THE HOMOTOPY TYPE OF G/TOP

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1. Definition of G/TOP

Recall TOP_n is the topological group of self-homeomorphisms of $\mathbb{R}^n$ fixing 0. Crossing with the identity on $\mathbb{R}$ gives stabilization maps for the topological group

$$\mathrm{TOP} \;:=\; \operatorname*{colim}_{n\to\infty} \mathrm{TOP}_n.$$

Recall G_n is the topological monoid of self-homotopy equivalences of S^{n-1}. Unreduced suspension of a self-map gives stabilization maps for the topological monoid

$$G \;:=\; \operatorname*{colim}_{n\to\infty} G_n.$$

Note $\pi_i G \cong \pi_i^s$ for all $i > 0$. Reversing stereographic projection $S^n - point \to \mathbb{R}^n$, one-point compactification gives inclusions $\mathrm{TOP}_n \hookrightarrow G_{n+1}$ of topological monoids. The homogenous space G/TOP of cosets fits into a fibration of topological spaces

$$\mathrm{TOP} \longrightarrow G \longrightarrow G/\mathrm{TOP}. \tag{1.1}$$

Remark 1. Via a contractible free G-space EG, it deloops to a homotopy fibration

$$G/\mathrm{TOP} \longrightarrow B\mathrm{TOP} \simeq EG/\mathrm{TOP} \longrightarrow BG = EG/G.$$

2. Its homotopy groups

Theorem 2. *For all $n > 0$, the group $\pi_n(G/\mathrm{TOP})$ is isomorphic to $L_n(1)$.*

Lemma 3. G/TOP *is 1-connected, so it's a* simple *space: π_1 acts trivially on π_*.*

Proof. The fibration (1.1) induces an exact sequence of abelian groups:

$$\begin{array}{ccccccccccc} \pi_1 O & \overset{J_1}{\rightarrowtail\!\!\twoheadrightarrow} & \pi_1^s & & & & & & & & \\ \downarrow & & \| & & & & & & & & \\ \pi_1\mathrm{TOP} & \longrightarrow & \pi_1 G & \longrightarrow & \pi_1(G/\mathrm{TOP}) & \xrightarrow{\partial} & \pi_0\mathrm{TOP} & \rightarrowtail\!\!\twoheadrightarrow & \pi_0 G & \twoheadrightarrow & \pi_0(G/\mathrm{TOP}). \end{array}$$

Recall $\pi_1 O = \mathbb{Z}/2 = \pi_1^s$ generated by the $\mathbb{C}$-Hopf map $S^3 \longrightarrow S^2$, with J_1 an isomorphism. Note $\pi_0 G = \mathbb{Z}/2 = \pi_0\mathrm{TOP}$ generated by complex conjugation the circle; the latter equality is a corollary of Kirby's Stable Homeomorphism Conjecture. □

In 4 and 5, we implicitly use the Kervaire–Milnor braid for $O \subset \mathrm{PL} \subset \mathrm{TOP} \subset G$. By Cerf, PL/O is 6-connected. By Kirby–Siebenmann, TOP/PL models $K(\mathbb{Z}/2, 3)$.

Lemma 4. $\pi_2(G/\mathrm{TOP}) \equiv \mathcal{N}_{\mathrm{TOP}}(S^2) \cong \Omega_2^{fr} = \mathbb{Z}/2.$

The generator is a degree-one normal map $T^2 \longrightarrow S^2$, with the Lie framing on T^2.

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Proof. The fibration (1.1) induces an exact sequence of abelian groups:

$$\begin{array}{ccccccccc}
\pi_2 O & \xrightarrow{J_2} & \pi_2^s & & & & & & \\
\big\downarrow\!\!\!\downarrow & & \Vert & & & & & & \\
\pi_2 \mathrm{TOP} & \longrightarrow & \pi_2 G & \longrightarrow & \pi_2(G/\mathrm{TOP}) & \xrightarrow{\partial} & \pi_1 \mathrm{TOP} & \rightarrowtail\!\!\!\twoheadrightarrow & \pi_1 G.
\end{array}$$

The isomorphism on the right uses the proof of Lemma 3. The epimorphism on the left uses $\pi_2(\mathrm{TOP}/O) = 0$. Note $\pi_2 O = 0$, and $\pi_2^s \cong \Omega_2^{fr}$ by Pontryagin–Thom. □

Lemma 5. $\pi_3(G/\mathrm{TOP}) = 0$.

