L-GROUPS

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1. Rings with involution

An *involution* on a unital associative ring R is an order-two ring map $^- : R \to R^{op}$:

$$\overline{\overline{r}} = r \quad \text{and} \quad \overline{r+s} = \overline{r} + \overline{s} \quad \text{and} \quad \overline{rs} = \overline{s}\,\overline{r}.$$

In particular, note $\overline{0} = 0$ and $\overline{1} = 1$, since

$$\overline{0} = \overline{0+0} - \overline{0} = \overline{0} - \overline{0} = 0$$
$$\overline{1} = 1\overline{1} = \overline{\overline{1}}\,\overline{1} = \overline{1\overline{1}} = \overline{\overline{1}} = 1.$$

Example 1. Below are some frequently occurring rings with involution $(R, ^-)$.

1. (commutative ring, identity map)
2. (complex numbers $\mathbb{C}$, complex conjugation: $\overline{x+iy} = x - iy$)
3. ($n \times n$ matrices $M_n(\mathbb{C})$, conjugate transpose $[a_{ij}]^* = [\overline{a_{ji}}]$): $(AB)^* = B^*A^*$
4. $\mathbb{Z}G^\omega$ =(group ring $\mathbb{Z}G$, geometric involution: $\overline{g} = \omega(g)g^{-1}$), where the given homomorphism $\omega : G \longrightarrow \{\pm 1\}$ is called an *orientation character*.

2. Symmetric & quadratic forms

Let M be a based left R-module. A *sesquilinear form* is a bi-additive function

$$\lambda : M \times M \longrightarrow R \quad \text{satisfying} \quad \lambda(rx, sy) = r\,\lambda(x,y)\,\overline{s}.$$

It is $(-1)^k$*-symmetric* means $\lambda(y,x) = (-1)^k\overline{\lambda(x,y)}$. It is *nonsingular* means

$$M \longrightarrow M^* := \mathrm{Hom}_R(M,R) \ ; \ y \longmapsto \lambda(-,y)$$

is an isomorphism with zero torsion, with respect to the dual basis of M^*, in the reduced K-group $\widetilde{K}_1(R) := K_1(R)/K_1(\mathbb{Z})$.

Exercise 2. Turn the right R-module structure on M^* into a left one, using $^-$.

Exercise 3. Show $\overline{\mathrm{ev}} : M \to M^{**}; x \mapsto (f \mapsto \overline{f(x)})$ is an isomorphism (M f.g. free).

A *quadratic refinement* of (M,λ) is a function that is 'quadratic' and 'refinement':

$$\mu : M \longrightarrow \frac{R}{\{r - (-1)^k\,\overline{r}\}} \quad \text{such that} \quad \begin{cases} \mu(rx) = r\,\mu(x)\,\overline{r} \\ \mu(x+y) = \mu(x) + \mu(y) + [\lambda(x,y)] \\ \lambda(x,x) = \mu(x) + (-1)^k\overline{\mu(x)} \in R. \end{cases}$$

Exercise 4. (M,λ) admits a unique quadratic refinement if 2 is a unit in R.

Example 5. A *hyperbolic form* is the triple $\mathcal{H}(M) = (M \oplus M^*, \left(\begin{smallmatrix} 0 & I \\ (-1)^k I & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right))$.

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Example 6 (quadratic forms in classic linear algebra)**.** Let A be an $n \times n$ matrix over $\mathbb{C}$ such that the symmetrization $A + A^*$ is invertible. Take $k = 0$ and $M = \mathbb{C}^n$ and $\lambda(x, y) = x^t(A + A^*)\,\overline{y}$ and $\mu(x) = [x^t A\,\overline{x}] \in \mathbb{R} = \mathbb{C}/\{z - \overline{z}\}$.

3. Definition of $L_{2k}(R)$

A *sublagrangian* F in a $(-1)^k$-quadratic form $Q = (M, \lambda, \mu)$ is a free submodule $F \subset M$ with a basis that extends to M which is simple-isomorphic to the preferred one such that λ and μ vanish on F. It is a *lagrangian* if it is maximal such object.

Example 7. $\Delta := \{(x, x) \mid x \in M\}$ is a lagrangian in $(M \oplus M, \lambda \oplus -\lambda, \mu \oplus -\mu)$.

Exercise 8. A nonsingular Q admits a lagrangian F if and only if Q is isomorphic to the hyperbolic form $\mathcal{H}(F)$. The image of F^* in M is a *complementary lagrangian*.

The abelian monoid $L^s_{2k}(R)$ consists of the stable isomorphism classes of (simple) nonsingular $(-1)^k$-quadratic forms over $(R, ^-)$. Here, the sum operation is $Q + Q' = (M \oplus M', \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu \\ \mu' \end{smallmatrix}\right))$, and *stably isomorphic* means $Q + \mathcal{H}(R^m) \cong Q' + \mathcal{H}(R^n)$.

Exercise 9. It is an abelian group. (Hint: Use the diagonal Δ.)

Note $L_0 = L_4 = L_8 = \dots$ and $L_2 = L_6 = L_{10} = \dots$. Write $L^s_*(G^\omega) = L^s_*(\mathbb{Z}[G]^\omega)$.

4. Signature

Any $(+1)$-symmetric invertible matrix over $\mathbb{Z}$ has all eigenvalues real and nonzero. Moreover, by the spectral theorem and taking square roots, this matrix is *congruent* $(\exists C : C^t AC)$ over $\mathbb{R}$ to a diagonal matrix of $+1$'s (the number of which is the positive inertia i_+) and -1's (the number of which is the negative inertia i_-).

