SURGERY IN THE MIDDLE DIMENSION

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1. Review of surgery kernels

Let $f : M^n \longrightarrow X$ be a k-connected degree-one normal map to a simple Poincaré complex. The relative homotopy groups $\pi_{j+1}(X \simeq \mathrm{Cyl}(f), M)$ vanish for all $j < k$. By Exercise 7, $H_j(M; \mathbb{Z}\pi) \cong H_j(X; \mathbb{Z}\pi) \oplus K_j(M)$, $\pi := \pi_1(X)$, with *surgery kernel*

$$K_j(M) \ := \ \mathrm{Ker}\left(H_j(M; \mathbb{Z}\pi) \xrightarrow{f_*} H_j(X; \mathbb{Z}\pi) \right).$$

So, by the relative Hurewicz theorem and homology exact sequence of a pair, note

$$\pi_{k+1}(X, M) \xrightarrow[\cong]{} H_{k+1}(X, M; \mathbb{Z}\pi) \xrightarrow[\cong]{\partial} K_k(M).$$

By homological algebra, $K_k(M)$ is a f.g. $\mathbb{Z}\pi$-module. If $n = 2k$, it is stably based.

Select an element in $\pi_{k+1}(X, M)$, which is represented by a commutative diagram

$$\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ S^k & \hookrightarrow & D^{k+1}. \end{array}$$

As we learned in Lecture 06, by the Hirsch–Smale theorem, this element is represented using a unique regular homotopy class of immersion $S^k \times D^{n-k} \longrightarrow M$. Indeed, the normal bundle $\nu(S^k \longrightarrow M)$ is stably framed by cancellation, because:

$$\begin{aligned} \nu(S^k \longrightarrow M) \oplus \quad \nu(M \hookrightarrow S^N)\big|_{S^k} \quad &= \ \nu(S^k \hookrightarrow S^N) \\ \nu(S^k \longrightarrow M) \oplus \qquad f^*\xi|_{S^k} \qquad &= \ \underline{\mathbb{R}}^{N-k} \end{aligned}$$

and using the canonical framing $f^*\xi|_{S^k} = (\xi|_{D^{k+1}})|_{S^k} = \underline{\mathbb{R}}^{N-n}$; see Exercise 11.

2. The equivariant intersection form $(n = 2k)$

Let $\alpha, \beta : S^k \times D^k \longrightarrow M^{2k}$ be immersions that intersect transversally in double-points. Assume the cores $\alpha_0, \beta_0 : S^k \longrightarrow M$ are pointed, as occur in Section 1. There are unique pointed lifts $\widetilde{\alpha}_0, \widetilde{\beta}_0 : S^k \longrightarrow \widetilde{M}$ to the universal cover. Define

$$\lambda(\alpha, \beta) \ := \ \sum_{g \in \pi} (\widetilde{\alpha}_0 g \cdot \widetilde{\beta}_0) g \ \in \ \mathbb{Z}\pi$$

where π has a right-action on $\widetilde{M}$ and $\cdot$ is the usual $\mathbb{Z}$-valued intersection product.

This defines the *equivariant intersection form* $\lambda : K_k(M) \times K_k(M) \longrightarrow \mathbb{Z}\pi$, which is $(-1)^k$-symmetric and is nonsingular by Poincaré duality on surgery kernels.

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3. Equivariant self-intersection $(n = 2k > 2)$

The quadratic refinement $\mu : K_k(M) \longrightarrow \mathbb{Z}\pi/\{r - (-1)^k\overline{r}\}$ is defined geometrically as follows. Let $\alpha_0 : S^k \longrightarrow M^{2k}$ be as above, intersecting itself transversally in double-points p. For each such p, assign $g_p \in \pi$ as the pointed loop obtained from applying α_0 to the oriented arc joining the two points of $(\alpha_0)^{-1}(p)$ through the basepoint in S^k. Since $k > 1$, this arc in S^k is well-defined up to homotopy relative endpoints. Also, the orientation on S^k gives local orientation to the two intersecting sheets near p, so there is an intersection product $n_p \in \{\pm 1\}$. Define

$$\mu(\alpha) \;:=\; \sum_p n_p g_p.$$

However, the value of n_p depended on taking an arbitrary order of the two sheets, and reversing the order introduces a sign of $(-1)^k$ and also reverses the arc and loop to g_p^{-1}. So μ is well-defined by dividing by this effect.

We ignored the normal framings for both λ and μ, and instead we only used the uniqueness of the regular homotopy class of immersion. The normal framing will be remembered in the Whitney trick (Section 5) and when performing surgery.

4. The surgery obstruction map $(n = 2k > 2)$

The *even-dimensional surgery obstruction map* is now defined as

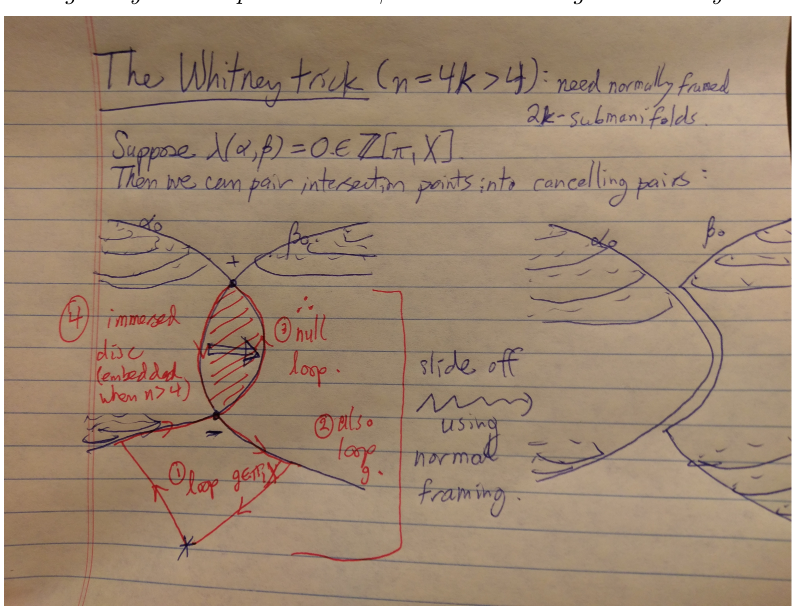
$$\mathcal{N}_{\mathrm{DIFF}}(X^{2k}) \xrightarrow{\;\sigma\;} L^s_{2k}(\pi_1 X)\ ;\ [M, f, \overline{f}] \longmapsto [K_k(M), \lambda, \mu].$$

