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Davis, James F. [Davis, James Frederic] (1-IN); **Khan, Qayum** (1-NDM); **Ranicki, Andrew** [Ranicki, Andrew A.] (4-EDIN-SM)

Algebraic K-theory over the infinite dihedral group: an algebraic approach. (English summary)

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The infinite dihedral group D_∞ can be described both as a free product and as a semidirect product. More precisely,

$$D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$$

where the generator of the cyclic group $\mathbb{Z}_2$ acts on $\mathbb{Z}$ by $-\mathrm{id}$ on the right-hand side. A discrete group G is said to be over D_∞ if it admits a surjective homomorphism $G \to D_\infty$. Such a homomorphism induces an amalgamated free product decomposition $G \cong G_1 *_H G_2$, and at the same time an HNN-structure $\overline{G} \cong H \rtimes_\alpha \mathbb{Z}$ for some automorphism $\alpha \in \mathrm{Aut}(H)$ where $\overline{G} \subset G$ is a subgroup of index 2. Groups over D_∞ show up naturally in the study of virtually cyclic groups.

Now let R be a ring. If G is a group over D_∞ then the algebraic K-theory decomposition theorems of Waldhausen for injective amalgamated free products and HNN-extensions yield a description of the algebraic K-theory of the group rings $R[G]$ and $R[\overline{G}]$, respectively. The main theorem of this paper shows that the Nil-groups in these decompositions are closely related.

Several consequences of this result are discussed. Let us mention just some of them. A theorem of J. F. Davis, F. Quinn and H. Reich [J. Topol. **4** (2011), no. 3, 505–528; MR2832565] shows that the Farrell-Jones isomorphism conjecture in algebraic K-theory can be reduced from the family of virtually cyclic groups to the family of finite-by-cyclic groups. The authors present an alternative proof of this result in degrees $n < 1$. Apart from this, an example of a group ring of an amalgamated free product with nonvanishing Nil-group in its K-theory is presented, and the algebraic K-theory of the group ring $R[\Gamma]$ of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is computed. Finally, a topological result on semisplitness of certain separating subcomplexes of finite CW-complexes in the situation of amalgamated free products is obtained. *Christian Voigt*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.