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Cancellation for 4-manifolds with virtually abelian fundamental group. **(English summary)**
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Understanding the stable homeomorphism classification of topological 4-manifolds, namely compact connected 4-manifolds up to homeomorphisms and connected sums with $S^2 \times S^2$, is a fundamental problem in 4-manifold topology. Here, orientability is not required.

The paper under review gives a contribution to the following problem. Suppose that there are given 4-manifolds X and Y, and assume that there is a homeomorphism

$$X \# r(S^2 \times S^2) \cong Y \# r(S^2 \times S^2)$$

for some $r > 0$. Is there a similar homeomorphism with fewer $S^2 \times S^2$ connected summands?

Building on work of S. E. Cappell and J. L. Shaneson [Comment. Math. Helv. **46** (1971), 500–528; MR0301750] and of I. Hambleton and M. Kreck [J. Reine Angew. Math. **443** (1993), 21–47; MR1241127], the author gives in Theorem 2.1 a certain technical algebraic condition on the fundamental group (which is assumed to be good in the sense of M. H. Freedman and F. S. Quinn III [*Topology of 4-manifolds*, Princeton Math. Ser., 39, Princeton Univ. Press, Princeton, NJ, 1990; MR1201584]), under which the number of stabilizations (that is, the number r of the $S^2 \times S^2$ connected summands) can be bounded from above, under the extra assumption that X itself is of the form $X \cong X' \# (S^2 \times S^2)$. This bound is expressed as the dimension (if it is finite) of a certain (commutative) subring R_0 of the center of the fundamental group ring $\mathbb{Z}[\pi_1]$, such that $\mathbb{Z}[\pi_1]$ is a finitely generated R_0-module.

Then, the author proves in Proposition 2.2 that every finitely presented and virtually abelian fundamental group satisfies the hypotheses of Theorem 2.1, and if π_1 contains $\mathbb{Z}^n$ as a finite index subgroup, then the number of stabilizations can be bounded by $n+1$.

These results are applied to nonorientable topological 4-manifolds with fundamental group of order two (they have been classified up to homeomorphisms by Hambleton, Kreck and P. Teichner [Trans. Amer. Math. Soc. **344** (1994), no. 2, 649–665; MR1234481]). For example, the following is proved (in part 2 of Theorem 3.4). Let $X = M \# M'$, with M and M' closed nonorientable 4-manifolds with π_1 of order two, and let X_ϑ be tangential homotopy equivalent to X (meaning that there is a homotopy equivalence $h_\vartheta : X_\vartheta \simeq X$ that pulls back the tangent microbundle of X to that of X_ϑ). Then,

$$X_\vartheta \# 3(S^2 \times S^2) \cong X \# 3(S^2 \times S^2).$$

In particular, this is true for $X = \mathbb{R}P^4 \# \mathbb{R}P^4$.

It is remarkable that the proofs take some ideas from algebraic geometry.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.