skew $(-\epsilon)$ -even morphisms, as one-half of the ϵ -symmetric morphism. So it is equivalent to show the vanishing of the ϵ -symmetric Witt class

$$(1 + T_\epsilon)(\vartheta) = [P[x], \lambda_0 + x\lambda_1].$$

Here, for each $i = 0, 1$, the ϵ -symmetrizations are defined as

$$\lambda_i := f_i + \epsilon f_i^* : P[x] \rightarrow P[x].$$

After stabilization if necessary, there exists a lagrangian P_0 of the ϵ -symmetric form (P, λ_0) over A , since

$$\text{eval}_0(\vartheta) = 0 \in L_{2k}^S(A).$$

Since $\lambda_0 + x\lambda_1$ is invertible, we obtain a nilpotent element of the ring $\text{End}_{A[x]}(P[x])$:

$$x\nu := \lambda_0^{-1} \circ x\lambda_1.$$

Then by Lemma 14, there exists

$$V \in \text{End}_{A[x]}(P[x])^\times$$

commuting with $x\nu$ such that $V^2 = (1 + x\nu)^{-1}$. Observe that $x\nu \in \text{End}_{A[x]}(P[x])$ is a self-adjoint operator with respect to the nonsingular form $(P[x], \lambda_0)$:

$$(x\nu)^* \circ \lambda_0 = (\lambda_0^{-1} \circ x\lambda_1)^* \circ \lambda_0 = x\lambda_1 \circ \lambda_0^{-1} \circ \lambda_0 = x\lambda_1 = \lambda_0 \circ (x\nu).$$

Hence the automorphism V defined in Proof 14 is also self-adjoint:

$$V^* \circ \lambda_0 = \lambda_0 \circ V.$$

Then note the pullback is $V^*(P[x], \lambda_0 + x\lambda_1) = (P[x], \lambda_0)$, since

$$V^* \circ (\lambda_0 + x\lambda_1) \circ V = V^* \circ \lambda_0 \circ (1 + x\nu) \circ V = \lambda_0 \circ V \circ V \circ (1 + x\nu) = \lambda_0.$$

Hence the form $(P[x], \lambda_0 + x\lambda_1)$ is homotopy equivalent to the symplectic form $(P[x], \lambda_0)$. That is, $V(P_0[x])$ is a lagrangian for the ϵ -symmetric form $(P[x], \lambda_0 + x\lambda_1)$. Therefore

$$\vartheta = (1 + T_\epsilon)^{-1}[P[x], \lambda_0 + x\lambda_1] = 0 \in NL_{2k}^h(A).$$

Thus we have shown $NL_{2k}^h(A) = 0$. By the fundamental theorem of algebraic L -theory [31], we inductively obtain that $NL_{2k}^{(i)}(A) = 0$ for all $i \leq 1$. Hence $NL_{2k}^{(-\infty)}(A) = 0$.

Finally, the case $n = 2k - 1$ follows from the Ranicki–Shaneson sequence [29, Theorem 1.1] for ultimate decorations:

$$0 \rightarrow NL_{2k}^{(-\infty)}(A) \rightarrow NL_{2k}^{(-\infty)}(A[C_\infty]) \rightarrow NL_{2k-1}^{(-\infty)}(A) \rightarrow 0. \quad \square$$

Next, we show that there are excision sequences for certain cartesian squares.

Lemma 15. *Suppose Φ is a cartesian square of rings with involution:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

such that Φ is a localization-completion square or that one of the maps to D is surjective. Then, for all $n \in \mathbb{Z}$, there are Mayer–Vietoris exact sequences:

(i) of 2-periodic Tate cohomology groups

$$\dots \rightarrow \widehat{H}^{n+1}(D) \xrightarrow{\partial} \widehat{H}^n(A) \rightarrow \widehat{H}^n(B) \oplus \widehat{H}^n(C) \rightarrow \widehat{H}^n(D) \rightarrow \dots$$

(ii) and of 4-periodic ultimate NL-groups

$$\dots \rightarrow NL_{n+1}(D) \xrightarrow{\partial} NL_n(A) \rightarrow NL_n(B) \oplus NL_n(C) \rightarrow NL_n(D) \rightarrow \dots$$

Proof. Given a finite group Γ , consider a short exact sequence of $\mathbb{Z}[\Gamma]$ -modules:

$$0 \rightarrow M_0 \rightarrow M_- \oplus M_+ \rightarrow M \rightarrow 0.$$

Let C be a contractible complex of f.g. free Γ -modules such that

$$\text{Cok}(\partial : C_1 \rightarrow C_0) = \mathbb{Z},$$

which is the trivial $\mathbb{Z}[\Gamma]$ -module. Recall, for any coefficient $\mathbb{Z}[\Gamma]$ -module N , the definition of Tate cohomology:

$$\widehat{H}^j(\Gamma; N) = H^j(\text{Hom}_{\mathbb{Z}[\Gamma]}(C, N)).$$

Then we obtain the Bockstein sequence:

$$\dots \xrightarrow{\partial} \widehat{H}^j(\Gamma; M_0) \rightarrow \widehat{H}^j(\Gamma; M_-) \oplus \widehat{H}^j(\Gamma; M_+) \rightarrow \widehat{H}^j(\Gamma; M) \xrightarrow{\partial} \dots$$

Therefore Part (1) is proven by substitution: $(\Gamma, M_0, M_-, M_+, M) = (C_2, A, B, C, D)$.

In order to prove Part (2), let $i \leq 0$. Consider the cartesian square κ of $*$ -invariant subgroups [27, p. 498]:

$$\kappa_i := \begin{pmatrix} I_i & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{pmatrix} \subseteq NK_i(\Phi)$$

where

$$I_i := \text{Ker}(NK_i(A) \longrightarrow NK_i(B) \oplus NK_i(C)).$$

Recall, for any ring R , that the fundamental theorem of algebraic K -theory [4, XII] states

$$K_i(R[x, x^{-1}]) = K_i(R) \times NK_i(R) \times NK_i(R) \times K_{i-1}(R).$$

Consider the cartesian squares $\Phi[x]$, $\Phi[x^{-1}]$, $\Phi[x, x^{-1}]$ of the polynomial extensions and the decoration squares $\lambda_i, \lambda_i, \mu_i$ defined by

$$\lambda_i := \begin{pmatrix} J_i & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{pmatrix} \subseteq N NK_i(\Phi) \subseteq NK_i(\Phi[x]) \cong NK_i(\Phi[x^{-1}])$$

$$\mu_i := \kappa_i \times \lambda_i \times \lambda_i \times \kappa_{i-1} \subseteq NK_i(\Phi[x, x^{-1}])$$

where

$$J_i := \text{Ker}(N NK_i(A) \longrightarrow N NK_i(B) \oplus N NK_i(C)).$$

Then, by the fundamental theorem of algebraic L -theory ($\bar{x} = x$) [31, Corollary 4.4], we obtain a decomposition of decorated triad NL -groups:

$$NL_*^{\mu_i}(\Phi[x, x^{-1}]) = NL_*^{\kappa_i}(\Phi) \times NNL_*^{\lambda_i}(\Phi) \times NNL_*^{\lambda_i}(\Phi) \times NL_*^{\kappa_{i-1}}(\Phi).$$

