Countable approximation of topological G-manifolds

by Qayum Khan (Saint Louis U)

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Conjecture (Hilbert–Smith)

Let M be a connected topological manifold. Any locally compact subgroup of $\mathrm{Homeo}(M)$, with the compact-open topology, is Lie.

Definition

Let G be a Lie group. A G-space M is a **smooth G-manifold** if it is a smooth (C^∞) manifold and the continuous homomorphism $G \longrightarrow \mathrm{Homeo}(M)$ has image in the subgroup $\mathrm{Diffeo}(M)$.

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Definition (Matumoto 1971)

A G-space X is a **G-CW complex** if it is recursively a pushout of $\coprod G \times_H D^{k+1} \longleftarrow \coprod G \times_H S^k \longrightarrow X^{(k)}$ with quotient topology.
It is called **countable** if it has countably many G-cells.

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Theorem (Illman 2000)

Let G be a Lie group. Any smooth G-manifold is equivariantly homeomorphic to a countable G-CW complex. In particular, if the manifold is compact then there are only finitely many G-cells.

Definition (Khan 2018)

Let G be a locally compact, Hausdorff, topological group.
A G-space M is a **topological G-manifold** if, for each closed subgroup H of G, the H-fixed set is a topological manifold:

$$M^H := \{x \in M \mid \forall g \in H : gx = x\}.$$

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Theorem (Khan 2018)

Let G be a compact Lie group. Any topological G-manifold is equivariantly homotopy equivalent to a countable G-CW complex. If the manifold is compact then the complex is finite-dimensional.

Corollary (Khan 2018)

Let Γ be a virtually torsionfree, discrete group. Any topological Γ-manifold with properly discontinuous action has the equivariant homotopy type of a Γ-CW complex. If the action is cocompact then the complex is finite-dimensional.

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The Alexander horned 2-sphere A is embedded in the 3-sphere. The 3-cell side has closed complement E, the *solid horned sphere*. Bing showed that $E \cup_A E$ is homeomorphic to the 3-sphere.

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- *Equivariant triangulability of open G-subsets of euclidean space* — from smooth triangulation theorem [Illman 1983]
- *Equivariant Mather trick* — Any G-space G-dominated by a countable G-CW complex is G-homotopy equivalent to one.

Definition (Palais 1961)

Let G be a locally compact, Hausdorff, topological group.
A regular Hausdorff G-space X is **proper** if each $x \in X$ has a neighborhood U, such that any $y \in X$ has a neighborhood V with $\langle U, V \rangle := \{g \in G \mid gU \cap V \neq \varnothing\}$ having compact closure in G.

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A Lie group is **linear** if it is a closed subgroup of some $GL_n(\mathbb{R})$.

Theorem (Khan 2019)

Let G be a linear Lie group. Any proper topological G-manifold has the equivariant homotopy type of a countable G-CW complex.

Definition (Bredon 1972)

Let G be a Lie group. A **locally linear** G**-manifold** M is a proper G-space such that each point x has a G-neighborhood G-homeomorphic to $G \times_{G_x} \mathbb{R}^k$, for an orthogonal representation $G_x \longrightarrow O(k)$ of its **isotropy group** $G_x := \{g \in G \mid gx = x\}$.

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The above G-**tube** is a manifold, since it fits into a vector bundle sequence $\mathbb{R}^k \to G \times_{G_x} \mathbb{R}^k \to G/G_x$ with base space a manifold.

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Let H be a closed subgroup of G. Let $x \in M^H$. Since $H \subset G_x$, notice x has a neighborhood in M^H homeomorphic to $(\mathbb{R}^k)^H$. Hence any locally linear G-manifold is a topological G-manifold.

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Corollary (Elfving 1996 dissertation — under Søren Illman)

Let G be a linear Lie group. Any proper, locally linear G-manifold has the equivariant homotopy type of a G-CW complex.

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Thank you for your attention!

- *Countable approximation of topological G-manifolds, I: compact Lie groups G* by Qayum Khan
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- *Countable approximation of topological G-manifolds, II: linear Lie groups G* by Qayum Khan
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