Proof. The fibration (1.1) induces an exact sequence of abelian groups:

$$\begin{array}{ccccccccc}
\pi_3 O & \overset{J_3}{\twoheadrightarrow} & \pi_3^s & & & & & & \\
\big\downarrow & & \Vert & & & & & & \\
\pi_3 \mathrm{TOP} & \longrightarrow & \pi_3 G & \longrightarrow & \pi_3(G/\mathrm{TOP}) & \xrightarrow{\partial} & \pi_2 \mathrm{TOP} & \rightarrowtail & \pi_2 G.
\end{array}$$

The monomorphism on the right uses the proof of Lemma 4. The epimorphism $J_3 : \pi_3 O = \mathbb{Z} \longrightarrow \pi_3^s = \mathbb{Z}/24$ has source generated by the $\mathbb{H}$-Hopf map $S^7 \longrightarrow S^4$. □

Remark 6. In the rest of this section and in the next one, we shall use the fact that the topological surgery obstruction map σ is a homomorphism of abelian groups.

Proof of Theorem 2. We calculate the remaining homotopy groups ($n \geqslant 4$), using the topological surgery exact sequence, where the $n = 4$ case is due to Freedman:

$$\mathcal{S}_{\mathrm{TOP}}(S^n) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(S^n) \overset{\sigma}{\twoheadrightarrow} L_n(1).$$

The (split) epimorphism is due to the existence of the closed topological Milnor $4m$-manifold and Kervaire $(4m+2)$-manifold, and the vanishing of the target $L_{2k+1}(1)$. By the Generalized Poincaré Conjecture, $\mathcal{S}_{\mathrm{TOP}}(S^n) \equiv 0$. So σ is an isomorphism. By topological transversality and 3, $\mathcal{N}_{\mathrm{TOP}}(S^n) \equiv [S^n, G/\mathrm{TOP}] \equiv \pi_n(G/\mathrm{TOP})$. □

3. Application

Corollary 7. *For all* $n > 0$, $\mathcal{S}_{\mathrm{TOP}}(\mathbb{CP}_n) = L_{2n-2}(1) \oplus L_{2n-4}(1) \oplus \cdots \oplus L_2(1)$.

Proof. The topological surgery exact sequence of $\mathbb{CP}_n$ consists of abelian groups:

$$0 = L_{2n+1}(1) \xrightarrow{\partial} \mathcal{S}_{\mathrm{TOP}}(\mathbb{CP}_n) \xrightarrow{\eta} \mathcal{N}_{\mathrm{TOP}}(\mathbb{CP}_n) \overset{\sigma}{\twoheadrightarrow} L_{2n}(1) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}/2 & n \text{ odd.} \end{cases}$$

As in the proof of Theorem 2, σ is always a split epimorphism, where the splitting # is given by connect-sum of elements in $\mathcal{N}_{\mathrm{TOP}}(S^{2n})$ with the identity on $\mathbb{CP}_n$. So

$$\mathcal{N}_{\mathrm{TOP}}(\mathbb{CP}_n) = L_{2n}(1) \oplus \mathcal{S}_{\mathrm{TOP}}(\mathbb{CP}_n).$$

Consider the cofiber sequence, where the left arrow is quotient by a circle action:

$$\mathbb{C}^n \supset S^{2n-1} \xrightarrow{/U_1} \mathbb{CP}_{n-1} \longrightarrow \mathbb{CP}_n \longrightarrow S^{2n}.$$

The associated Puppe sequence consists of abelian groups:

$$[S^{2n}, G/\mathrm{TOP}] \xrightarrow{\#} [\mathbb{CP}_n, G/\mathrm{TOP}] \longrightarrow [\mathbb{CP}_{n-1}, G/\mathrm{TOP}] \longrightarrow [S^{2n-1}, G/\mathrm{TOP}] = 0.$$

Therefore, the restriction map $\mathcal{S}_{\mathrm{TOP}}(\mathbb{CP}_n) \longrightarrow \mathcal{N}_{\mathrm{TOP}}(\mathbb{CP}_{n-1})$ is an isomorphism. Geometrically, the map does transverse splitting along $\mathbb{CP}_{n-1}$ (see Exercise 15). Indeed, induction leads us down to $n = 2$ because of Freedman as $\pi_1(\mathbb{CP}_2) = 1$. $\square$

This calculation was rather special because of the recursive nature of $\mathbb{CP}_n$. In general, one needs more than the homotopy groups of G/TOP: one needs information involving the Postnikov k-invariants. This motivates the rest of the lecture.