If $(M, f, \overline{f}) = \partial(W, F, \overline{F})$, the image of $\partial : K_{k+1}(W, M) \to K_k(M)$ is a lagrangian.

Theorem 1. *Suppose $\sigma[M, f, \overline{f}] = 0 \in L^s_{2k}(\pi_1 X)$ for a k-connected degree-one normal map $f : M \to X$. Then f is normally bordant a simple homotopy equivalence.*

5. The Whitney trick $(n = 2k > 4)$

Lemma 2 (Whitney)**.** *Suppose α and β do not algebraically intersect: $\lambda(\alpha, \beta) = 0$. Then β is regularly homotopic to some β' that does not geometrically intersect α.*



Lemma 3 (Wall). *Similarly, α is regularly homotopic to an embedding if $\mu(\alpha) = 0$.*

6. Proof of the fundamental theorem ($n = 2k > 4$)

Proof. Stabilizing with a hyperbolic form via connected sum with copies of $S^k \times S^k$, we may assume that $(K_k(M), \lambda, \mu)$ has a lagrangian F, say with basis $\alpha_1, \ldots, \alpha_r$. Since $\lambda(\alpha_i, \alpha_j) = 0$ for each $i \neq j$, by Lemma 2, isotope the $\alpha_i : S^k \times D^k \longrightarrow M^{2k}$ to be disjoint. Then, since $\mu(\alpha_i) = 0$, by Lemma 3, isotope the α_i to be embeddings.

Now do surgery on $\alpha_1, \ldots, \alpha_r$ to obtain a degree-one normal map $f' : M' \longrightarrow X$. The effect is to kill $F \oplus F^* = K_k(M)$. So $K_k(M') = 0$. Since f' is also k-connected, note $K_*(M') = 0$, by Poincaré duality of surgery kernels: $K_{n-j}(M') \cong K^j(M)$.

Then $\widetilde{f'}$ is an integral homology equivalence. So, $\widetilde{M'}$ and $\widetilde{X}$ are simply connected, by Whitehead's theorem, $\widetilde{f'}$ is a weak homotopy equivalence. Therefore, since f' induces an isomorphism on π_1, f' is a weak homotopy equivalence. Hence, since M' and X have CW structures, by Whitehead's theorem, f' is a homotopy equivalence. Its Whitehead torsion turns out to be $\tau(f') = [y \mapsto \lambda(-, y)] = 0 \in \mathrm{Wh}(\pi_1 X)$. □

7. The surgery obstruction map ($n = 2k + 1$)

Let $f : M^{2k+1} \longrightarrow X$ be a k-connected degree-one normal map to a simple Poincaré complex. Again, each element of the surgery kernel $K_k(M)$, which is a stably based $\mathbb{Z}\pi$-module, is represented by a unique regular homotopy class of normally framed embedding $S^k \times D^{k+1} \hookrightarrow M^{2k+1}$ by general position. (However, the embedding may not be unique up to regular isotopy, as knotting often occurs.)

Choose such an embedding for each basis element $e_1, \ldots, e_r : S^k \times D^{k+1} \longrightarrow M$ of $K_k(M)$ such that the images are disjoint. (However, linking can easily occur.) Taking boundary-connected sum produces an embedded k-handlebody U in M:

$$U^{2k+1} := e_1 \natural \cdots \natural e_r \ : \ \natural_{i=1}^{r} S^k \times D^{k+1} \longrightarrow M^{2k+1}.$$

Then its boundary is a hypersurface (similar to a Heegard decomposition for $n = 3$):

$$\partial U \ : \ \#_{i=1}^{r} S^k \times S^k \longrightarrow M^{2k+1}.$$

However note ∂U bounds in two ways; it's also the boundary of the exterior $M - \mathring{U}$. This yields two lagrangians in the intersection form of the degree-one normal map

$$f| : \partial U \longrightarrow S^{2k} \subset D^{2k+1} \subset X;$$

see Exercise 15. Specifically, these lagrangians are the images

$$\begin{aligned} F &:= \mathrm{Im}\left(K_{k+1}(U, \partial U) \xrightarrow{\partial} K_k(\partial U)\right) \\ G &:= \mathrm{Im}\left(K_{k+1}(M - \mathring{U}, \partial U) \xrightarrow{\partial} K_k(\partial U)\right). \end{aligned}$$

The *odd-dimensional surgery obstruction* is now defined in terms of this formation:

$$\mathcal{N}_{\mathrm{DIFF}}(X^{2k+1}) \xrightarrow{\ \sigma\ } L^s_{2k+1}(\pi_1 X) \ ; \ [M, f, \overline{f}] \longmapsto [H_k(\partial U), \lambda_{\partial U}, \mu_{\partial U}; F, G].$$