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[Khan, Qayum](#) (1-IN)

Countable approximation of topological G -manifolds, III: arbitrary Lie groups G . (English summary)

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This is the third paper in a series in which the author studies the G -homotopy type of topological G -manifolds. In the first paper [Part I, Q. Khan, *Topology Appl.* **235** (2018), 14–21; [MR3760187](#)], he considered the case where G is a compact Lie group. In the second paper [Part II, Q. Khan, “Countable approximation of topological G -manifolds, II: linear Lie groups G ”, preprint, [arXiv:1806.06410](#), *J. Topol. Anal.*, to appear], G was assumed to be a linear Lie group. In the paper under review, the author considers the general case, where G is an arbitrary Lie group.

If G is a non-compact group, it is necessary to put some restrictions on how the group is allowed to act. This is done by assuming the action to be proper. There are several non-equivalent ways to define a proper action. The author assumes the action to be Palais-proper. A G -CW complex is called countable if it has countably many G -cells. The main result of the paper says that for any Lie group G , any Palais-proper topological G -manifold has the G -homotopy type of a countable proper G -CW complex. Moreover, a G -map between such spaces is a G -homotopy equivalence if and only if its restriction to their K -fixed sets is a homotopy equivalence for each $K \leq G$.

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