More generally, write $X_i := \{x_1, \dots, x_{1-i}\}$. Thus, we inductively obtain that $NL_*^{\kappa_{i-1}}(\Phi)$ is a summand of $NL_*^{\omega_0}(\Phi[X_i, X_i^{-1}])$ for a certain cartesian square $\omega_0 \subseteq NK_0(\Phi[X_i, X_i^{-1}])$ of $*$ -invariant subgroups. Observe that $\Phi[X_i, X_i^{-1}]$ is a cartesian square of rings with involution satisfying the same stated condition as Φ . Then, by Ranicki's excision theorem for intermediate projective decorations [27, Proposition 6.3.1], we obtain $NL_*^{\kappa_0}(\Phi) = 0$ and $NL_*^{\omega_0}(\Phi[X_i, X_i^{-1}]) = 0$. Hence $NL_*^{\kappa_i}(\Phi) = 0$ for all $i \leq 0$. In other words, we obtain a Mayer-Vietoris exact sequence

$$\dots \longrightarrow NL_{n+1}^{(i+1)}(D) \xrightarrow{\partial} NL_n^{(i+1)}(A) \longrightarrow NL_n^{(i+1)}(B) \oplus NL_n^{(i+1)}(C) \longrightarrow NL_n^{(i+1)}(D) \longrightarrow \dots$$

Now, observe the intermediate and full decorations intertwine by forget-decoration maps:

$$NL_n^{I_0}(A) \longrightarrow NL_n^{NK_0(A)}(A) \longrightarrow NL_n^{I_{-1}}(A) \longrightarrow NL_n^{NK_{-1}(A)}(A) \longrightarrow \dots$$

This cofinality implies that the induced map between the direct limits is an isomorphism:

$$\text{colim}_{i \leq 0} NL_n^{I_i}(A) = \text{colim}_{i \leq 0} NL_n^{NK_i(A)}(A) = NL_n^{(-\infty)}(A).$$

Therefore we obtain the desired exact sequence of $NL_*^{(-\infty)}$ -groups as a direct limit. \square

The following basic isomorphism shall be useful for ring decompositions of group rings.

Theorem 16. *Let A be a ring with involution, and let $n \in \mathbb{Z}$.*

(i) *For all N odd, the following induced maps are isomorphisms:*

$$\widehat{H}^n(A) \longrightarrow \widehat{H}^n\left(A \begin{bmatrix} 1 \\ N \end{bmatrix}\right) \quad \text{and} \quad NL_n(A) \longrightarrow NL_n\left(A \begin{bmatrix} 1 \\ N \end{bmatrix}\right).$$

(ii) *The following induced maps are isomorphisms:*

$$\widehat{H}^n(A) \longrightarrow \widehat{H}^n(\widehat{A}_{(2)}) \quad \text{and} \quad NL_n(A) \longrightarrow NL_n(\widehat{A}_{(2)}).$$

Proof. Let $r > 0$. Consider the localization-completion cartesian square Φ of rings with involution [27, pp. 197–8]:

$$\Phi := \left(\begin{array}{ccc} A & \longrightarrow & \widehat{A}_{(r)} \\ \downarrow & & \downarrow \\ A \left[\frac{1}{r} \right] & \longrightarrow & \widehat{A}_{(r)} \left[\frac{1}{r} \right] \end{array} \right).$$

Note that 2 is a unit in the localization $A \left[\frac{1}{2} \right]$ and in the completion $\widehat{A}_{(N)}$. Then, by Theorem 12, we obtain:

$$\begin{aligned} \widehat{H}^* \left(A \left[\frac{1}{2} \right] \right) &= \widehat{H}^* \left(\widehat{A}_{(2)} \left[\frac{1}{2} \right] \right) = 0 = \widehat{H}^* (\widehat{A}_{(N)}) = \widehat{H}^* \left(\widehat{A}_{(N)} \left[\frac{1}{2} \right] \right) \\ NL_* \left(A \left[\frac{1}{2} \right] \right) &= NL_* \left(\widehat{A}_{(2)} \left[\frac{1}{2} \right] \right) = 0 = NL_* (\widehat{A}_{(N)}) = NL_* \left(\widehat{A}_{(N)} \left[\frac{1}{2} \right] \right). \end{aligned}$$

Therefore we are done by the Mayer–Vietoris exact sequences of Lemma 15. \square

Now we establish useful Mayer–Vietoris sequences for certain normal subgroups.

Theorem 17. Let R be a ring with involution of characteristic zero. Let K be a finite normal subgroup of any group G . Its **norm** is defined as

$$\Sigma_K := \sum_{g \in K} g \in R[G].$$

(i) There is a cartesian square of rings with involution:

$$\begin{array}{ccc} R[G] & \longrightarrow & R[G]/\Sigma_K \\ \downarrow & & \downarrow \\ R[G/K] & \longrightarrow & \frac{R}{|K|}[G/K]. \end{array}$$

(ii) There is a Mayer–Vietoris exact sequence of 2-periodic Tate cohomology groups:

$$\begin{aligned} \cdots \rightarrow \widehat{H}^n(R[G]) &\rightarrow \widehat{H}^n(R[G/K]) \oplus \widehat{H}^n(R[G]/\Sigma_K) \\ &\rightarrow \widehat{H}^n \left(\frac{R}{|K|}[G/K] \right) \xrightarrow{\partial} \widehat{H}^{n-1}(R[G]) \rightarrow \cdots \end{aligned}$$

(iii) There is a Mayer–Vietoris exact sequence of 4-periodic ultimate NL-groups:

$$\begin{aligned} \cdots \rightarrow NL_n(R[G]) &\rightarrow NL_n(R[G/K]) \oplus NL_n(R[G]/\Sigma_K) \\ &\rightarrow NL_n \left(\frac{R}{|K|}[G/K] \right) \xrightarrow{\partial} NL_{n-1}(R[G]) \rightarrow \cdots \end{aligned}$$

Proof. It is straightforward to check that the diagram of Part (1) is commutative and a pushout-pullback square (that is, a cartesian square). Therefore we are done by the Mayer–Vietoris exact sequences of Lemma 15. \square

Finally, there is a useful localization result for integral group rings.

Proposition 18. Let G be a group. Then, for all $n \in \mathbb{Z}$, the prime 2 localization map is an isomorphism:

$$NL_n(\mathbb{Z}[G]) \longrightarrow \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} NL_n(\mathbb{Z}[G]); \quad m \longmapsto 1 \otimes m.$$

Proof. By Farrell’s exponent theorem [15, Theorem 1.3], the Connolly–Ranicki isomorphism [10, Theorem A], and the Ranicki–Shaneson sequence [29, Theorem 1.1], the abelian group $NL_n^h(\mathbb{Z}[G]) = \text{UNil}_n^h(\mathbb{Z}[G]; \mathbb{Z}[G], \mathbb{Z}[G])$ has exponent 8. Then, by the Ranicki–Rothenberg sequence [29, Theorem 4.1], we conclude the colimit $NL_n^{(-\infty)}(\mathbb{Z}[G])$ is 2-primary. The result now follows. \square

3. Hyperelementary induction for NL

In this section, we show that only 2-hyperelementary subgroups of a finite group F are required for induction (Theorem 20). Also, we decompose NL for a 2-hyperelementary group H into components involving cyclotomic number rings $\mathbb{Z}[\zeta_d]$ (Theorem 21).

For any prime p , recall that a finite group H is p -hyperelementary if it is a group extension

$$1 \rightarrow C_N \rightarrow H \rightarrow P \rightarrow 1$$

with P a finite p -group and N coprime to p ; the extension is necessarily split.

For every group G and prime p , define a category $\mathcal{H}_p(G)$ as follows. Its objects H are all the p -hyperelementary subgroups of G , and its morphisms are defined as conjugate-inclusions:

$$\varphi_{H,g,H'} : H \rightarrow H'; \quad x \mapsto gxg^{-1},$$

for all possible p -hyperelementary subgroups H, H' and elements g of G . It is a subcategory of the category FINITE GROUPS of finite groups and monomorphisms. For any normal subgroup S of G , the category $\mathcal{H}_p(G)$ contains a full subcategory $\mathcal{H}_p(G) \cap S$ whose objects are all the p -hyperelementary subgroups of S . The inclusion map also induces a functor

$$\text{incl}_* : \mathcal{H}_p(S) \rightarrow \mathcal{H}_p(G) \cap S,$$

which is bijective on object sets and injective on morphism sets.

