THE s/h-COBORDISM THEOREM

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1. Whitehead torsion

Let R be a (unital associative) ring. The *stable general linear group*

$$GL(R) \;:=\; \operatorname*{colim}_{n\to\infty} GL_n(R)$$

is the direct limit given by the stabilization homomorphisms

$$GL_n(R) \longrightarrow GL_{n+1}(R) \;;\; A \longmapsto \left[\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

The *n-th elementary subgroup* $E_n(R) < GL_n(R)$ is generated by those matrices with 1's along the diagonal and any element $r \in R$ at any (i,j)-th entry with $i \neq j$.

Lemma 1 (Whitehead)**.** *The elementary subgroup* $E(R) = \operatorname*{colim}_{n\to\infty} E_n(R)$ *equals the commutator subgroup of* $GL(R)$*.*

The 'generalized determinant' $[A]$ is an abelian invariant defined as the stable class of an invertible matrix $A \in GL_n(R)$ under these row and column operations:

$$[A] \;\in\; K_1(R) := GL(R)^{ab} \;=\; \frac{GL(R)}{[GL(R), GL(R)]} = \frac{GL(R)}{E(R)}.$$

Proposition 2. *The following two facts are easily verified. If R is commutative, then the determinant* $\det : K_1(R) \longrightarrow R^\times$ *is defined and a split epimorphism. Furthermore, if R is euclidean (in particular, a field), then* $\det$ *is an isomorphism.*

Let $C_\bullet = (C_*, d_*)$ be a contractible finite chain complex of based left R-modules. Here *based* means free with a chosen finite basis. Select a *chain contraction* $s_* : C_* \longrightarrow C_{*+1}$, which is a chain homotopy from id to 0; that is: $d \circ s + s \circ d = \mathrm{id} - 0$. The the *algebraic torsion* is well-defined by the formula

$$\tau(C_\bullet) \;:=\; [d + s : C_{even} \longrightarrow C_{odd}] \in K_1(R),$$

with $C_{even} := C_0 \oplus C_2 \oplus \cdots + C_{2N}$ and $C_{odd} := C_1 \oplus C_3 \oplus \cdots$ finite based modules.

Exercise 3. Verify that $(d+s)^{-1} = (d+s)(1-s^2+\cdots+(-1)^N s^{2N}) : C_{odd} \longrightarrow C_{even}$.

Let G be a group. Divide by trivial units in group ring for the *Whitehead group*

$$\mathrm{Wh}(G) \;:=\; K_1(\mathbb{Z}G)/\langle \mathbb{Z}^\times, G\rangle.$$

Conjecture 4 (Hsiang)**.** $\mathrm{Wh}(G) = 0$ *if G is torsion-free.*

Let $f : Y \longrightarrow X$ be a cellular homotopy equivalence of connected finite CW complexes. Write $\widetilde{f} : \widetilde{Y} \longrightarrow \widetilde{X}$ for the induced $\pi_1 X$-equivariant homotopy equivalence of universal covers. Select a lift and orientation in $\widetilde{X}$ of each cell in X. This gives a finite basis to the free $\mathbb{Z}[\pi_1 X]$-module complex $C_\bullet(\widetilde{X})$. Do the same for $\widetilde{Y}$.

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Dividing by these two sets of choices, the *Whitehead torsion* of f is well-defined in terms of the algebraic mapping cone of the cellular map induced by $\widetilde{f}$:

$$\tau(f) \ := \ [\tau(\mathrm{Cone}(C_\bullet \widetilde{f}))] \in \mathrm{Wh}(\pi_1 X).$$

If the homotopy equivalence f is not cellular, then $\tau(f) := \tau(f')$ is well-defined for any cellular approximation f' to f. The homotopy equivalence $f : Y \longrightarrow X$ is *simple* means that $\tau(f) = 0$. Clearly, any cellular homeomorphism is simple.

Theorem 5 (Chapman)**.** *Any homeomorphism of finite CW complexes is simple.*

This fundamental result is proven by showing that: $\tau(f) = 0$ if and only if $f \times \mathrm{id}_Q$ is homotopic to a homeomorphism, where $Q := [0,1]^{\mathbb{N}}$ is the Hilbert cube. Here, one uses a geometric characterization of 'simple' in terms of a finite sequence of elementary expansions and elementary collapses of cancelling cell-pairs.

2. Statement of the s-cobordism theorem

A homotopy cobordism (shortly, *h-cobordism*) is a cobordism $(W^{n+1}; M^n, M')$ such that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences; that is, M and M' are deformation retracts of W. A smooth h-cobordism $(W; M, M')$ is simple (shortly, *s-cobordism*) means that these inclusions are simple. We use the Whitehead triangulations induced by their smooth structures, in which simplices are smoothly embedded, to parse the formulas $\tau(M \hookrightarrow W) = 0 = \tau(M' \hookrightarrow W)$.