Theorem 10 (Sylvester's Law of Inertia, 1852)**.** *Congruent matrices over $\mathbb{R}$ have equal inertia (i_+, i_-). So the* signature $i_+ - i_-$ *is an invariant of congruence classes.*

Thus the signature function $\operatorname{sign} : L_0(\mathbb{Z}) \longrightarrow \mathbb{Z}$ is a well-defined homomorphism. Because of the existence of the quadratic refinement, all the diagonal entries of the symmetric matrix are even. In fact, sign is injective with image $8\mathbb{Z}$ via Gauss sums.

Remark 11. Signature injects from $L_0(\mathbb{R})$ or $L_0(\mathbb{C})$ onto $4\mathbb{Z}$, from $L_0(\mathbb{H})$ onto $2\mathbb{Z}$.

5. Some computations of $L_{2k}(R)$

Remark 12. An isomorphism $L_2(\mathbb{Z}) \to L_2(\mathbb{F}_2) \xrightarrow{\text{Arf}} \mathbb{Z}/2$ exists; see its minilecture.

Let G be a finite group. Via irreducible representations of G, by the theorems of Maschke and Artin–Wedderburn, there is an isomorphism of rings with involution:

$$\mathbb{R}G \;\cong\; M_{r_1}(\mathbb{R}) \times \cdots \times M_{c_1}(\mathbb{C}) \times \cdots \times M_{h_1}(\mathbb{H}) \times \cdots .$$

The complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ are equipped with their conjugations.

Example 13. Here are decompositions with representations indicated in subscript.

(1) $\mathbb{R}C_2 = \mathbb{R}_+ \times \mathbb{R}_-$

(2) $\mathbb{R}C_p = \mathbb{R}_+ \times \prod_{a=1}^{(p-1)/2} \mathbb{C}_{\zeta^a}$ where p is an odd prime and $\zeta := e^{2\pi i/p} \in \mathbb{C}$

(3) $\mathbb{R}Q_8 = \mathbb{R}_{++} \times \mathbb{R}_{+-} \times \mathbb{R}_{-+} \times \mathbb{R}_{--} \times \mathbb{H}$ with $1 \to \{\pm 1\} \to Q_8 \to C_2 \times C_2 \to 1$.

Proposition 14 (Morita equivalence)**.** *Any $L_*(M_n(R))$ is isomorphic to $L_*(R)$.*

Therefore, the *multisignature* of G is the induced homomorphism

$$\text{msign} : L_*(\mathbb{Z}G) \longrightarrow L_*(\mathbb{R}G) \longrightarrow L_*(\mathbb{R}) \oplus \cdots L_*(\mathbb{C}) \oplus \cdots L_*(\mathbb{H}) \oplus \cdots .$$

Theorem 15 (Wall)**.** *On* $L_{2k}(\mathbb{Z}G)$*,* msign *has kernel and cokernel finite 2-groups.*

The reduced L-groups $\widetilde{L}_*(R) := L_*(R)/L_*(\mathbb{Z})$ satisfy $L_*(\mathbb{Z}G) = L_*(\mathbb{Z}) \oplus \widetilde{L}_*(\mathbb{Z}G)$.

Theorem 16 (Bak)**.** *Suppose G has odd order. On* $\widetilde{L}_{2k}(\mathbb{Z}G)$*,* msign *is injective.*

6. Symmetric & quadratic formations

A $(-1)^k$*-quadratic formation* is a triple $(Q; F, G)$, consisting of a nonsingular $(-1)^k$-quadratic form $Q = (M, \lambda, \mu)$ over $(R, ^-)$, along with a lagrangian F and a sublagrangian G in Q. The formation is *nonsingular* means that G is a langrangian.

Example 17. A *hyperbolic formation* is the triple $(\mathcal{H}(M); M \oplus 0, 0 \oplus M^*)$.

Example 18. A *boundary formation* is a triple $(\mathcal{H}(M); M \oplus 0, \Gamma_f = \{(x, g(x))\})$, where $f : M \longrightarrow M^*$ is any homomorphism of left R-modules and $g = f - (-1)^k f^*$.

Exercise 19. The previous two examples are isomorphic if g is an isomorphism. So, if M has even rank, then boundary formations generalize hyperbolic formations.

7. Definition of $L_{2k+1}(R)$

The abelian monoid $L^s_{2k+1}(R)$ under component-wise sum consists of stabilized boundary isomorphism classes of nonsingular $(-1)^k$-quadratic formations over R. *Boundary isomorphic* is isomorphic after adding boundary formations to both sides. *Stably isomorphic* is isomorphic after adding hyperbolic formations to both sides.

Exercise 20. It is an abelian group. (Hint: Choose complements, as in Exercise 8.)

As functors there is 4-periodicity, $L_1 = L_5 = L_9 = \ldots$ and $L_3 = L_7 = L_{11} = \ldots$.

8. Some computations of $L_{2k+1}(R)$

Remark 21. An isomorphism $L_3(C_2) \xleftarrow{\partial} L_4(\mathbb{F}_2) \xrightarrow{\text{Arf}} \mathbb{Z}/2$ exists, by Rim's square.

Theorem 22 (Connolly–Hausmann)**.** *Let G be finite. Then* $16 \cdot L^s_{2k+1}(G) = 0$.

Theorem 23 (Bak)**.** *Suppose G has odd order. Then* $L^s_{2k+1}(G) = 0$.