Definition 19. Fix $n \in \mathbb{Z}$. Define a functor \mathcal{N} by

$$\begin{aligned} \mathcal{N} : \text{FINITE GROUPS} &\rightarrow \text{ABELIAN GROUPS} \\ \mathcal{N}(F) &:= NL_n^{(-\infty)}(\mathbb{Z}[F]) \\ \mathcal{N}\left(F \xrightarrow{\varphi} G\right) &:= \left(\mathcal{N}(F) \xrightarrow{\varphi_*} \mathcal{N}(\varphi F) \xrightarrow{\text{incl}_*} \mathcal{N}(G)\right). \end{aligned}$$

Let F be a finite group and S a normal subgroup. We shall reduce the computation of $\mathcal{N}(F)$ to colimits and coinvariants over S , as follows. The group F has a covariant action on the category $\mathcal{H}_2(S)$ defined by pushforward via left conjugation:

$$\text{conj} : F \rightarrow \text{Aut}(\mathcal{H}_2(S)).$$

Recall that \mathcal{N} transforms inner automorphisms into the identity, by Taylor’s Lemma [32, Cor. 1.1], since we are assuming that all our groups are equipped with trivial orientation character. Then the quotient group F/S has an induced action on the colimit. The **group of coinvariants** is defined by

$$\left(\text{colim}_{\mathcal{H}_2(S)} \mathcal{N}\right)_{F/S} := \left(\text{colim}_{\mathcal{H}_2(S)} \mathcal{N}\right) / \langle x_H - \text{conj}_*^g(x_H) \mid x_H \in \mathcal{N}(H) \text{ and } gS \in F/S \rangle.$$

Recall that the direct limit of a functor is defined in this case by

$$\text{colim}_{\mathcal{H}_2(S)} \mathcal{N} := \left(\prod_{H \in \text{Ob } \mathcal{H}_2(S)} \mathcal{N}(H)\right) / \langle x_H - \mathcal{N}_*(\varphi)(x_H) \mid \varphi \in \text{Mor } \mathcal{H}_2(S) \rangle,$$

and similarly over the finite category $\mathcal{H}_2(F) \cap S$.

Here is the main theorem of this section.

Theorem 20. Let F be a finite group and S a normal subgroup.

(i) The following induced map from the direct limit is an isomorphism:

$$\text{incl}_* : \text{colim}_{\mathcal{H}_2(F)} \mathcal{N} \rightarrow \mathcal{N}(F).$$

(ii) The following map, from the group of coinvariants, is an induced isomorphism:

$$\text{incl}_* : \left(\text{colim}_{\mathcal{H}_2(S)} \mathcal{N}\right)_{F/S} \rightarrow \text{colim}_{\mathcal{H}_2(F) \cap S} \mathcal{N}.$$

Proof of Theorem 20(1). Observe that \mathcal{N} transforms inner automorphisms into the identity, by Taylor’s Lemma [32, Cor. 1.1], since we are assuming that all our groups are equipped with trivial orientation character. For the commutative rings $R = \mathbb{Z}, \mathbb{Z}[x]$, C.T.C. Wall shows that a certain functor $U_F(R)$ from the Burnside category $\Omega(F)$ to commutative rings is a Green functor and that, for every $n \in \mathbb{Z}$, the localization $L_n^h(R[-])$ is a module over $U_F(R)$ [35, Theorem 2.1.2, Proposition 2.1.3]. Since the module structure is natural in the commutative ring R , it follows that $L_n^{(-\infty)}(R[-])$ is a module over $U_F(R)$. By naturality of aug_0 , we conclude that \mathcal{N} is a Mackey functor.

Then Dress Induction [14, Theorem 1] is applicable in its covariant form [24, Theorem 11.1]. So the functor $\mathbb{Z}_{(2)} \otimes \mathcal{N}$ is \mathcal{H}_2 -computable. That is, the following induced map is an isomorphism:

$$\text{incl}_* : \text{colim}_{\mathcal{H}_2(F)} \mathbb{Z}_{(2)} \otimes \mathcal{N} \longrightarrow \mathbb{Z}_{(2)} \otimes \mathcal{N}(F).$$

Therefore we are done by Proposition 18. \square

Proof of Theorem 20(2). For existence and surjectivity of the map, it suffices to show that the following commutative diagram exists:

$$\begin{array}{ccc} \text{colim}_{\mathcal{H}_2(S)} \mathcal{N} & \xrightarrow{\text{incl}_*} & \text{colim}_{\mathcal{H}_2(F) \cap S} \mathcal{N}. \\ \text{quotient} \downarrow & \nearrow \text{incl}_* & \\ \left(\text{colim}_{\mathcal{H}_2(S)} \mathcal{N} \right)_{F/S} & & \end{array}$$

In order to show that the induced map exists, let

$$z := x_H - \text{conj}_*^g(x_H) \in \prod \mathcal{N}(H)$$

represent a generator of the kernel of the quotient map, where $x_H \in \mathcal{N}(H)$ and $g \in F$. But note

$$z = x_H - \mathcal{N}_*(\varphi_{H,g,gHg^{-1}})(x_H).$$

Hence it maps to zero in the direct limit over $\mathcal{H}_2(F) \cap S$. Thus the desired map exists and is surjective.

In order to show that the induced map is injective, suppose $[x_H]$ is an equivalence class in the coinvariants which maps to zero in the direct limit over $\mathcal{H}_2(F) \cap S$. Then there exists an expression

$$(x_H) = \sum_{i=1}^r (w_i - \mathcal{N}_*(\varphi_i)(w_i)) \in \prod_{H \in \text{Ob } \mathcal{H}_2(S)} \mathcal{N}(H)$$

for some

$$H_1, \dots, H_r \in \text{Ob } \mathcal{H}_2(S) \quad \text{and} \quad w_i \in \mathcal{N}(H_i) \quad \text{and} \quad \varphi_i \in \text{Mor } \mathcal{H}_2(F) \cap S.$$

But each monomorphism $\varphi_i = \varphi_{H_i, g_i, H'_i}$ admits a factorization

$$\varphi_i = \varphi'_i \circ \text{conj}_*^{g_i}$$

into an isomorphism $\text{conj}_*^{g_i}$ and an inclusion

$$\varphi'_i := \varphi_{g_i H_i g_i^{-1}, 1, H'_i} \in \text{Mor } \mathcal{H}_2(S).$$

Then note

$$[w_i - \mathcal{N}_*(\varphi_i)(w_i)] = [w_i - \text{conj}_*^{g_i}(w_i)] + [v_i - \mathcal{N}_*(\varphi'_i)(v_i)] = 0 \in \left(\text{colim}_{\mathcal{H}_2(S)} \mathcal{N} \right)_{F/S}$$

where

$$v_i := \text{conj}_*^{g_i}(w_i).$$

Hence $[x_H] = 0$ in the coinvariants. Thus the desired map incl_* is injective. \square

Therefore, we are reduced to the computation of $\mathcal{N}(H)$ for all 2-hyerelementary groups H . The next theorem reduces it further.

Theorem 21. Suppose H is a 2-hyerelementary group:

$$H = C_N \rtimes_{\tau} P$$

where N is odd and P is a finite 2-group. Consider the ring $R := \mathbb{Z}[\frac{1}{N}]$. Then for all $n \in \mathbb{Z}$, there is an induced isomorphism

$$NL_n(\mathbb{Z}[H]) \longrightarrow NL_n \left(\left(\bigoplus_{d|N} R[\zeta_d] \right) \circ_{\tau} P \right),$$

where the action τ' is induced by τ .