4. Periodicity

Above, we used a homotopy-everything H-space structure on G/TOP, so that homotopy classes of maps to it form an abelian group. However, we did not use the classic H-space structure given by Whitney sum. Instead, we used the one given by the fact that G/TOP can be delooped twice. This follows from 4-fold periodicity:

Theorem 8 (Casson–Sullivan). *$A := \mathbb{Z} \times G/\mathrm{TOP}$ is homotopy equivalent to $\Omega^4 A$.*

Theorem 2 predicted this 4-periodicity: $\pi_n(A) \cong L_n(1)$ for all $n \geqslant 0$.

The aforementioned abelian group structure on topological normal invariants is

$$\mathcal{N}_{\mathrm{TOP}}(M) \;\equiv\; [M, G/\mathrm{TOP}] \;\equiv\; [M, A]_0 \;\equiv\; [M, \Omega^2(\Omega^2 A)]_0$$

where M is a nonempty connected closed topological manifold.

More, the homotopy equivalence $\pi : A \longrightarrow \Omega^4$ yields a 0-connective Ω-spectrum

$$\mathbf{L}\langle 0\rangle \;:\; A,\ \Omega^3 A,\ \Omega^2 A,\ \Omega A,\ A,\ \ldots.$$

Its 1-connective cover $\mathbf{L}\langle 1\rangle$ is a 1-connective Ω-spectrum with 0-th space G/TOP. (This yields a generalized cohomology theory.) Since it is an Ω-spectrum, note:

$$\mathcal{N}_{\mathrm{TOP}}(M) \;\equiv\; [M, G/\mathrm{TOP}] \;=\; H^0(M; \mathbf{L}\langle 1\rangle).$$

Remark 9. When M^n is oriented, a sophisticated form of Poincaré duality gives $\mathcal{N}_{\mathrm{TOP}}(M) \;\cong\; H_n(M; \mathbf{L}\langle 1\rangle)$. Then σ becomes a $\pi_1(M)$-equivariant assembly map.

5. Localization of spaces

Let S be a multiplicatively closed subset of the positive integers containing 1, so that the S-localization ring $S^{-1}\mathbb{Z}$ of the integers $\mathbb{Z}$ satisfies $\mathbb{Z} \overset{l}{\hookrightarrow} S^{-1}\mathbb{Z} \subseteq \mathbb{Q}$.

Let X be a simply connected CW complex. The *S-localization of X* is a topological space $S^{-1}X$ equipped with a map $L : X \longrightarrow S^{-1}X$ with induced isomorphisms:

$$\pi_*(X) \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} \xrightarrow[\cong]{L_* \otimes \mathrm{id}} \pi_*(S^{-1}X) \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} \xleftarrow{\mathrm{id} \otimes l} \pi_*(S^{-1}X) \otimes_{\mathbb{Z}} \mathbb{Z}.$$

Remark 10. If X is an H-space, then $S^{-1}X$ is also and $S^{-1}[-, X] \cong [-, S^{-1}X]$.

6. Its 2-local, odd-local, and rational homotopy types

We abbreviate three localizations of particular interest:

$$X_{(2)} \;:=\; \langle \text{odd primes}\rangle^{-1} X, \quad X[\tfrac{1}{2}] \;:=\; \langle 2\rangle^{-1} X, \quad X_{(0)} \;:=\; \langle \text{primes}\rangle^{-1} X.$$

Observe that X is recovered as the homotopy limit of $X_{(2)} \longrightarrow X_{(0)} \longleftarrow X[\frac{1}{2}]$.

Theorem 11 (Sullivan). *$(G/\mathrm{TOP})_{(2)} \;\simeq\; \prod_{m=1}^{\infty} K(\mathbb{Z}/2, 4m-2) \times K(\mathbb{Z}_{(2)}, 4m)$.*

Theorem 12 (Sullivan). *$(G/\mathrm{TOP})[\frac{1}{2}] \;\simeq\; BO[\frac{1}{2}]$ and $(G/\mathrm{TOP})_{(0)} \;\simeq\; BO_{(0)}$.*