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On smoothable surgery for 4-manifolds. (English summary)
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Surgery theory was intensively developed in the early 1970's, culminating in the solution of the Hauptvermutung for n-manifolds with $n > 4$ and the classification of differentiable structures on these manifolds. One of the cornerstones of surgery theory is the exact surgery sequence. Let $(X, \partial X)$ be a simple, n-dimensional Poincaré space whose boundary may be empty. In particular, X is homotopy equivalent to a finite CW complex which satisfies Poincaré duality for any coefficients, with a twist in the non-orientable case, and simple means that there is a chain map

$$[X, \partial X] \cap \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C_*(X), \mathbb{Z}[\pi_1(X)]) \to C_{n-*}(X)$$

which is a simple isomorphism between based chain complexes. Let CAT stand for either TOP, the topological category, or DIFF, the differential category. Fix a CAT-manifold L^{n-1} without boundary and a simple homotopy equivalence $h\colon L \to \partial X$. Define the set $\mathcal{S}_{\mathrm{CAT}}(X, \mathrm{rel}\, h)$ as the set of all simple homotopy equivalences of pairs, $f\colon (M, \partial M) \to (X, \partial X)$, where $(M, \partial M)$ is a CAT-manifold, and for which there exists a CAT-equivalence $g\colon L \to \partial M$ such that the composition $L \to \partial M \to \partial X$ is homotopic to h. Let G be the limit of the classifying spaces of $G(m)$, the space of homotopy automorphisms of S^{m-1}. If $n \geq 5$ then the following sequence (the surgery sequence) is exact:

$$\mathcal{S}_{\mathrm{CAT}}(X, \mathrm{rel}\, h) \to \mathcal{N}_{\mathrm{CAT}}(X) \to L^h_n(\mathbb{Z}[\pi]^\omega),$$

with the normal invariant set $\mathcal{N}_{\mathrm{CAT}} \simeq [X/\partial X, G/\mathrm{CAT}]_0$ and the abbreviated notation $\pi = \pi_1(X)$ and $\omega = w_1(X)$. The surgery obstruction group $L^h_4(\mathbb{Z}[\pi]^\omega)$ consists of Witt classes of nonsingular quadratic forms over the group ring $\mathbb{Z}[\pi]$ with involution $(g \mapsto \omega(g)g^{-1})$.

The extension of that sequence to 4-manifolds for all possible fundamental groups is very problematic. Freedman proved the corresponding sequence for "good" fundamental groups. Later on that result was extended by Freedman and Teichner to groups of subexponential growth. See the nice paper of R. C. Kirby and L. R. Taylor [in *Surveys on surgery theory, Vol. 2*, 387–421, Ann. of Math. Stud., 149, Princeton Univ. Press, Princeton, NJ, 2001; MR1818779] for an overview. In the paper under review this result is extended to the special case of the smooth surgery sequence at the normal invariants for the 4-torus T^4 and the real projective 4-space $\mathbb{R}P^4$ as well as to a broader class of non-orientable 4-manifolds.

The simple structure set $\mathcal{S}^s_{\mathrm{CAT}}(X)$ consists of CAT s-bordism classes in $\mathbb{R}^\infty$ of simple homotopy equivalences $h\colon Y \to X$ such that $\partial h\colon \partial Y \to \partial X$ is the identity. Indeed, transversality in the TOP category holds for all dimensions and codimensions, then the normal invariants map $\eta\colon \mathcal{S}^s_{\mathrm{CAT}}(X) \to \mathcal{N}_{\mathrm{CAT}}(X)$ is a forgetful map. The surgery obstruction map $\sigma^h_*\colon \mathcal{N}_{\mathrm{CAT}}(X) \to L^h_4(\mathbb{Z}[\pi]^\omega)$ vanishes on the image of η. The main result of the paper is the following theorem:

The author considers the following three hypotheses:

1. Let X be orientable. Suppose that the homomorphism $\kappa_2\colon H_2(\pi, \mathbb{Z}_2) \to L^h_4(\mathbb{Z}[\pi]^\omega)$ is injective on the subgroup $u_2(\ker v_2(X))$ with induced homomorphism $u_2\colon H_2(X, \mathbb{Z}_2) \to H_2(\pi, \mathbb{Z}_2)$ from the classifying map $u\colon X \to B\pi$ of the universal

cover and $v_2(X) \in H^2(X, \mathbb{Z}_2)$ as second Wu class.

2. Let X be non-orientable such that π contains an orientation-reversing element of finite order, and if CAT = DIFF, then suppose that the orientation-reversing element has order two. Suppose that κ_2 is injective on all $H_2(\pi, \mathbb{Z}_2)$, and suppose that $\ker(u_2) \subseteq \ker(v_2)$.
3. Let X be non-orientable such that there exists an epimorphism $\pi^\omega \to \mathbb{Z}^-$. Suppose that κ_2 is injective on the subgroup $u_2(\ker v_2(X))$.

Then the author obtains (Theorem 4.1): Let $(X, \partial X)$ be a based, compact, connected, CAT 4-manifold with fundamental group $\pi = \pi_1(X)$ and orientation character ω.

- Suppose Hypothesis 1 or 2. Then the surgery sequence of based sets is exact at the smooth normal invariants:

$$\mathcal{S}^s_{\mathrm{DIFF}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{DIFF}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega).$$

- Suppose Hypothesis 1 or 2 or 3. Then the surgery sequence of based sets is exact at the stably smoothable normal invariants:

$$\mathcal{S}^s_{\mathrm{TOP0}}(X) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP0}}(X) \xrightarrow{\sigma_*} L^h_4(\mathbb{Z}[\pi]^\omega),$$

  where TOP0 refers to manifolds with the same smoothing invariant as X.

The author derives many corollaries of this theorem for orientable as well as for non-orientable 4-manifolds. *Torsten Asselmeyer-Maluga*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.