Moreover if τ is trivial, then it can be lifted to an induced isomorphism

$$NL_n(\mathbb{Z}[C_N \times P]) \longrightarrow \bigoplus_{d|N} NL_n(\mathbb{Z}[\zeta_d][P]).$$

Suppose A is a ring, P is a group, and $\tau : P \rightarrow \text{Aut}(A)$ is a homomorphism. So P acts on A by ring automorphisms. Then $A \circ_\tau P$ denotes the twisted group ring of P with coefficients from A .

Proof. For each divisor d of N , let ρ_d be the cyclotomic \mathbb{Q} -representation of C_N defined by $\rho_d(T) := \zeta_d$. The ρ_d represent all the distinct isomorphism classes of irreducible \mathbb{Q} -representations of the group C_N . An elementary argument using the Chinese remainder theorem and the trace of right multiplication (cf. [17, Theorem 7.1]) shows that there exists a restricted isomorphism of R -algebras with involution:

$$\rho = \bigoplus_{d|N} \rho_d : R[C_N] \longrightarrow \bigoplus_{d|N} R[\zeta_d].$$

Therefore, using Theorem 16, we obtain a composition of isomorphisms:

$$NL_n(\mathbb{Z}[C_N] \circ_\tau P) \longrightarrow NL_n(R[C_N] \circ_\tau P) \xrightarrow{\rho_*} NL_n\left(\left(\bigoplus_{d|N} R[\zeta_d]\right) \circ_{\tau'} P\right).$$

The assertion for τ trivial follows from Theorem 16 and the additivity of $L_*^{(-\infty)}$ (hence $NL_*^{(-\infty)}$) for finite products of rings with involution [19, Corollary 5.13]. \square

4. Basic reductions

Continuing on, we establish four reductions (orientable, hyper elementary, nilpotent, homological) that we shall use to prove Theorem 1.

4.1. Orientable reduction

Proposition 22. Suppose R is a ring with involution and G is a group with trivial orientation character. Then there is a natural decomposition

$$NL_n(R[G]) = NL_n(R) \oplus \tilde{NL}_n(R[G]),$$

where the **reduced L-group** is defined by

$$\tilde{NL}_n(R[G]) := \text{Ker}(\text{aug}_1 : NL_n(R[G]) \rightarrow NL_n(R[1])).$$

Proof. The covariant morphism on $NL_*(R[-])$ -groups induced by the map $\text{incl} : 1 \rightarrow G$ is a monomorphism split by the morphism of $NL_*(R[-])$ -groups induced by the augmentation $\text{aug}_1 : G \rightarrow 1$ of groups with orientation character. \square

4.2. Hyper elementary reduction

For simplicity, from Section 3, we shall continue the notation for fixed $n \in \mathbb{Z}$:

$$\mathcal{N}(G) := NL_n^{(-\infty)}(\mathbb{Z}[G]).$$

Theorem 23. Let F be a finite group and S a normal subgroup. Suppose for all 2-hyper elementary subgroups H of F that the following induced map is an isomorphism:

$$\text{incl}_* : \mathcal{N}(H \cap S) \longrightarrow \mathcal{N}(H).$$

Then the following induced map, from the group of coinvariants, is an isomorphism:

$$\text{incl}_* : \mathcal{N}(S)_{F/S} \longrightarrow \mathcal{N}(F).$$

Proof. Observe that the following diagram commutes:

$$\begin{array}{ccc}
 \operatorname{colim}_{\mathcal{H}_2(F) \cap S} \mathcal{N} & & \\
 \uparrow \operatorname{incl}_* & \searrow \operatorname{incl}_* & \\
 \left(\operatorname{colim}_{\mathcal{H}_2(S)} \mathcal{N} \right)_{F/S} & \xrightarrow{\operatorname{incl}_*} & \operatorname{colim}_{\mathcal{H}_2(F)} \mathcal{N} \\
 \downarrow \operatorname{incl}_* & & \downarrow \operatorname{incl}_* \\
 \mathcal{N}(S)_{F/S} & \xrightarrow{\operatorname{incl}_*} & \mathcal{N}(F).
 \end{array}$$

The vertical maps are isomorphisms by Hyperelementary Induction 20. It follows from the hypothesis that the diagonal map is an isomorphism. Therefore the bottom map is an isomorphism. \square

4.3. Nilpotent reduction

The following is a specialization of Wall’s theorem for complete semilocal rings [34, Theorem 6], which was applied extensively in [35]. In the classical L -theory of finite groups, Wall’s theorem was applied to the Jacobson radical [34, Section 3] of the 2-adic integral group ring of finite 2-groups [35, Section 5.2].

In our case, a theorem of Amitsur [1, Theorem 1] states that the Jacobson radical of $R[x]$ for any ring R is a two-sided ideal $N[x]$, where N is a nil ideal of R containing the locally nilpotent radical. Recall for left artinian rings R that its locally nilpotent radical, nilradical, and Jacobson radical all coincide. In our applications, we limit ourselves to rings of nonzero characteristic.

Proposition 24. *Let R be a ring with involution.*

(i) *Suppose that I is a nilpotent, involution-invariant, two-sided ideal of R . Then for all $n \in \mathbb{Z}$, the map induced by the quotient map $\pi : R \rightarrow R/I$ is an isomorphism:*

$$\pi_* : NL_n^h(R) \rightarrow NL_n^h(R/I).$$

(ii) *Suppose for some prime p that \mathbb{F} is a finite field of characteristic p and P is a finite p -group. Then for all $n \in \mathbb{Z}$, the following induced map is an isomorphism:*

$$\operatorname{aug}_1 : NL_n^h(\mathbb{F}[P] \otimes_{\mathbb{Z}} R) \rightarrow NL_n^h(\mathbb{F} \otimes_{\mathbb{Z}} R).$$

Proof. For Part (1), observe that $I[x]$ is nilpotent implies that the map

$$R[x] \rightarrow \widehat{R[x]}_{I[x]},$$

to the $I[x]$ -adic completion, is an isomorphism of rings with involution. Then, since $\widetilde{K}_1(R[x]) = (\pi_*)^{-1} \widetilde{K}_1(R[x]/I[x])$, by [34, Theorem 6], we have induced isomorphisms:

$$\begin{aligned}
 \pi_* : L_n^h(R[x]) &\rightarrow L_n^h(R[x]/I[x]) \\
 \pi_* : L_n^h(R) &\rightarrow L_n^h(R/I).
 \end{aligned}$$

Therefore we obtain that $\pi_* : NL_n^h(R) \rightarrow NL_n^h(R/I)$ is an isomorphism.

For Part (2), observe that the involution-invariant, two-sided ideal

$$J := (\{g - 1 \mid g \in P\})$$

of $\mathbb{F}[P]$ is its Jacobson radical. Since $\mathbb{F}[P]$ is finite hence left artinian, we must have that J is nilpotent. So we are done by Part (1) using $I = J \otimes 1_R$. \square

4.4. Homological reduction

The following little observation shall be applied to algebraic number ring \mathcal{O} with a Galois involution in the next section.

