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Stable existence of incompressible 3-manifolds in 4-manifolds. **(English summary)**

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The Kneser Conjecture in 3-dimensional topology states that if the fundamental group of a 3-manifold X is a free product $G_- * G_+$, then X is homeomorphic to the connected sum $X_- \# X_+$, where $X_\pm$ have fundamental group $G_\pm$, respectively. This conjecture was proved by J. R. Stallings [*Group theory and three-dimensional manifolds*, Yale Univ. Press, New Haven, CT, 1971; MR0415622]. Later, C. D. Feustel [Pacific J. Math. **46** (1973), 123–130; MR0328908] and G. A. Swarup [Math. Z. **132** (1973), 305–317; MR0322883] proved a generalized version of the conjecture when the fundamental group admits an injective amalgamated product along a surface group.

In the paper under review, the authors study the problem of stably realizing injective amalgamated product decomposition of the given group. A pair of 4-manifolds are called *bistably diffeomorphic* if they become diffeomorphic after connect summing each with finitely many copies of the complex-projective plane $\mathbb{CP}_2$ and its orientation reversal $\overline{\mathbb{CP}_2}$. Given a nonempty connected CW-complex A, by a continuous map $u: a \to B\Gamma$ *classifying the universal cover* $\widetilde{A}$, we mean the induced map $u_\#$ on the fundamental groups is an isomorphism. By a connected subcomplex being *incompressible*, we mean that the inclusion induces a monomorphism on fundamental groups. Then one of the main results is the following:

Theorem 1.1. Let X be an oriented closed smooth 4-manifold with fundamental group G. Let $c: X \to BG$ classify its universal cover. Let X_0 be a connected oriented closed 3-manifold with fundamental group G_0. Suppose $G = G_- *_{G_0} G_+$ with $G_0 \subset G_\pm$. There exists an incompressible embedding of X_0 in some bistabilization of X that induces the given injective amalgamation of fundamental group, if and only if there exists a map $d: X_0 \to BG_0$ classifying its universal cover, that satisfies: $d_*[X_0] = \partial c_*[X] \in H_3(G_0; \mathbb{Z})$, where ∂ is the boundary map in the Mayer-Vietoris sequence.

Analogous results are obtained for cases with additional conditions on X, that is, Theorem 1.3 treats the case when the universal cover of X has no spin structure, Theorem 1.4 treats the case when X admits a spin structure, and Theorem 1.7 treats the case when the universal cover of X admits a spin structure. *Tsuyoshi Kobayashi*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.