Example 6. The product s-cobordism on M is $(M^n \times [0,1]; M \times \{0\}, M \times \{1\})$.

Theorem 7 (Mazur–Stallings–Barden, the s-cobordism theorem)**.** *Let $n > 4$. Any smooth s-cobordism $(W^{n+1}; M, M')$ is diffeomorphic to the product, relative to M.*

Corollary 8 (Smale, the h-cobordism theorem)**.** *Let $n > 4$. Any simply connected smooth h-cobordism $(W^{n+1}; M, M')$ is diffeomorphic to the product, relative to M.*

(S Donaldson demonstrated this statement is false when $n = 4$.) More generally:

Theorem 9 (realization)**.** *Let M a connected closed smooth manifold of dimension $n > 4$. Under Whitehead torsion of the inclusion of M, the set of diffeomorphism classes rel M of smooth h-cobordisms on M corresponds bijectively to* $\mathrm{Wh}(\pi_1 M)$.

3. Application

Corollary 10 (the generalized Poincaré conjecture)**.** *Let $m > 5$. Any closed smooth manifold in the homotopy type of the m-dimensional sphere is homeomorphic to it.*

This is true for topological manifolds. By other means, the GPC holds for $m \leqslant 5$.

Proof. Let Σ^m be a smooth homotopy m-sphere. Consider the smooth cobordism $(W^m; M^{m-1}, M')$ where $W := \Sigma - \mathring{D}^m_- - \mathring{D}^m_+$ and $M := \partial D_-$ and $M' := \partial D_+$. Since $m > 2$, by the Seifert–vanKampen theorem, W is simply connected, as well as M and M'. Using excision, the relative homology with integer coefficients is

$$H_*(W, M) \xrightarrow{\cong} H_*(\Sigma - \mathring{D}_+, D_-) \ = \ \widetilde{H}_*(\Sigma - point) = 0.$$

Then, by the Whitehead theorem, the inclusion $M \hookrightarrow W$ is a homotopy equivalence, and similarly $M' \hookrightarrow W$ is also. So, since $n := m - 1 > 4$, by the h-cobordism theorem, $(W; S^n_-, S^n_+)$ is diffeomorphic to the product $(S^n \times [0,1]; S^n \times \{0\}, S^n \times \{1\})$, relative to the identification $S^n_- = S^n \times \{0\}$, which extends to $D^{n+1}_- = D^{n+1} \times \{0\}$.

Hence $\Sigma - \mathring{D}_+ = D_- \cup W$ is diffeomorphic to the disc $D^m = D^m \times \{0\} \cup S^n \times [0,1]$. The restricted exotic diffeomorphism $S_+ \longrightarrow S^n$ extends to a homeomorphism $D_+ \longrightarrow D^{n+1}$ by coning (the so-called Alexander trick). Therefore, Σ is homeomorphic to the standard sphere $S^m = D^m \cup_{homeo} D^m$. □

The proof shows more: Σ is diffeomorphic to a *twisted double* $D^m \cup_{diffeo} D^m$.

4. Proof outline of the h-cobordism theorem

A good reference is page 87 of the monograph of C Rourke and B Sanderson.

(1) Consider a 'nice' handle decomposition of W relative to M, say via a so-called *nice* Morse function: handles arranged in increasing index and different handles having different critical values. It exists for all dimensions.
(2) Since $\pi_0(M) \longrightarrow \pi_0(W)$ is surjective (nonexample: $W = M \times I \sqcup S^{n+1}$), we can cancel each 0-handle with a corresponding 1-handle.
(3) Since $\pi_1(M) \longrightarrow \pi_1(W)$ is surjective (nonexample: $W = m \times I \# S^1 \times S^n$), we can trade each remaining 1-handle for a new 3-handle. This part works for the non-simply connected case as well.
(4) Dually eliminate the $(n+1)$-handles and n-handles, working relative to M'.
(5) Similarly, since $\pi_k(M) \longrightarrow \pi_k(W)$ is surjective, we can trade each k-handle for a new $(k+2)$-handle. Only $(n-1)$-handles and $(n-2)$-handles remain.
(6) Flip the resulting handle decomposition upside down: only 2-handles and 3-handles relative to M'. Since $\pi_1(M') = 1$ and $H_2(W, M'; \mathbb{Z}) = 0$, we can cancel each such 2-handle with a 3-handle.
(7) Thus we obtain only 3-handles relative to M'. But $H_3(W, M'; \mathbb{Z}) = 0$, so actually there are no 3-handles remaining! Therefore, we can conclude that W is diffeomorphic to $M \times I$ relative to $M \times \{0\}$.

Above, the canceling and trading of handles necessitates the Whitney trick ($n > 4$).