Proposition 25. *Let R be a Dedekind domain with involution. Suppose the Tate cohomology groups vanish: $\widehat{H}^*(R) = 0$. Then the nilpotent L -groups vanish: $NL_*^h(R) = 0$.*

Proof. Let $n \in \mathbb{Z}$. By [10, Proposition 20], there is an isomorphism

$$NL_n^h(R) \cong NQ_n^h(R) := \text{Ker}(Q_n(B^{R[x]}, \beta^{R[x]}) \longrightarrow Q_n(B^R, \beta^R)).$$

The pair (B^A, β^A) is the so-called universal chain bundle of Michael S. Weiss [37, 2.A.4]. These are constructed for any ring A with involution, by first selecting free A -modules F_k and A -module epimorphisms $F_k \rightarrow \widehat{H}^k(A)$ for every $k \in \mathbb{Z}$. Here, the left A -module structure is defined by

$$A \times \widehat{H}^k(A) \longrightarrow \widehat{H}^k(A); \quad (\lambda, [a]) \longmapsto [\lambda a \bar{\lambda}].$$

However, we have assumed that $\widehat{H}^*(R) = 0$, which implies $\widehat{H}^*(R[x]) = 0$ by direct calculation. Here, the involution on $R[x]$ is extended from R by $\bar{x} = x$. Then we can take $F_k = 0$ in Weiss’s construction for both rings R and $R[x]$ with involution. So the \mathbb{Z} -graded free module chain complexes vanish: $B^R = 0$ and $B^{R[x]} = 0$. Therefore $NL_n^h(R) = 0$. \square

5. Finite groups with normal abelian Sylow 2-subgroup

The goal of this section is to prove Theorem 1. Observe that any finite group F satisfying the hypothesis is a group extension

$$1 \longrightarrow S \longrightarrow F \longrightarrow E \longrightarrow 1$$

for a unique finite, abelian 2-group S and odd order group E . Since $H^2(E; S) = 0$ by a transfer argument [5, Cor. 3.13], F must be of the form $F = S \rtimes E$.

Many techniques [35, 18] used to compute the quadratic L -theory of finite groups, namely: hyperelementary induction, the Mayer–Vietoris sequence for cartesian squares, nilradical quotients, maximal involuted orders, and Morita equivalence, along with our new technique of homological reduction (Section 4.4), are employed in combination to determine the quadratic NL -theory of certain finite groups, up to extension issues.

The first lemma is a vanishing result for cyclotomic number rings.

Lemma 26. Let $\zeta_r := e^{2\pi\sqrt{-1}/r}$ be a primitive r -th root of unity for some $r > 0$. Write $r = d 2^e$ for some $d > 0$ odd and $e \geq 0$. Note that

$$\mathcal{O} := \mathbb{Z}[\zeta_r] = \mathbb{Z}[\zeta_d, \zeta_{2^e}]$$

as rings whose involution is complex conjugation. Then, for all $n \in \mathbb{Z}$, we have

$$NL_n^h(\mathcal{O}) = \begin{cases} NL_n^h(\mathbb{Z}) & \text{if } d = 1 \text{ and } e = 0, 1 \\ 0 & \text{if } d > 1. \end{cases}$$

Proof. If $d = 1$ and $e = 0, 1$ then $\mathcal{O} = \mathbb{Z}$. So we may assume $d > 1$.

Write $R := \mathbb{Z}[\zeta_{2^e}]$, and consider d as a divisor of some odd $N > 0$. By the ring decomposition ρ of Proof 21 and the isomorphisms of Theorem 16, the following induced upper map is an isomorphism:

$$\begin{array}{ccc} \widehat{H}^j(R[C_N]) & \longrightarrow & \bigoplus_{d|N} \widehat{H}^j(R[\zeta_d]) \\ \downarrow & & \downarrow \\ \widehat{H}^j\left(R\left[\frac{1}{N}\right][C_N]\right) & \xrightarrow{\rho_*} & \bigoplus_{d|N} \widehat{H}^j\left(R\left[\frac{1}{N}\right][\zeta_d]\right). \end{array}$$

But a direct computation shows that the upper map has image in the $d = 1$ factor. That is, the following induced map is an isomorphism:

$$\text{incl}_* : \widehat{H}^j(R[1]) \longrightarrow \widehat{H}^j(R[C_N]).$$

Then, since $d > 1$, the corresponding Tate cohomology groups vanish:

$$\widehat{H}^j(R[\zeta_d]) = 0.$$

Therefore, since $\mathcal{O} = R[\zeta_d]$, by Homological Reduction 25, we obtain $NL_n^h(\mathcal{O}) = 0$. \square

Its analogue in characteristic two is the following lemma.

Lemma 27. Let P be a finite 2-group, and let $d > 0$ be odd. Consider the ring $R = \mathbb{F}_2[P] \otimes \mathbb{Z}[\zeta_d]$ with involution. If $d = 1$, then for all $n \in \mathbb{Z}$, the induced map $NL_n(R) \rightarrow NL_n(\mathbb{F}_2)$ is an isomorphism. Otherwise if $d > 1$, then the groups $NL_n(R)$ vanish.

Proof. By Nilpotent Reduction 24, the following induced map is an isomorphism:

$$NL_n(R) \longrightarrow NL_n(\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d]).$$

Recall, in terms of the d -th cyclotomic polynomial $\Phi_d(x) \in \mathbb{Z}[x]$, that

$$\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d] = \mathbb{F}_2[x]/(\Phi_d(x)).$$

Note, by taking formal derivative of $x^d - 1$ with d odd, that $\Phi_d(x)$ is separable over \mathbb{F}_2 . Then, by the Chinese remainder theorem, the ring $\mathbb{F}_2 \otimes \mathbb{Z}[\zeta_d]$ is a finite product of fields¹ hence is 0-dimensional.

Therefore, by Homological Reduction 25, it suffices to show that its Tate cohomology vanishes. But, as in the previous proof, this follows from the fact that the induced map

$$\text{incl}_* : \widehat{H}^*(\mathbb{F}_2[1]) \longrightarrow \widehat{H}^*(\mathbb{F}_2[C_N])$$

is an isomorphism for all odd N . \square

Remark 28. It seems appropriate to mention here that the techniques of Connolly–Ranicki [10] and of Connolly–Davis [11] can be used to generalize their computations of $\text{UNil}_*(\mathbb{F}_2)$. Namely, let \mathbb{F} be a perfect field of characteristic two with identity involution. Here perfect means that the (Frobenius) squaring endomorphism is surjective. For example, any finite field \mathbb{F}_{2^e} of characteristic two is perfect. Consider the squaring monomorphisms ψ and $\psi[x]$ defined by

$$\psi : \mathbb{F} \longrightarrow \mathbb{F}, \quad \psi[x] : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]; \quad f \longmapsto f^2.$$

Suppose $n = 2k - 1$ is odd. Since \mathbb{F} is perfect, by [10, Proposition 20, Lemma 21, Equation 27], there is an isomorphism

$$NL_{2k-1}(\mathbb{F}, \text{id}) \longrightarrow NQ_{2k}(\mathbb{F}, \text{id}) = \text{Ker}(\text{aug}_0 : \text{Ker}(\psi[x] - 1) \longrightarrow \text{Ker}(\psi - 1)).$$

Note that $f \in \mathbb{F}[x]$ and $f^2 = f$ imply $f \in \mathbb{F}$. Therefore, we obtain $NL_{2k-1}(\mathbb{F}, \text{id}) = 0$.

Otherwise, suppose $n = 2k$ is even. The surjectivity of the Arf invariant below was established earlier by Connolly–Kozniowski [9, Proposition 5.7]. They asked if Arf is injective [9, Open Question (c)]. We answer their question in the affirmative, as follows.

Consider the \mathbb{Z} -module $\mathbb{F}' := \text{Cok}(\psi - 1)$. For example, note $\mathbb{F}'_2 = \mathbb{F}_2$. Then, by generalizing the proof of [11, Lemma 4.6(2), page 1061] and using the calculus of [11, Lemma 4.3], it can be shown that the Arf invariant of symplectic forms over the function field $\mathbb{F}(x)$ defines an isomorphism

$$\text{Arf} : NL_{2k}(\mathbb{F}, \text{id}) \longrightarrow \text{Cok}(\psi[x] - 1)/\mathbb{F}' = \bigoplus_{d \text{ odd}} x^d \mathbb{F}'.$$

The next lemma is a vanishing result for cyclic 2-groups C .

Lemma 29. Let C be a cyclic 2-group, and let $d > 1$ be odd. Consider the ring $R = \mathbb{Z}[\zeta_d]$ whose involution is complex conjugation. Then the groups $NL_*(R[C])$ vanish.

Proof. We induct on the exponent of C . If $e(C) = 1$ then

$$NL_*(R[C]) = NL_*(R) = 0,$$

by Lemma 26(2). Otherwise suppose the lemma is true for all cyclic 2-groups C' with $e(C') < e(C)$. Then we may define a ring extension R' of R and a group quotient C' of C by

$$R' := R[\zeta_{e(C)}] \quad \text{and} \quad C' := C_{e(C)/2},$$

and there is a cartesian square 17 of rings with involution:

$$\begin{array}{ccc} R[C] & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R[C'] & \longrightarrow & \mathbb{F}_2[C'] \otimes R. \end{array}$$

Note, by Lemmas 26(2) and 27, that

$$NL_*(R') = 0 \quad \text{and} \quad NL_*(\mathbb{F}_2[C'] \otimes R) = 0.$$

Therefore, by the Mayer–Vietoris sequence 17, the map induced by the left column is an isomorphism:

$$NL_*(R[C]) \xrightarrow{\cong} NL_*(R[C']).$$

But $NL_*(R[C']) = 0$ by inductive hypothesis. This concludes the argument. \square

¹ It is in fact a product of $\phi(d)/n(d)$ copies of the finite field $\mathbb{F}_{2^{n(d)}}$, where $\phi(d)$ is the Euler ϕ -function and $n(d) > 0$ is minimal with respect to the congruence $2^{n(d)} \equiv 1 \pmod{d}$.

Now we use induction to generalize this vanishing result from cyclic 2-groups C to finite abelian 2-groups P .

Lemma 30. *Let R be a ring with involution, and let P be a finite abelian 2-group. If the groups $NL_*(R[C])$ vanish for all cyclic 2-groups C of exponent $e(C) \leq e(P)$, then the groups $NL_*(R[P])$ vanish.*

Proof. We induct on the order of P . If $|P| = 1$ then

$$NL_*(R[P]) = NL_*(R[1]) = 0,$$

by hypothesis. Otherwise suppose the lemma is true for all R and P'' with $|P''| < |P|$. Since P is a nontrivial abelian 2-group, we can write an internal direct product

$$P = P' \times C_{e(P)}.$$

Then we can define a ring extension R' of R and a group quotient P_0 of P by

$$R' := R[\zeta_{e(P)}] = R[x] / (x^{e(P)/2} + 1) \quad \text{and} \quad P_0 := P' \times C_{e(P)/2}.$$

Consider the cartesian square 17 of rings with involution:

$$\begin{array}{ccc} R[P] & \longrightarrow & R'[P'] \\ \downarrow & & \downarrow \\ R[P_0] & \longrightarrow & \mathbb{F}_2[P_0] \otimes R. \end{array}$$

Note, by Nilpotent Reduction 24 and by hypothesis using both 2-groups C with $e(C) \leq 2$, that

$$NL_*(\mathbb{F}_2[P_0] \otimes R) \xrightarrow{\cong} NL_*(\mathbb{F}_2 \otimes R) = 0.$$

Also $NL_*(R[P_0]) = 0$, by inductive hypothesis. Then, by the Mayer–Vietoris sequence 17, the map induced by the top row is an isomorphism:

$$NL_*(R[P]) \xrightarrow{\cong} NL_*(R'[P']).$$

We are done by induction if we can show that R' and P' also satisfy the hypothesis of the lemma.

Let C be any cyclic 2-group satisfying

$$1 \leq e(C) \leq e(P') \leq e(P).$$

We now induct on $e(C)$. If $e(C) = 1$ then

$$NL_*(R'[C]) = NL_*(R') = 0.$$

The latter equality follows from the Mayer–Vietoris sequence of the cartesian square of ring with involution:

$$\begin{array}{ccc} R[C_{e(P)}] & \longrightarrow & R' \\ \downarrow & & \downarrow \\ R[C_{e(P)/2}] & \longrightarrow & \mathbb{F}_2[C_{e(P)/2}] \otimes R \end{array}$$

and from Nilpotent Reduction, as in the above argument, using the hypothesis of the lemma.

Otherwise suppose $e(C) > 1$. Then we may define a quotient group

$$C' := C_{e(C)/2}$$

of C , and there is a cartesian square of rings with involution:

$$\begin{array}{ccc} R'[C] & \longrightarrow & R'[\zeta_{e(C)}] \\ \downarrow & & \downarrow \\ R'[C'] & \longrightarrow & \mathbb{F}_2[C'] \otimes R'. \end{array}$$

We are again done by Nilpotent Reduction and induction on $e(C)$ if we show that

$$NL_*(R'[\zeta_{e(C)}]) = 0.$$

Consider the primitive root of unity:

$$\omega := (\zeta_{e(P)})^{e(P)/e(C)} \in R'.$$

Observe the quotient and factorization

$$R'[\zeta_{e(C)}] = R[\zeta_{e(P)}][x] / (x^{e(C)/2} + 1) \quad \text{and} \quad x^{e(C)/2} + 1 = \prod_{\text{odd } d=1}^{e(C)-1} (x - \omega^d).$$

Then, by the Chinese remainder theorem, we obtain an isomorphism of rings with involution:

$$R'[\zeta_{e(C)}] \xrightarrow{\cong} \prod_{\text{odd } d=1}^{e(C)-1} R'.$$

Hence it induces an isomorphism

$$NL_*(R'[\zeta_{e(C)}]) \xrightarrow{\cong} \bigoplus_{\text{odd } d=1}^{e(C)-1} NL_*(R').$$

But we have already shown that $NL_*(R') = 0$. This concludes the induction on both $e(C)$ and $|P|$. \square

The last lemma reduces our computation from abelian 2-hyerelementary groups H to abelian 2-groups P .

Lemma 31. Consider any abelian 2-hyerelementary group $H = C_N \times P$. Then for all $n \in \mathbb{Z}$, the following induced map is an isomorphism:

$$\text{incl}_* : NL_n(\mathbb{Z}[P]) \rightarrow NL_n(\mathbb{Z}[H]).$$

Proof. Recall, by Theorem 21 and additivity of L -groups, that the following induced map is an isomorphism:

$$NL_n(\mathbb{Z}[H]) \rightarrow \bigoplus_{d|N} NL_n(\mathbb{Z}[\zeta_d][P]).$$

But all the $d \neq 1$ factors vanish by Lemmas 29 and 30. The result now follows. \square

We are finally in a position to prove the main theorem.

Proof of Theorem 1. Let H be a 2-hyerelementary subgroup of F . Since S is normal abelian, the group H is abelian. Then we can write

$$H = C_N \times P$$

for some odd N , and $P = H \cap S$ a finite abelian 2-group. So, by Lemma 31, the following map is an isomorphism:

$$\text{incl}_* : \mathcal{N}(H \cap S) \rightarrow \mathcal{N}(H).$$

Therefore, by Hyerelementary Reduction 23, the induced map is an isomorphism:

$$\text{incl}_* : \mathcal{N}(S)_{F/S} \rightarrow \mathcal{N}(F). \quad \square$$

Proof of Corollary 2. This is immediate from the theorem, since F has odd order implies $S = 1$, and since F/S acts trivially by inner automorphisms on S . \square

6. On abelian 2-groups

Our main Theorem 1 reduces the computation of $UNil = NL$ for certain finite groups to their maximal abelian 2-subgroup. The following result shows that the NL -theory of abelian 2-groups is determined up to iterated extensions from the Dedekind domains:

$$\mathbb{F}_2 \quad \text{and} \quad \mathbb{Z} = \mathbb{Z}[\zeta_2] \quad \text{and} \quad \mathbb{Z}[i] = \mathbb{Z}[\zeta_4] \quad \text{and} \quad \mathbb{Z}[\zeta_8] \quad \text{and} \quad \mathbb{Z}[\zeta_{16}] \quad \text{and} \quad \dots$$

The $UNil$ -groups of the rings \mathbb{F}_2 and \mathbb{Z} with identity involution and have been calculated [9–11,3]. The dyadic cyclotomic number rings $\mathbb{Z}[\zeta_{2^k}]$ for all $k > 1$ have involution given by complex conjugation, and their $UNil$ -groups shall be calculated in another paper.

Proposition 32. Let P be a nontrivial, finite abelian 2-group, and let $n \in \mathbb{Z}$. Write

$$P = P' \times C_{e(P)} \quad \text{and} \quad P_0 := P' \times C_{e(P)/2}.$$

(i) The Weiss boundary map is an isomorphism:

$$NQ_{n+1}(\mathbb{Z}[P]) \xrightarrow{\partial} NL_n(\mathbb{Z}[P]).$$

(ii) There is an exact sequence

$$\cdots \rightarrow NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[P]) \rightarrow NL_n(\mathbb{Z}[P_0]) \oplus A \rightarrow NL_n(\mathbb{F}_2) \xrightarrow{\partial} \cdots$$

where

$$A := \begin{cases} NL_n(R_0[P']) & \text{if } P' \neq 1 \\ NL_n(\mathbb{Z}[\zeta_{e(P)}]) & \text{if } P' = 1 \text{ and } e(P) > 1 \end{cases} \quad \text{where } R_0 := \mathbb{Z}[\zeta_{e(P)}].$$

(iii) Suppose R is of the form

$$R = \mathbb{Z}[\zeta_e]$$

for some $e \geq e(P)$ a power of 2. There is an exact sequence

$$\cdots \rightarrow NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(R[P]) \rightarrow \bigoplus_2 NL_n(R[P_0]) \rightarrow NL_n(\mathbb{F}_2) \xrightarrow{\partial} \cdots$$

Proof. The above sequences are derived from the Mayer–Vietoris exact sequences 17 of the cartesian squares

$$\begin{array}{ccc} \mathbb{Z}[P] & \longrightarrow & R_0[P'] \\ \downarrow & & \downarrow \\ \mathbb{Z}[P_0] & \longrightarrow & \mathbb{F}_2[P_0] \end{array} \quad \text{and} \quad \begin{array}{ccc} R[P] & \longrightarrow & R[P'][\zeta_{e(P)}] \\ \downarrow & & \downarrow \\ R[P_0] & \longrightarrow & \mathbb{F}_2[P_0]. \end{array}$$

Part (1) follows by induction on $|P|$ and the five-lemma using these exact sequences, along with a similar exact sequence for $\mathbb{F}_2[P]$. Since $\mathbb{Z}[\zeta]$ and \mathbb{F}_2 are Dedekind domains with involution, the basic cases for the induction are indeed isomorphisms:

$$NQ_{n+1}(\mathbb{Z}[\zeta]) \xrightarrow[\cong]{\partial} NL_n(\mathbb{Z}[\zeta]) \quad \text{and} \quad NQ_{n+1}(\mathbb{F}_2) \xrightarrow[\cong]{\partial} NL_n(\mathbb{F}_2).$$

For Parts (2) and (3), recall Nilpotent Reduction 24 shows that the following induced map is an isomorphism:

$$NL_n(\mathbb{F}_2[P_0]) \rightarrow NL_n(\mathbb{F}_2).$$

Finally, for Part (3), since $\zeta_e \in R$, observe that there exists an isomorphism of rings with involution:

$$f : R[\zeta_{e(P)}] \rightarrow R[C_{e(P)}]; \quad \zeta_{e(P)} \mapsto (\zeta_e)^{e/e(P)} T,$$

where T is a generator of the cyclic group $C_{e(P)}$. Therefore we obtain an induced isomorphism

$$f_* : NL_n(R[P'][\zeta_{e(P)}]) \rightarrow NL_n(R[P_0]). \quad \square$$

7. On special 2-groups

The next step beyond the study of cyclic 2-groups (hence abelian 2-groups, involved in Theorem 1) is the study of special 2-groups. A finite group is **special** if every normal abelian subgroup is cyclic.

Proposition 33 ([20, 2.2.1]). *A finite 2-group P is special if and only if it is either:*

(0) for some $e \geq 0$, **cyclic**

$$\mathbf{C}_e := \langle T \mid T^{2^e} = 1 \rangle$$

(1) for some $e > 3$, **dihedral**

$$\mathbf{D}_e := \langle T, R \mid T^{2^{e-1}} = 1 = R^2, RTR^{-1} = T^{-1} \rangle$$

(2) for some $e > 3$, **semidihedral**

$$\mathbf{SD}_e := \langle T, R \mid T^{2^{e-1}} = 1 = R^2, RTR^{-1} = T^{2^{e-2}-1} \rangle$$

(3) for some $e \geq 3$, **quaternionic**

$$\mathbf{Q}_e := \langle T, R \mid T^{2^{e-1}} = 1, R^2 = T^{2^{e-2}}, RTR^{-1} = T^{-1} \rangle.$$

The Mayer–Vietoris exact sequence for cyclic 2-groups \mathbf{C}_e is provided in the previous section; the Mayer–Vietoris exact sequence for the other special 2-groups $P \in \{ \mathbf{D}_e, \mathbf{SD}_e, \mathbf{Q}_e \}$ is provided in the following proposition. The main ingredient is that P has an index two dihedral quotient \mathbf{D}_{e-1} .

Proposition 34. Consider any noncyclic special 2-group P , and let $n \in \mathbb{Z}$. Let $e \geq 3$, and write $\zeta := \zeta_{2^{e-1}}$ a dyadic root of unity. Denote $\circ_{\pm c}$ as twisting a quadratic extension by \pm complex conjugation. Then there are the following long exact sequences.

$$\begin{aligned} \cdots &\rightarrow NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[\mathbf{D}_e]) \\ &\rightarrow NL_n(\mathbb{Z}[\mathbf{D}_{e-1}]) \oplus NL_n(\mathbb{Z}[\zeta] \circ_c C_2) \rightarrow NL_n(\mathbb{F}_2) \rightarrow \cdots \\ \cdots &\rightarrow NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[\mathbf{SD}_e]) \\ &\rightarrow NL_n(\mathbb{Z}[\mathbf{D}_{e-1}]) \oplus NL_n(\mathbb{Z}[\zeta] \circ_{-c} C_2) \rightarrow NL_n(\mathbb{F}_2) \rightarrow \cdots \\ \cdots &\rightarrow NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[\mathbf{Q}_e]) \\ &\rightarrow NL_n(\mathbb{Z}[\mathbf{D}_{e-1}]) \oplus NL_n(\mathbb{Z}[\zeta] \circ_c [i]) \rightarrow NL_n(\mathbb{F}_2) \rightarrow \cdots \end{aligned}$$

In the next subsections, we shall examine these twisted quadratic extensions.

Proof. By Theorem 17, we obtain the Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots &\rightarrow NL_{n+1}(\mathbb{F}_2[G/K]) \xrightarrow{\partial} NL_n(\mathbb{Z}[G]) \\ &\rightarrow NL_n(\mathbb{Z}[G/K]) \oplus NL_n(\mathbb{Z}[G]/\Sigma_K) \rightarrow NL_n(\mathbb{F}_2[G/K]) \rightarrow \cdots \end{aligned}$$

using the following order two subgroup K of the group $G = \mathbf{D}_e, \mathbf{SD}_e, \mathbf{Q}_e$:

$$K = \langle T^{2^{e-2}} \rangle.$$

Observe the quotient group $G/K = \mathbf{D}_{e-1}$ and the quotient rings

$$\mathbb{Z}[\mathbf{D}_e]/\Sigma_K = \mathbb{Z}[\zeta] \circ_c C_2, \quad \mathbb{Z}[\mathbf{SD}_e]/\Sigma_K = \mathbb{Z}[\zeta] \circ_{-c} C_2, \quad \mathbb{Z}[\mathbf{Q}_e]/\Sigma_K = \mathbb{Z}[\zeta] \circ_c [i].$$

Finally, Nilpotent Reduction 24 shows that the following map is an isomorphism:

$$NL_*(\mathbb{F}_2[\mathbf{D}_{e-1}]) \rightarrow NL_*(\mathbb{F}_2). \quad \square$$

7.1. Dihedral and semidihedral 2-groups

Let $e > 3$ and write $\zeta := \zeta_{2^e}$. We now show that a definite chunk of the UNil-groups of the above twisted quadratic extensions

$$R := \mathbb{Z}[\zeta] \circ_{\pm c} C_2 = \mathbb{Z}\{\zeta, x\}/(x^2 - 1, x\zeta x^{-1} \mp \zeta^{-1})$$

consists of the UNil-groups (really NQ-groups) of the Dedekind domains

$$\mathcal{O} := \mathbb{Z}[\zeta \pm \zeta^{-1}].$$

Here, the involution on \mathcal{O} is given by complex conjugation.

Proposition 35. For all $n \in \mathbb{Z}$, there is an isomorphism

$$NL_n(R) \longrightarrow NL_n(\mathcal{O}) \oplus NL_n(\mathcal{O} \rightarrow R).$$

Proof. The inclusion $\mathcal{O} \rightarrow R$ of rings with involution induces an exact sequence of a pair [27, Proposition 2.2.2]:

$$\cdots \xrightarrow{\partial} NL_n(\mathcal{O}) \xrightarrow{\text{incl}_*} NL_n(R) \rightarrow NL_n(\mathcal{O} \xrightarrow{\text{incl}} R) \xrightarrow{\partial} NL_{n-1}(\mathcal{O}) \rightarrow \cdots$$

Now, W. Pardon [25, Proof 4.14] constructs an embedding of rings with involution:

$$f : R \rightarrow M_2(\mathcal{O})$$

whose restriction to the center \mathcal{O} is the diagonal embedding. Here, the involution $\bar{}$ on the matrix ring $M_2(\mathcal{O})$ is defined by scaling [20, Defn. 2.5.5] the conjugate-transpose involution $*$:

$$\bar{B} := AB^*A^{-1},$$

where $A \in SL_2(R)$ is a certain hermitian matrix (i.e. $A = A^*$). For any quadratic complex (C, ψ) of f.g. projective left R -modules, right-multiplication by A gives an isomorphism from $\text{Hom}_R(C_i, R)$ with left R -module structure given by $*$ to $\text{Hom}_R(C_i, R)$ with left R -module structure given by $-$. Then we obtain an induced isomorphism

$$(\text{id}, A)_\# : NL_n(M_2(\mathcal{O}), *) \rightarrow NL_n(M_2(\mathcal{O}), -).$$

There is a commutative square

$$\begin{array}{ccc} NL_n(\mathcal{O}) & \xrightarrow{\text{incl}_*} & NL_n(R) \\ \downarrow (\mathcal{O} \oplus \mathcal{O}) \otimes_R (-)_\# = \text{diag}_* & & \downarrow f_* \\ NL_n(M_2(\mathcal{O}), *) & \xrightarrow{(\text{id}, A)_\#} & NL_n(M_2(\mathcal{O}), -). \end{array}$$

The left-hand vertical map is also an isomorphism, by quadratic Morita equivalence (see [20, Section 2.4–5] for a discussion). Therefore incl_* is a split monomorphism, and we obtain the desired left-split short exact sequence. \square

7.2. Quaternionic 2-groups

Let $e \geq 3$ and write $\zeta := \zeta_{2^e}$. An argument, similar to the previous subsection, is used to decompose the UNil-groups of the above twisted quadratic extension

$$S := \mathbb{Z}[\zeta] \circ_c [i] = \mathbb{Z}\{\zeta, y\} / (y^2 + 1, y\zeta y^{-1} - \zeta^{-1}).$$

Again consider the Dedekind domain with identity involution:

$$\mathcal{O} := \mathbb{Z}[\zeta + \zeta^{-1}].$$

Proposition 36. For all $n \in \mathbb{Z}$, there is an isomorphism

$$NL_n(S) \longrightarrow NL_n(\mathcal{O}) \oplus NL_n(\mathcal{O} \rightarrow S).$$

Proof. The inclusion $\mathcal{O} \rightarrow S$ of rings with involution induces an exact sequence of a pair [27, Proposition 2.2.2]:

$$\dots \xrightarrow{\partial} NL_n(\mathcal{O}) \xrightarrow{\text{incl}_*} NL_n(S) \rightarrow NL_n(\mathcal{O} \xrightarrow{\text{incl}} S) \xrightarrow{\partial} NL_{n-1}(\mathcal{O}) \rightarrow \dots$$

By a classical theorem of Weber [19, Theorem 2.2.4], our ring \mathcal{O} of algebraic integers in $F := \mathbb{Q}(\zeta + \zeta^{-1})$ is totally ramified over $2\mathbb{Z}$ by a principal prime ideal \mathfrak{p} . That is, $2\mathcal{O} = \mathfrak{p}^r$ where $r := [F : \mathbb{Q}]$. Performing completions at \mathfrak{p} , Pardon [25, Proof 4.14] constructs an embedding of rings with involution:

$$f| : \hat{S}_{\mathfrak{p}} \subset \hat{\mathcal{N}}_{\mathfrak{p}} \rightarrow M_2(\hat{\mathcal{O}}_{\mathfrak{p}})$$

whose restriction to the center $\hat{\mathcal{O}}_{\mathfrak{p}}$ is the diagonal embedding. (There, $\mathcal{N} \supset S$ is a certain maximal order with involution in the quaternion algebra $\mathbb{Q} \otimes S$ over F , so that f is an isomorphism of $\hat{\mathcal{O}}_{\mathfrak{p}}$ -algebras with involution.) Here, the involution $-$ on the matrix ring $M_2(\hat{\mathcal{O}}_{\mathfrak{p}})$ is defined by scaling [20, Defn. 2.5.5] the conjugate-transpose involution $*$:

$$\bar{B} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

The above Proof 35 shows that $\text{incl}_* : NL_n(\hat{\mathcal{O}}_{\mathfrak{p}}) \rightarrow NL_n(\hat{S}_{\mathfrak{p}})$ is a split monomorphism.

Consider the commutative diagram induced by inclusions:

$$\begin{array}{ccc} NL_n(\mathcal{O}) & \xrightarrow{\text{incl}_*} & NL_n(S) \\ \downarrow & & \downarrow \\ NL_n(\hat{\mathcal{O}}_{(2)}) & \longrightarrow & NL_n(\hat{S}_{(2)}). \end{array}$$

Observe that the completion of an \mathcal{O} -algebra at \mathfrak{p} equals its completion at $(2) = 2\mathcal{O} = \mathfrak{p}^r$. Then the bottom map is a split monomorphism. The vertical maps are isomorphisms by Theorem 16(2). Therefore, the top map incl_* is a split monomorphism, and we obtain the desired left-split short exact sequence. \square

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